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*Research Article*

# The Two-Variable $(G'/G, 1/G)$-Expansion Method for Solving the Nonlinear KdV-mKdV Equation

**E. M. E. Zayed and M. A. M. Abdelaziz**

*Mathematics Department, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*

Correspondence should be addressed to E. M. E. Zayed, e.m.e.zayed@hotmail.com

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We apply the two-variable $(G'/G, 1/G)$-expansion method to construct new exact traveling wave solutions with parameters of the nonlinear $(1+1)$-dimensional KdV-mKdV equation. This method can be thought of as the generalization of the well-known $(G'/G)$-expansion method given recently by M. Wang et al. When the parameters are replaced by special values, the well-known solitary wave solutions of this equation are rediscovered from the traveling waves. It is shown that the proposed method provides a more general powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

## 1. Introduction

In the recent years, investigations of exact solutions to nonlinear PDEs play an important role in the study of nonlinear physical phenomena. Many powerful methods have been presented, such as the inverse scattering transform method [1], the Hirota method [2], the truncated Painlevē expansion method [3–6], the Backlund transform method [7, 8], the exp-function method [9–13], the tanh function method [14–17], the Jacobi elliptic function expansion method [18–20], the original $(G'/G)$-expansion method [21–29], the two-variable $(G'/G, 1/G)$-expansion method [30], and the first integral method [31]. The key idea of the original $(G'/G)$-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable $(G'/G)$ in which $G = G(\xi)$ satisfies the second ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\lambda$ and $\mu$ are constants. In this paper, we will use the two-variable $(G'/G, 1/G)$-expansion method, which can be considered as an extension of the original $(G'/G)$-expansion method. The key idea of the two-variable $(G'/G, 1/G)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in the two variables $(G'/G)$ and $(1/G)$, in which $G = G(\xi)$ satisfies a second-order linear ODE, namely $G''(\xi) + \lambda G(\xi) = \mu$, where $\lambda$ and $\mu$ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs, while the coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. Recently, Li et al. [30] have applied the two-variable $(G'/G, 1/G)$-expansion method and determined the exact solutions of Zakharov equations.

The objective of this paper is to apply the two-variable $(G'/G, 1/G)$-expansion method to find the exact traveling wave solutions of the following nonlinear (1+1)-dimensional KdV-mKdV equation:

$$u_t + \alpha u u_x + \beta u^2 u_x + u_{xxx} = 0, \tag{1.1}$$

where $\alpha$ and $\beta$ are nonzero constants. This equation may describe the wave propagation of the bound particle, sound wave, and thermal pulse. It is the most popular soliton equation and often exists in practical problems, such as fluid physics and quantum field theory. Recently, Zayed and Gepreel [23] have found the exact solutions of (1.1) using the original $(G'/G)$-expansion method.

## 2. Description of the Two-Variable $(G'/G, 1/G)$-Expansion Method

Before we describe the main steps of this method, we need the following remarks (see [30]):

*Remark 2.1.* If we consider the second-order linear ODE

$$G''(\xi) + \lambda G(\xi) = \mu \tag{2.1}$$

and set $\phi = G'/G$ and $\psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi. \tag{2.2}$$

*Remark 2.2.* If $\lambda < 0$, then the general solution of (2.1) is

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{2.3}$$

where $A_1$ and $A_2$ are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{2.4}$$

where $\sigma = A_1^2 - A_2^2$.

*Remark 2.3.* If $\lambda > 0$, then the general solution of (2.1) is

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{2.5}$$

and hence

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{2.6}$$

where $\sigma = A_1^2 + A_2^2$.

*Remark 2.4.* If $\lambda = 0$, then the general solution of (2.1) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{2.7}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{2.8}$$

Suppose we have the following NLPDEs in the form:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0, \tag{2.9}$$

where $F$ is a polynomial in $u$ and its partial derivatives. In the following, we give the main steps of the two-variable $(G'/G, 1/G)$-expansion method [30].

*Step 1.* The traveling wave variable

$$u(x,t) = u(\xi), \quad \xi = x - Vt \tag{2.10}$$

reduces (2.9) to an ODE in the form

$$P(u, u', u'', \ldots) = 0, \tag{2.11}$$

where $V$ is a constant and $P$ is a polynomial in $u$ and its total derivatives, while $' = d/d\xi$.

*Step 2.* Suppose that the solutions of (2.11) can be expressed by a polynomial in the two variables $\phi$ and $\psi$ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{2.12}$$

where $a_i (i = 0, 1, \ldots, N)$ and $b_i (i = 1, \ldots, N)$ are constants to be determined later.

*Step 3.* Determine the positive integer $N$ in (2.12) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in (2.11).

*Step 4.* Substituting (2.12) into (2.11) along with (2.2) and (2.4), the left-hand side of (2.11) can be covered into a polynomial in $\phi$ and $\psi$, in which the degree of $\psi$ is not longer than 1. Equating each coefficient of this polynomial to zero yields a system of algebraic equations that can be solved by using the Maple or Mathematica to get the values of $a_i, b_i, V, \mu, A_1, A_2$, and $\lambda$ where $\lambda < 0$. Thus, we get the exact solutions in terms of the hyperbolic functions.

*Step 5.* Similar to Step 4, substituting (2.12) into (2.11) along with (2.2) and (2.6) for $\lambda > 0$ (or (2.2) and (2.8) for $\lambda = 0$), we obtain the exact solutions of (2.11) expressed by trigonometric functions (or by rational functions), respectively.

## 3. An Application

In this section, we apply the method described in Section 2 to find the exact traveling wave solutions of the nonlinear (1+1)-dimensional KdV-mKdV Equation (1.1). To this end, we see that the traveling wave variable (2.10) permits us to convert (1.1) into the following ODE:

$$-Vu' + \alpha uu' + \beta u^2 u' + u''' = 0. \tag{3.1}$$

By balancing $u'''$ with $u^2 u'$ in (3.1), we get $N = 1$. Consequently, we get

$$u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi), \tag{3.2}$$

where $a_0$, $a_1$, and $b_1$ are constants to be determined later. There are three cases to be discussed as follows.

*Case 1.* Hyperbolic function solutions ($\lambda < 0$).

If $\lambda < 0$, substituting (3.2) into (3.1) and using (2.2) and (2.4), the left-hand side of (3.1) becomes a polynomial in $\phi$ and $\psi$. Setting the coefficients of this polynomial to zero yields a system of algebraic equations in $a_0, a_1, b_1, \mu, \sigma$, and $\lambda$ as follows:

$$\begin{aligned}
\phi^4 :\ & 12a_1\lambda^2\sigma\mu^2 + 6a_1\mu^4 + \beta a_1^3\lambda^4\sigma^2 + 2\beta a_1^3\lambda^2\sigma\mu^2 + \beta a_1^3\mu^4 + 6a_1\lambda^4\sigma^2 \\
& - 3\beta a_1 b_1^2\lambda^3\sigma - 3\beta a_1 b_1^2\lambda\mu^2 = 0, \\
\phi^3 :\ & 4\beta a_0 a_1^2\lambda^2\sigma\mu^2 + 2\beta a_0 a_1^2\lambda^4\sigma^2 + \alpha a_1^2\lambda^4\sigma^2 - \alpha b_1^2\lambda\mu^2 - 2\beta b_1^3\lambda^2\mu \\
& - \alpha b_1^2\lambda^3\sigma - 2\beta a_0 b_1^2\lambda^3\sigma + 6b_1\mu^3\lambda + 2\beta a_1^2 b_1\lambda^3\mu\sigma + 2\beta a_1^2 b_1\lambda\mu^3 + \alpha a_1^2\mu^4 \\
& + 2\alpha a_1^2\lambda^2\sigma\mu^2 + 6b_1\mu\lambda^3\sigma - 2\beta a_0 b_1^2\lambda\mu^2 + 2\beta a_0 a_1^2\mu^4 = 0,
\end{aligned}$$

$$
\begin{aligned}
&\phi^3\psi: 6b_1\lambda^4\sigma^2 - \beta b_1^3\lambda^3\sigma + 3\beta a_1^2 b_1\mu^4 + 6\beta a_1^2 b_1\lambda^2\sigma\mu^2 + 6b_1\mu^4 + 12b_1\lambda^2\sigma\mu^2\\
&\quad - \beta b_1^3\lambda\mu^2 + 3\beta a_1^2 b_1\lambda^4\sigma^2 = 0,\\
&\phi^2: -Va_1\lambda^4\sigma^2 + \alpha b_1\lambda^3 a_1\mu\sigma + \alpha a_0 a_1\mu^4 + \beta a_0^2 a_1\mu^4 + \beta a_1^3\lambda^5\sigma^2 + \beta a_1^3\lambda\mu^4\\
&\quad + 13a_1\lambda^3\sigma\mu^2 - Va_1\mu^4 - 2Va_1\lambda^2\sigma\mu^2 + 5a_1\lambda\mu^4 + 8a_1\lambda^5\sigma^2 + 2\beta a_1^3\lambda^3\sigma\mu^2\\
&\quad + \beta a_0^2 a_1\lambda^4\sigma^2 + \alpha a_0 a_1\lambda^4\sigma^2 + \alpha b_1\lambda a_1\mu^3 - 2\beta b_1^2\lambda^2 a_1\mu^2 + 2\beta a_0 b_1\lambda a_1\mu^3\\
&\quad - 4\beta b_1^2\lambda^4 a_1\sigma + 2\alpha a_0 a_1\lambda^2\sigma\mu^2 + 2\beta a_0^2 a_1\lambda^2\sigma\mu^2 + 2\beta a_0 b_1\lambda^3 a_1\mu\sigma = 0,\\
&\phi^2\psi: -\beta a_1^3\mu^5 + 7\beta b_1^2\lambda^3 a_1\mu\sigma - 12a_1\mu\lambda^4\sigma^2 - 2\beta a_1^3\mu^3\lambda^2\sigma - \beta a_1^3\mu\lambda^4\sigma^2\\
&\quad + 2\alpha a_1 b_1\mu^4 + 4\beta a_0 a_1 b_1\lambda^4\sigma^2 + 8\beta a_0 a_1 b_1\lambda^2\sigma\mu^2 - 24a_1\mu^3\lambda^2\sigma\\
&\quad + 4\alpha a_1 b_1\lambda^2\sigma\mu^2 + 4\beta a_0 a_1 b_1\mu^4 + 7\beta b_1^2\lambda a_1\mu^3 + 2\alpha a_1 b_1\lambda^4\sigma^2 - 12a_1\mu^5 = 0,\\
&\phi^1: 6b_1\mu^3\lambda^2 - \alpha b_1^2\lambda^4\sigma + 2\beta a_0 a_1^2\lambda^5\sigma^2 + \alpha a_1^2\lambda^5\sigma^2 + \alpha a_1^2\lambda\mu^4 + 2\alpha a_1^2\lambda^3\sigma\mu^2\\
&\quad - \alpha b_1^2\lambda^2\mu^2 - 2\beta b_1^3\lambda^3\mu + 2\beta a_1^2 b_1\lambda^2\mu^3 + 2\beta a_0 a_1^2\lambda\mu^4 + 6b_1\mu\lambda^4\sigma - 2\beta a_0 b_1^2\lambda^2\mu^2\\
&\quad + 4\beta a_0 a_1^2\lambda^3\sigma\mu^2 - 2\beta a_0 b_1^2\lambda^4\sigma + 2\beta a_1^2 b_1\lambda^4\mu\sigma = 0,\\
&\phi^1\psi: -\beta b_1^3\lambda^4\sigma + \alpha a_0 b_1\mu^4 + \beta a_0^2 b_1\mu^4 - Vb_1\lambda^4\sigma^2 + 3\beta b_1^3\lambda^2\mu^2 - 2b_1\lambda^3\sigma\mu^2\\
&\quad - 2\beta a_0 a_1^2\mu^5 + 2\alpha b_1^2\lambda\mu^3 - \alpha a_1^2\mu^5 - Vb_1\mu^4 + 5b_1\lambda^5\sigma^2 - 7b_1\lambda\mu^4 + \alpha a_0 b_1\lambda^4\sigma^2\\
&\quad + \beta a_0^2 b_1\lambda^4\sigma^2 + 2\alpha a_0 b_1\lambda^2\sigma\mu^2 - 2\alpha a_1^2\mu^3\lambda^2\sigma - \alpha a_1^2\mu\lambda^4\sigma^2 - 2Vb_1\lambda^2\sigma\mu^2\\
&\quad - 2\beta a_0 a_1^2\mu\lambda^4\sigma^2 - 4\beta a_0 a_1^2\mu^3\lambda^2\sigma + 2\beta a_0^2 b_1\lambda^2\sigma\mu^2 + 2\beta a_1^2 b_1\lambda^5\sigma^2 - 2\beta a_1^2 b_1\lambda\mu^4\\
&\quad + 2\alpha b_1^2\lambda^3\mu\sigma + 4\beta a_0 b_1^2\lambda^3\mu\sigma + 4\beta a_0 b_1^2\lambda\mu^3 = 0,\\
&\phi^0: a_1\lambda^4\sigma\mu^2 - Va_1\lambda^5\sigma^2 - Va_1\lambda\mu^4 + \alpha a_0 a_1\lambda^5\sigma^2 + \alpha a_0 a_1\lambda\mu^4 - a_1\lambda^2\mu^4\\
&\quad + 2a_1\lambda^6\sigma^2 + \alpha b_1\lambda^4 a_1\mu\sigma + \beta b_1^2\lambda^3 a_1\mu^2 - \beta b_1^2\lambda^5 a_1\sigma + \beta a_0^2 a_1\lambda^5\sigma^2 - 2Va_1\lambda^3\sigma\mu^2\\
&\quad + 2\beta a_0^2 a_1\lambda^3\sigma\mu^2 + \alpha b_1\lambda^2 a_1\mu^3 + 2\beta a_0 b_1\lambda^2 a_1\mu^3 + 2\alpha a_0 a_1\lambda^3\sigma\mu^2 + 2\beta a_0 b_1\lambda^4 a_1\mu\sigma\\
&\quad + \beta a_0^2 a_1\lambda\mu^4 = 0,\\
&\phi^0\psi: -\alpha b_1 a_1\lambda\mu^4 - \beta a_0^2 a_1\mu\lambda^4\sigma^2 - \beta b_1^2\lambda^2 a_1\mu^3 + a_1\mu^5\lambda - \alpha a_0 a_1\mu^5 + 3\beta b_1^2\lambda^4 a_1\mu\sigma\\
&\quad + \alpha b_1 a_1\lambda^5\sigma^2 - 2\alpha a_0 a_1\mu^3\lambda^2\sigma - \beta a_0^2 a_1\mu^5 + 2Va_1\mu^3\lambda^2\sigma + Va_1\mu\lambda^4\sigma^2 + Va_1\mu^5\\
&\quad - 2\beta a_0^2 a_1\mu^3\lambda^2\sigma - 4a_1\mu^3\lambda^3\sigma + 2\beta a_0 b_1 a_1\lambda^5\sigma^2 - 2\beta a_0 b_1 a_1\lambda\mu^4 - 5a_1\mu\lambda^5\sigma^2\\
&\quad - \alpha a_0 a_1\mu\lambda^4\sigma^2 = 0.
\end{aligned}
\tag{3.3}
$$

Solving the algebraic equations (3.3) by the Maple or Mathematica, we get the following results.

*Result 1.* We have

$$a_0 = -\frac{1}{2\beta}\left(\alpha \pm \mu\sqrt{\frac{6\beta\lambda}{\lambda^2\sigma+\mu^2}}\right), \qquad a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{3(\lambda^2\sigma+\mu^2)}{2\beta\lambda}},$$
$$V = -\frac{4\sigma\beta\lambda^3+\sigma\alpha^2\lambda^2-2\beta\lambda\mu^2+\alpha^2\mu^2}{4\beta(\lambda^2\sigma+\mu^2)}, \qquad \sigma = A_1^2 - A_2^2. \tag{3.4}$$

From (2.3) and (3.2) and (3.4), we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-1}{2\beta}\left(\alpha \pm \mu\sqrt{\frac{6\beta\lambda}{\lambda^2\sigma+\mu^2}}\right) \pm \sqrt{\frac{3(\lambda^2\sigma+\mu^2)}{2\beta\lambda}}$$
$$\times\left(\frac{1}{A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\mu/\lambda}\right), \tag{3.5}$$

where

$$\xi = x + \left(\frac{4\sigma\beta\lambda^3+\sigma\alpha^2\lambda^2-2\beta\lambda\mu^2+\alpha^2\mu^2}{4\beta(\lambda^2\sigma+\mu^2)}\right)t. \tag{3.6}$$

In particular, by setting $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ in (3.5), we have the solitary solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-3\lambda}{2\beta}}\operatorname{sech}\left(\xi\sqrt{-\lambda}\right), \tag{3.7}$$

while, if $A_2 = 0$, $A_1 > 0$, and $\mu = 0$, then we have the solitary solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{3\lambda}{2\beta}}\operatorname{csch}\left(\xi\sqrt{-\lambda}\right). \tag{3.8}$$

*Result 2.* We have

$$a_0 = \frac{-\alpha}{2\beta}, \qquad a_1 = \pm\sqrt{\frac{-3}{2\beta}}, \qquad b_1 = \pm\sqrt{\frac{6(\lambda^2\sigma+\mu^2)}{\beta\lambda}},$$
$$V = \frac{2\beta\lambda-\alpha^2}{4\beta}, \qquad \sigma = A_1^2 - A_2^2. \tag{3.9}$$

In this result, we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \frac{1}{\sqrt{\beta}\left[A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \mu/\lambda\right]} \times \left\{\sqrt{\frac{3\lambda}{2}}\left[A_1 \cosh\left(\xi\sqrt{-\lambda}\right) + A_2 \sinh\left(\xi\sqrt{-\lambda}\right)\right] + \sqrt{\frac{6(\lambda^2\sigma + \mu^2)}{\lambda}}\right\}, \quad (3.10)$$

where

$$\xi = x - \left(\frac{2\beta\lambda - \alpha^2}{4\beta}\right)t. \quad (3.11)$$

In particular, by setting $A_1 = 0, A_2 > 0$, and $\mu = 0$ in (3.10), we have the solitary solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{3\lambda}{2\beta}} \tanh\left(\xi\sqrt{-\lambda}\right) \pm \sqrt{\frac{-6\lambda}{\beta}} \operatorname{sech}\left(\xi\sqrt{-\lambda}\right), \quad (3.12)$$

while, if $A_2 = 0, A_1 > 0$, and $\mu = 0$, then we have the solitary solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{3\lambda}{2\beta}} \coth\left(\xi\sqrt{-\lambda}\right) \pm \sqrt{\frac{6\lambda}{\beta}} \operatorname{csch}\left(\xi\sqrt{-\lambda}\right). \quad (3.13)$$

*Case 2.* Trigonometric function solutions ($\lambda > 0$).

If $\lambda > 0$, substituting (3.2) into (3.1) and using (2.2) and (2.6), we get a polynomial in $\phi$ and $\psi$. We vanish each coefficient of this polynomial to get the following algebraic equations.

$$\begin{aligned}
\phi^4 &: -\beta a_1^3\mu^4 - \beta a_1^3\lambda^4\sigma^2 + 2\beta a_1^3\lambda^2\sigma\mu^2 - 6a_1\mu^4 - 6a_1\lambda^4\sigma^2 + 12a_1\lambda^2\sigma\mu^2 \\
&\quad - 3\beta a_1 b_1^2\lambda^3\sigma + 3\beta a_1 b_1^2\lambda\mu^2 = 0, \\
\phi^3 &: 6b_1\mu\lambda^3\sigma + \alpha b_1^2\lambda\mu^2 - 2\beta a_1^2 b_1\lambda\mu^3 - 6b_1\mu^3\lambda + 4\beta a_0 a_1^2\lambda^2\sigma\mu^2 + 2\beta b_1^3\lambda^2\mu \\
&\quad - 2\beta a_0 b_1^2\lambda^3\sigma + 2\beta a_0 b_1^2\lambda\mu^2 - \alpha a_1^2\lambda^4\sigma^2 - \alpha a_1^2\mu^4 - \alpha b_1^2\lambda^3\sigma - 2\beta a_0 a_1^2\lambda^4\sigma^2 \\
&\quad - 2\beta a_0 a_1^2\mu^4 + 2\alpha a_1^2\lambda^2\sigma\mu^2 + 2\beta a_1^2 b_1\lambda^3\mu\sigma = 0,
\end{aligned}$$

$$
\begin{aligned}
\phi^3\psi &: -3\beta a_1^2 b_1 \mu^4 + 6\beta a_1^2 b_1 \lambda^2 \sigma \mu^2 - 6 b_1 \lambda^4 \sigma^2 - 6 b_1 \mu^4 - 3\beta a_1^2 b_1 \lambda^4 \sigma^2 \\
&\quad + 12 b_1 \lambda^2 \sigma \mu^2 - \beta b_1^3 \lambda^3 \sigma + \beta b_1^3 \lambda \mu^2 = 0, \\
\phi^2 &: V a_1 \lambda^4 \sigma^2 - \alpha a_0 a_1 \mu^4 - \beta a_0^2 a_1 \mu^4 - \beta a_1^3 \lambda^5 \sigma^2 - \beta a_1^3 \lambda \mu^4 - 5 a_1 \lambda \mu^4 \\
&\quad + 13 a_1 \lambda^3 \sigma \mu^2 + V a_1 \mu^4 - 8 a_1 \lambda^5 \sigma^2 - \beta a_0^2 a_1 \lambda^4 \sigma^2 + 2\beta a_1^3 \lambda^3 \sigma \mu^2 - \alpha a_0 a_1 \lambda^4 \sigma^2 \\
&\quad + 2\beta a_0^2 a_1 \lambda^2 \sigma \mu^2 + 2\beta b_1^2 \lambda^2 a_1 \mu^2 + \alpha b_1 \lambda^3 a_1 \mu \sigma - \alpha b_1 \lambda a_1 \mu^3 + 2\beta a_0 b_1 \lambda^3 a_1 \mu \sigma \\
&\quad + 2\alpha a_0 a_1 \lambda^2 \sigma \mu^2 - 2 V a_1 \lambda^2 \sigma \mu^2 - 4\beta b_1^2 \lambda^4 a_1 \sigma - 2\beta a_0 b_1 \lambda a_1 \mu^3 = 0, \\
\phi^2\psi &: \beta a_1^3 \mu^5 - 24 a_1 \mu^3 \lambda^2 \sigma + 12 a_1 \mu^5 + 12 a_1 \mu \lambda^4 \sigma^2 + \beta a_1^3 \mu \lambda^4 \sigma^2 + 7\beta b_1^2 \lambda^3 a_1 \mu \sigma \\
&\quad - 4\beta a_0 a_1 b_1 \lambda^4 \sigma^2 - 2\beta a_1^3 \mu^3 \lambda^2 \sigma - 4\beta a_0 a_1 b_1 \mu^4 - 2\alpha a_1 b_1 \mu^4 + 4\alpha a_1 b_1 \lambda^2 \sigma \mu^2 \\
&\quad + 8\beta a_0 a_1 b_1 \lambda^2 \sigma \mu^2 - 7\beta b_1^2 \lambda a_1 \mu^3 - 2\alpha a_1 b_1 \lambda^4 \sigma^2 = 0, \\
\phi^1 &: -\alpha b_1^2 \lambda^4 \sigma - \alpha a_1^2 \lambda^5 \sigma^2 - \alpha a_1^2 \lambda \mu^4 - 2\beta a_0 a_1^2 \lambda \mu^4 - 2\beta a_1^2 b_1 \lambda^2 \mu^3 + \alpha b_1^2 \lambda^2 \mu^2 \\
&\quad + 2\beta b_1^3 \lambda^3 \mu + 2\beta a_1^2 b_1 \lambda^4 \mu \sigma + 6 b_1 \mu \lambda^4 \sigma - 6 b_1 \mu^3 \lambda^2 + 4\beta a_0 a_1^2 \lambda^3 \sigma \mu^2 - 2\beta a_0 a_1^2 \lambda^5 \sigma^2 \\
&\quad + 2\beta a_0 b_1^2 \lambda^2 \mu^2 - 2\beta a_0 b_1^2 \lambda^4 \sigma + 2\alpha a_1^2 \lambda^3 \sigma \mu^2 = 0, \\
\phi^1\psi &: -\beta b_1^3 \lambda^4 \sigma - \beta a_0^2 b_1 \mu^4 + V b_1 \lambda^4 \sigma^2 - \alpha a_0 b_1 \mu^4 - 3\beta b_1^3 \lambda^2 \mu^2 - 2 b_1 \lambda^3 \sigma \mu^2 \\
&\quad + 2\beta a_0 a_1^2 \mu^5 - 2\alpha b_1^2 \lambda \mu^3 + V b_1 \mu^4 + \alpha a_1^2 \mu^5 - 5 b_1 \lambda^5 \sigma^2 + 7 b_1 \lambda \mu^4 - \beta a_0^2 b_1 \lambda^4 \sigma^2 \\
&\quad - \alpha a_0 b_1 \lambda^4 \sigma^2 + \alpha a_1^2 \mu \lambda^4 \sigma^2 + 2\beta a_0^2 b_1 \lambda^2 \sigma \mu^2 + 2\beta a_0 a_1^2 \mu \lambda^4 \sigma^2 - 4\beta a_0 a_1^2 \mu^3 \lambda^2 \sigma \\
&\quad - 2\alpha a_1^2 \mu^3 \lambda^2 \sigma - 2 V b_1 \lambda^2 \sigma \mu^2 + 2\alpha a_0 b_1 \lambda^2 \sigma \mu^2 - 2\beta a_1^2 b_1 \lambda^5 \sigma^2 + 2\beta a_1^2 b_1 \lambda \mu^4 \\
&\quad + 2\alpha b_1^2 \lambda^3 \mu \sigma + 4\beta a_0 b_1^2 \lambda^3 \mu \sigma - 4\beta a_0 b_1^2 \lambda \mu^3 = 0, \\
\phi^0 &: a_1 \lambda^4 \sigma \mu^2 + V a_1 \lambda^5 \sigma^2 + V a_1 \lambda \mu^4 + \alpha b_1 \lambda^4 a_1 \mu \sigma - \beta b_1^2 \lambda^3 a_1 \mu^2 + a_1 \lambda^2 \mu^4 \\
&\quad - 2 a_1 \lambda^6 \sigma^2 - \beta a_0^2 a_1 \lambda^5 \sigma^2 - \beta a_0^2 a_1 \lambda \mu^4 - \alpha a_0 a_1 \lambda^5 \sigma^2 - \alpha a_0 a_1 \lambda \mu^4 + 2\beta a_0^2 a_1 \lambda^3 \sigma \mu^2 \\
&\quad - 2 V a_1 \lambda^3 \sigma \mu^2 - \alpha b_1 \lambda^2 a_1 \mu^3 + 2\alpha a_0 a_1 \lambda^3 \sigma \mu^2 - 2\beta a_0 b_1 \lambda^2 a_1 \mu^3 - \beta b_1^2 \lambda^5 a_1 \sigma \\
&\quad + 2\beta a_0 b_1 \lambda^4 a_1 \mu \sigma = 0, \\
\phi^0\psi &: -V a_1 \mu^5 - V a_1 \mu \lambda^4 \sigma^2 - a_1 \mu^5 \lambda - \alpha b_1 a_1 \lambda^5 \sigma^2 + \alpha b_1 a_1 \lambda \mu^4 + \beta a_0^2 a_1 \mu^5 \\
&\quad + \beta b_1^2 \lambda^2 a_1 \mu^3 + 5 a_1 \mu \lambda^5 \sigma^2 + \beta a_0^2 a_1 \mu \lambda^4 \sigma^2 + 3\beta b_1^2 \lambda^4 a_1 \mu \sigma + \alpha a_0 a_1 \mu^5 + 2 V a_1 \mu^3 \lambda^2 \sigma \\
&\quad - 4 a_1 \mu^3 \lambda^3 \sigma + \alpha a_0 a_1 \mu \lambda^4 \sigma^2 - 2\beta a_0 b_1 a_1 \lambda^5 \sigma^2 + 2\beta a_0 b_1 a_1 \lambda \mu^4 - 2\alpha a_0 a_1 \mu^3 \lambda^2 \sigma \\
&\quad - 2\beta a_0^2 a_1 \mu^3 \lambda^2 \sigma = 0.
\end{aligned}
\tag{3.14}
$$

Solving the algebraic equations (3.14) by the Maple or Mathematica, we obtain the following results.

*Result 1.* We have

$$a_0 = \frac{-1}{\beta}\left(\frac{\alpha}{2} \pm 3\mu\sqrt{\frac{\beta\lambda}{6(\mu^2-\lambda^2\sigma)}}\right), \qquad a_1 = 0, \qquad b_1 = \pm\sqrt{\frac{6(\mu^2-\lambda^2\sigma)}{\beta\lambda}},$$
$$V = -\frac{4\sigma\beta\lambda^3 + \sigma\alpha^2\lambda^2 + 2\beta\lambda\mu^2 - \alpha^2\mu^2}{4\beta(\lambda^2\sigma - \mu^2)}, \qquad \sigma = A_1^2 + A_2^2. \tag{3.15}$$

From (2.5), (3.2), and (3.15), we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-1}{\beta}\left(\frac{\alpha}{2} \pm 3\mu\sqrt{\frac{\beta\lambda}{6(\mu^2-\lambda^2\sigma)}}\right) \pm \sqrt{\frac{6(\mu^2-\lambda^2\sigma)}{\beta\lambda}}$$
$$\times\left(\frac{1}{A_1\sin\left(\xi\sqrt{\lambda}\right) + A_2\cos\left(\xi\sqrt{\lambda}\right) + \mu/\lambda}\right), \tag{3.16}$$

where

$$\xi = x + \left(\frac{4\sigma\beta\lambda^3 + \sigma\alpha^2\lambda^2 + 2\beta\lambda\mu^2 - \alpha^2\mu^2}{4\beta(\lambda^2\sigma - \mu^2)}\right)t. \tag{3.17}$$

In particular, by setting $A_1 = 0, A_2 > 0$, and $\mu = 0$ in (3.16), we have the periodic solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-6\lambda}{\beta}}\sec\left(\xi\sqrt{\lambda}\right), \tag{3.18}$$

while, if $A_2 = 0, A_1 > 0$, and $\mu = 0$, then we have the periodic solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-6\lambda}{\beta}}\csc\left(\xi\sqrt{\lambda}\right). \tag{3.19}$$

*Result 2.* We have

$$a_0 = \frac{-\alpha}{2\beta}, \qquad a_1 = \pm\sqrt{\frac{-3}{2\beta}}, \qquad b_1 = \pm\sqrt{\frac{3(\mu^2-\lambda^2\sigma)}{2\beta\lambda}},$$
$$V = \frac{2\beta\lambda - \alpha^2}{4\beta}, \qquad \sigma = A_1^2 + A_2^2. \tag{3.20}$$

In this result, we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-3}{2\beta}} \left( \frac{1}{A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \mu/\lambda} \right) \times \left\{ \sqrt{\lambda}\left[A_1 \cos\left(\xi\sqrt{\lambda}\right) - A_2 \sin\left(\xi\sqrt{\lambda}\right)\right] + \sqrt{\frac{\lambda^2\sigma - \mu^2}{\lambda}} \right\}, \tag{3.21}$$

where

$$\xi = x - \frac{2\beta\lambda - \alpha^2}{4\beta} t. \tag{3.22}$$

In particular, by setting $A_1 = 0, A_2 > 0$, and $\mu = 0$ in (3.21), we have the periodic solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-3\lambda}{2\beta}} \left[- \tan\left(\xi\sqrt{\lambda}\right) + \sec\left(\xi\sqrt{\lambda}\right)\right], \tag{3.23}$$

while, if $A_2 = 0, A_1 > 0$, and $\mu = 0$, then we have the periodic solution

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-3\lambda}{2\beta}} \left[\cot\left(\xi\sqrt{\lambda}\right) + \csc\left(\xi\sqrt{\lambda}\right)\right]. \tag{3.24}$$

*Case 3.* Rational function solutions ($\lambda = 0$).

If $\lambda = 0$, substituting (3.2) into (3.1) and using (2.2) and (2.8), we get a polynomial in $\phi$ and $\psi$. Setting each coefficient of this polynomial to zero yields the following algebraic equations:

$$\begin{aligned}
\phi^4 &: \beta a_1^3 A_1^4 - 4\beta a_1^3 A_1^2 \mu A_2 - 24 a_1 A_1^2 \mu A_2 + 24 a_1 \mu^2 A_2^2 + 4\beta a_1^3 \mu^2 A_2^2 + 6 a_1 A_1^4 \\
&\quad + 3\beta a_1 b_1^2 A_1^2 - 6\beta a_1 b_1^2 \mu A_2 = 0, \\
\phi^3 &: -4\beta a_0 b_1^2 \mu A_2 - 6 b_1 \mu A_1^2 + 4\alpha a_1^2 \mu^2 A_2^2 + 2\beta a_0 a_1^2 A_1^4 + 8\beta a_0 a_1^2 \mu^2 A_2^2 - 2\alpha b_1^2 \mu A_2
\end{aligned}$$

$$
\begin{aligned}
&+12b_1\mu^2A_2+2\beta a_0b_1^2A_1^2-2\beta b_1^3\mu+\alpha a_1^2A_1^4-2\beta a_1^2b_1\mu A_1^2-8\beta a_0a_1^2A_1^2\mu A_2+\alpha b_1^2A_1^2\\
&+4\beta a_1^2b_1\mu^2A_2-4\alpha a_1^2A_1^2\mu A_2=0,\\
\phi^3\psi:\ &12\beta a_1^2b_1\mu^2A_2^2+6b_1A_1^4+\beta b_1^3A_1^2-2\beta b_1^3\mu A_2+3\beta a_1^2b_1A_1^4-12\beta a_1^2b_1A_1^2\mu A_2\\
&-24b_1A_1^2\mu A_2+24b_1\mu^2A_2^2=0,\\
\phi^2:\ &\alpha a_0a_1A_1^4+\beta a_0^2a_1A_1^4+2\beta b_1^2a_1\mu^2-4Va_1\mu^2A_2^2-Va_1A_1^4+3a_1\mu^2A_1^2\\
&-6a_1\mu^3A_2-\alpha b_1a_1\mu A_1^2+4Va_1A_1^2\mu A_2-4\beta a_0^2a_1A_1^2\mu A_2+4\beta a_0^2a_1\mu^2A_2^2\\
&-4\alpha a_0a_1A_1^2\mu A_2+4\alpha a_0a_1\mu^2A_2^2+2\alpha b_1a_1\mu^2A_2-2\beta a_0b_1a_1\mu A_1^2\\
&+4\beta a_0b_1a_1\mu^2A_2=0,\\
\phi^2\psi:\ &-48a_1\mu^3A_2^2+48a_1\mu^2A_1^2A_2-12a_1\mu A_1^4-7\beta b_1^2a_1\mu A_1^2+4\beta a_1^3\mu^2A_1^2A_2\\
&-\beta a_1^3\mu A_1^4+8\alpha a_1b_1\mu^2A_2^2+14\beta b_1^2a_1\mu^2A_2+2\alpha a_1b_1A_1^4-8\alpha a_1b_1A_1^2\mu A_2-4\beta a_1^3\mu^3A_2^2\\
&+4\beta a_0a_1b_1A_1^4-16\beta a_0a_1b_1A_1^2\mu A_2+16\beta a_0a_1b_1\mu^2A_2^2=0,\\
\phi^1\psi:\ &\alpha a_0b_1A_1^4-\alpha a_1^2\mu A_1^4+\beta a_0^2b_1A_1^4-4Vb_1\mu^2A_2^2-4\alpha a_1^2\mu^3A_2^2-2\alpha b_1^2\mu A_1^2\\
&+4\alpha b_1^2\mu^2A_2-Vb_1A_1^4+4\beta b_1^3\mu^2+12b_1\mu^2A_1^2-24b_1\mu^3A_2+4Vb_1A_1^2\mu A_2\\
&+4\alpha a_0b_1\mu^2A_2^2-4\alpha a_0b_1a_1^2\mu A_2-2\beta a_0a_1^2\mu A_1^4-4\beta a_0^2b_1A_1^2\mu A_2+4\beta a_0^2b_1\mu^2A_2^2\\
&+4\alpha a_1^2\mu^2A_1^2A_2-4\beta a_0b_1^2\mu A_1^2+8\beta a_0b_1^2\mu^2A_2+4\beta a_1^2b_1\mu^2A_1^2+8\beta a_0a_1^2\mu^2A_1^2A_2\\
&-8\beta a_0a_1^2\mu^3A_2^2-8\beta a_1^2b_1\mu^3A_2=0,\\
\phi^0\psi:\ &Va_1\mu A_1^4-4\beta b_1^2a_1\mu^3+4Va_1\mu^3A_2^2-6a_1\mu^3A_1^2+12a_1\mu^4A_2+4\alpha a_0a_1\mu^2A_1^2A_2\\
&-\alpha a_0a_1\mu A_1^4-\beta a_0^2a_1\mu A_1^4-4\alpha a_0a_1\mu^3A_2^2+4\beta a_0^2a_1\mu^2A_1^2A_2-4\beta a_0^2a_1\mu^3A_2^2\\
&-4Va_1\mu^2A_1^2A_2-4\alpha b_1a_1\mu^3A_2+4\beta a_0b_1a_1\mu^2A_1^2+2\alpha b_1a_1\mu^2A_1^2\\
&-8\beta a_0b_1a_1\mu^3A_2=0.
\end{aligned}
\tag{3.25}
$$

Solving the algebraic equations (3.25) by the Maple or Mathematica, we obtain the following results.

*Result 1.* We have

$$
\begin{gathered}
a_0=\frac{-1}{\beta}\left(\frac{\alpha}{2}\pm 3\mu\sqrt{\frac{\beta}{6(2\mu A_2-A_1^2)}}\right),\qquad a_1=0,\qquad b_1=\pm\sqrt{\frac{6(2\mu A_2-A_1^2)}{\beta}},\\
V=-\frac{\alpha^2A_1^2+6\beta\mu^2-2\alpha^2\mu A_2}{4\beta(A_1^2-2\mu A_2)}.
\end{gathered}
\tag{3.26}
$$

From (2.7), (3.2) and (3.26), we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-1}{\beta}\left(\frac{\alpha}{2} \pm 3\mu\sqrt{\frac{\beta}{6(2\mu A_2 - A_1^2)}}\right) \pm \sqrt{\frac{6(2\mu A_2 - A_1^2)}{\beta}}\left(\frac{1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right), \tag{3.27}$$

where

$$\xi = x + \left(\frac{\alpha^2 A_1^2 + 6\beta\mu^2 - 2\alpha^2\mu A_2}{4\beta(A_1^2 - 2\mu A_2)}\right)t. \tag{3.28}$$

*Result 2.* We have

$$a_0 = \frac{-\alpha}{2\beta}, \qquad a_1 = \pm\sqrt{\frac{-3}{2\beta}}, \qquad b_1 = \pm\sqrt{\frac{3(2\mu A_2 - A_1^2)}{2\beta}}$$
$$V = \frac{-\alpha^2}{4\beta}. \tag{3.29}$$

In this result, we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = \frac{-\alpha}{2\beta} \pm \sqrt{\frac{-3}{2\beta}}\left(\frac{\mu\xi + A_1 + \sqrt{A_1^2 - 2\mu A_2}}{(\mu/2)\xi^2 + A_1\xi + A_2}\right), \tag{3.30}$$

where

$$\xi = x + \frac{\alpha^2}{4\beta}t. \tag{3.31}$$

*Remark 3.1.* All solutions of this paper have been checked with Maple by putting them back into the original equation (1.1).

## 4. Conclusions

In this paper, the $(G'/G, 1/G)$-expansion method was employed to obtain some new as well as some known solutions of a selected nonlinear equation, namely, the (1+1)-dimensional KdV-mKdV equation. As the two parameters $A_1$ and $A_2$ take special values, we obtain the solitary wave solutions. When $\mu = 0$ and $b_i = 0$ in (2.1) and (2.12), the two-variable $(G'/G, 1/G)$-expansion method reduces to the original $(G'/G)$-expansion method. So, the two-variable $(G'/G, 1/G)$-expansion method is an extension of the original $(G'/G)$-expansion method. The proposed method in this paper is more effective and more general than the original $(G'/G)$-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the two-variable $(G'/G, 1/G)$-expansion method over the original

$(G'/G)$-expansion method is that the solutions using the first method recover the solutions using the second one.

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