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Research Article

The Analytical and a Higher-Accuracy Numerical Solution of a Free Boundary Problem in a Class of Discontinuous Functions

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A new method is suggested for obtaining the exact and numerical solutions of the initial-boundary value problem for a nonlinear parabolic type equation in the domain with the free boundary. With this aim, a special auxiliary problem having some advantages over the main problem and being equivalent to the main problem in a definite sense is introduced. The auxiliary problem allows us to obtain the weak solution in a class of discontinuous functions. Moreover, on the basis of the auxiliary problem a higher-resolution numerical method is developed so that the solution accurately describes all physical properties of the problem. In order to extract the significance of the numerical solutions obtained by using the suggested auxiliary problem, some computer experiments are carried out.

1. Introduction

It is known that many practical problems such as distribution of heat waves, melting glaciers, and filtration of a gas in a porous medium, and so forth, are described by nonlinear equations of the parabolic type.

In [1], at first the effect of localization of the solution of the equation describing the motion of perfect gas in a porous medium is observed and the solution in the traveling wave form is structured. Then, the mentioned properties of the solution for the nonlinear parabolic type equation are studied in [2, 3], and so forth.

These problems are also called free boundary problems. Therefore, it is necessary to obtain the moving unknown boundary together with the solution of a differential problem. Its nature raises several difficulties for finding analytical as well as numerical solutions of this problem.

The questions of the existence and uniqueness of the solutions of the free boundary problems are studied in [4, 5]. In [5], Oleĭnik introduced the notion of a generalized solution of the Stefan problem whose uniqueness and existence were guaranteed in the class of measurable bounded functions. In [4], Kamenomostskaya considered the classical quasilinear heat conduction equation and constructed the generalized solution by the use of an explicit difference scheme.

In the literature there are some numerical algorithms (homogeneous schemes) which are approximated by finite differences of the differential problem without taking into account the properties occurring in the exact solution [6–8].

2. Traveling Wave Solution of the Main Problem

We consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2}, \quad \text{in } R_+^2 \tag{2.1}$$

with following initial

$$u(x,0) = u_0(x) = 0, \quad \text{in } I = [0,\infty) \tag{2.2}$$

and boundary

$$u(0,t) = u_1(t) = \mu_0 t^n, \quad t > 0 \tag{2.3}$$

conditions, where $R_+^2 = I \times [0,T)$. Here, μ_0 and n are real known constants. In order to study the properties of the exact solution of the problem (2.1)–(2.3) for the sake of simplicity the case $\varphi(u) = u^\sigma$ is considered. It is clear that the function $\varphi(u)$ satisfies the following conditions:

(i) $\varphi(u) \in C^2(R_+^2)$,

(ii) $\varphi'(u) \geq 0$, for $u \geq 0$ and $\sigma \geq 2$,

(iii) for $\sigma \geq 2$, $\varphi''(u)$ have alternative signs on the domain when $u(x,t) \neq 0$.

It is easily shown that the problem (2.1)–(2.3) has the solution in the traveling wave form as

$$u(x,t) = \begin{cases} \left(D\dfrac{\sigma-1}{\sigma}\right)^{1/(\sigma-1)} (Dt-x)^{1/(\sigma-1)}, & 0 < x < Dt, \\ 0, & x \geq Dt. \end{cases} \tag{2.4}$$

Via simple calculation we get the following:

(1) the function $u(x,t)$ and $w(x,t) = -\partial u^\sigma(x,t)/\partial x = Du(x,t)$ are continuous in $D_{\ell(t)} = \{(x,t) \mid 0 \leq x \leq Dt,\ 0 \leq t \leq T\}$, but u_t and u_x do not exist when $\sigma > 2$;

(2) when $\sigma = 2$, u_t and u_x are finite;

(3) for $1 < \sigma < 3/2$, all derivatives u_t, u_x, and u_{xx} exist;

(4) for $3/2 < \sigma < 2$ the u_t and u_x exist, but u_{xx} does not exist.

Therefore when $\sigma > 2$, we must only consider the weak solution for the problem (2.1)–(2.3). As it is seen from the formula (2.4), when $n = 1/(\sigma - 1)$, the solution is equal to zero at $x = \ell(t) = Dt$. Here, $D = \pm\sqrt{(\sigma/(\sigma-1))\mu_0^{\sigma-1}}$ is the speed of the front of the traveling wave and

$$u(\ell(t), t) = 0. \tag{2.5}$$

Taking into account (2.5), we get

$$\frac{dx}{dt} = -\frac{\partial u/\partial t}{\partial u/\partial x} = \frac{D\partial u(x,t)/\partial x}{\partial u/\partial x} = D. \tag{2.6}$$

It is clear that when $u = 0$, (2.1) degenerates to the first-order equation. The weak solution is defined as follows.

Definition 2.1. A nonnegative function $u(x,t)$ which is satisfying the initial condition (2.2) and boundary conditions (2.3) and (2.5) is called a weak solution of the problem (2.1)–(2.3), and (2.5), if the following integral relation

$$\iint_{D_{\ell(t)}} \left\{ u(x,t)\frac{\partial f(x,t)}{\partial t} - \frac{\partial u^{\sigma}(x,t)}{\partial x}\frac{\partial f(x,t)}{\partial x} \right\} dx\,dt = 0 \tag{2.7}$$

holds for every test functions $f(x,t)$ from $\overset{o}{C}_{1,1}(D_{\ell(t)})$ and $f(x,T) = 0$.

Because the function $u(x,t)$ and $w(x,t) = -\partial u^{\sigma}(x,t)/\partial x$ are continuous in the domain $D_{\ell(t)}$, the integral involving (2.7) exists in the Riemann sense.

Now, we show that the solution defined by the formula (2.4) satisfies the integral equality (2.7); that is, $u(x,t)$ is the weak solution of the problem (2.1)–(2.3).

According to the definition of the weak solution, we can write

$$\begin{aligned} 0 &= \int_0^T \int_0^{Dt} \left\{ u(x,t)\frac{\partial f(x,t)}{\partial t} + Du(x,t)\frac{\partial f(x,t)}{\partial x} \right\} dx\,dt \\ &= \int_0^T \int_0^{Dt} u(x,t)\frac{\partial f(x,t)}{\partial t} dx\,dt + D\int_0^T \int_0^{Dt} u(x,t)\frac{\partial f(x,t)}{\partial x} dx\,dt. \end{aligned} \tag{2.8}$$

Changing the order of integration in the first integral and then applying the integration by parts to the inner integrals of the first and second terms in the last expression with respect to t and x, respectively, we have

$$\int_0^{DT} \int_{x/D}^{T} u(x,t)\frac{\partial f(x,t)}{\partial t} dt\,dx + D\int_0^T \int_0^{Dt} u(x,t)\frac{\partial f(x,t)}{\partial x} dx\,dt = 0. \tag{2.9}$$

If we reapply integration by parts to the inner integrals in the first term and second term by t and x, respectively, we prove that the solution in the form (2.4) satisfies the integral relation (2.7).

3. Auxiliary Problem and Its Exact Solution

In order to find the weak solution of the problem (2.1)–(2.3), according to [9, 10] the special auxiliary problem

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)^{\sigma}, \tag{3.1}$$

$$v(x,0) = v_0(x), \tag{3.2}$$

$$u(0,t) = \frac{\partial v(0,t)}{\partial x} = \mu_0 t^n \tag{3.3}$$

is introduced. Here, the function $v_0(x)$ is any solution of the equation

$$\frac{dv_0(x)}{dx} = 0. \tag{3.4}$$

The problem (3.1)–(3.3) has the solution

$$v(x,t) = \begin{cases} -D^{1/(\sigma-1)}\left(\dfrac{\sigma-1}{\sigma}\right)^{\sigma/(\sigma-1)}(Dt-x)^{\sigma/(\sigma-1)}, & x < Dt, \\ 0, & x \geq Dt \end{cases} \tag{3.5}$$

in the traveling wave form [9]. As is seen from (3.5), the differentiability property of the function $v(x,t)$ is more than the differentiability property of the solution $u(x,t)$. In addition to this, from (3.5) we get

$$u(x,t) = \frac{\partial v(x,t)}{\partial x}. \tag{3.6}$$

Theorem 3.1. *If the function $v(x,t)$ is a soft solution of the problem* (3.1)–(3.3), *then the function $u(x,t)$ obtained by* (3.6) *is a weak solution of the main problem* (2.1)–(2.3) *in sense of* (2.7).

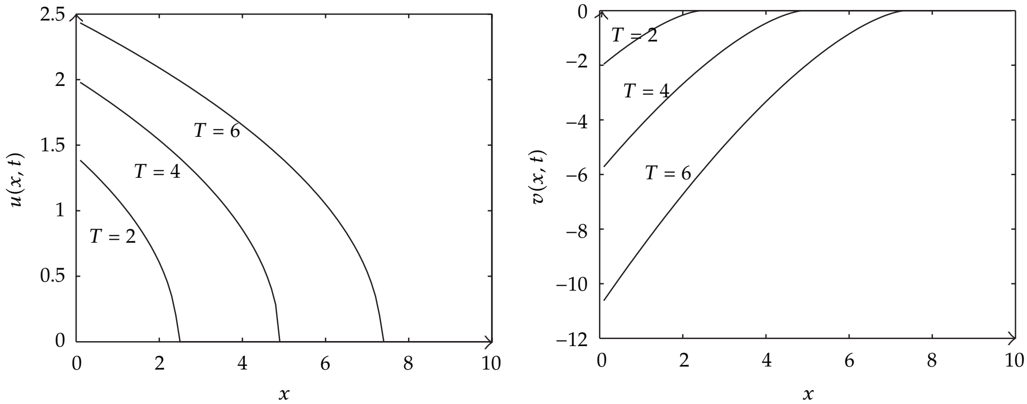
The auxiliary problem has the following advantages.

(i) The function $v(x,t)$ is smoother than $u(x,t)$.

(ii) The function $v(x,t)$ is an absolutely continuous function.

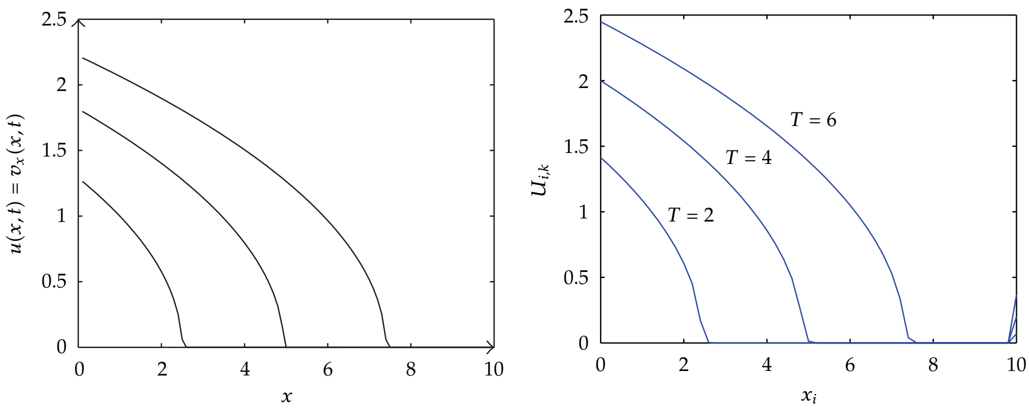
(iii) In the process of finding the solution $u(x,t)$, one does not need to use the first and second derivatives of $u(x,t)$ with respect to x.

The graphs of the function structured by the formulas (2.4), (3.5), and (3.6) are shown in the Figures 1(a), 1(b), and 2(a), respectively.



(a) (b)

Figure 1: (a) The exact solution of the main problem. (b) The exact solution of the auxiliary problem.



(a) (b)

Figure 2: (a) The function $u(x,t) = \partial v(x,t)/\partial x$. (b) Numerical solution obtained by using the classical algorithm (4.6)–(4.7).

4. Developing a Numerical Algorithm in a Class of Discontinuous Functions

In this section we investigate an algorithm for finding a numerical solution of the problem (2.1)–(2.3). At first, we cover the region $D_{\ell(t)}$ by a special grid:

$$\omega_\tau = \{(x_i, t_k) \mid x_i = Dt_i,\ t_k = k\tau,\ i,k = 0,1,2,\ldots\}, \tag{4.1}$$

where τ is the step of the grid with respect to t variable.

Now, we will develop a numerical algorithm as follows. Since the function $\partial u^{\sigma}/\partial x$ is continuous, we can approximate the problem (3.1)–(3.3) by the following finite differences schemes:

$$V_{i,k+1} = V_{i,k} + \frac{\tau}{h^{\sigma+1}}\left[(V_{i+1,k+1} - V_{i,k+1})^{\sigma} - (V_{i,k+1} - V_{i-1,k+1})^{\sigma}\right], \tag{4.2}$$

$$V_{i,0} = v_0(x_i), \tag{4.3}$$

$$V_{0,k+1} = V_{1,k+1} - h\mu_0 t_{k+1}^{n} \tag{4.4}$$

$(i = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots)$. Here, $h = D\tau$ and $v_0(x_i)$ is any grid function of (3.4).

It can be easily shown that

$$U_{i,k+1} = \frac{V_{i+1,k+1} - V_{i,k+1}}{h}, \tag{4.5}$$

and the grid function $U_{i,k+1}$ defined by (4.5) is a solution of the nonlinear system of algebraic equations:

$$U_{i,k+1} = U_{i,k} + \frac{\tau}{h^2}\left(U_{i+1,k+1}^{\sigma} - 2U_{i,k+1}^{\sigma} + U_{i-1,k+1}^{\sigma}\right) \tag{4.6}$$

$(i = 1, 2, \ldots; k = 0, 1, 2, \ldots)$. The initial and boundary conditions for (4.6) are

$$\begin{gathered} U_{i,0} = 0 \quad (i = 0, 1, 2, \ldots), \\ U_{0,k+1} = \mu_0 t_{k+1}^{n} \quad (k = 0, 1, 2, \ldots). \end{gathered} \tag{4.7}$$

Here, $V_{i,k}$ and $U_{i,k}$ are the approximation values of $v(x,t)$ and $u(x,t)$ at any point (x_i, t_k) of the grid ω_{τ}. For the sake of simplicity we introduce the notations $V_{i,k} = V$, $V_{i,k+1} = \widehat{V}$, and $V_{i\pm1,k+1} = \widehat{V}_{\pm}$.

4.1. Convergence

In this section we will investigate some properties of the numerical solution and of the question of convergence of the numerical solution to the weak exact solution. Suppose that $\varepsilon_{i,k}, \delta_{i,k}$, and $\eta_{i,k}$ are the errors of approximations of the functions $\partial v/\partial x, \partial v/\partial t$, and $\partial \varphi/\partial x$ by finite differences, respectively. Then, we can write (3.1) in the following form:

$$v_t + \delta_{i,k+1} = \varphi_x(\widehat{v}_{\overline{x}} + \varepsilon_{i,k+1}) + \eta_{i,k+1} \tag{4.8}$$

or

$$v_t = \varphi_x(\widehat{v}_{\overline{x}}) + \gamma_{i,k+1}, \tag{4.9}$$

where

$$\gamma_{i,k+1} = \delta_{i,k+1} + \varphi_{xx}(y_{\overline{x}})\varepsilon_{i,k+1} + \eta_{i,k+1}. \tag{4.10}$$

At first, we show that the finite difference scheme (4.2) is consistent; that is, $\gamma_{i,k+1}$ approaches zero when $\tau \to 0$. It is known that the suitable characteristic of continuity of any function $f(x)$ on the any interval $[a, b]$ is its modulus continuity:

$$\begin{aligned}
\omega(\delta, f) &\equiv \pi(f) = \sup_{|t-x|<\delta} \left|f(t) - f(x)\right|, \\
\varepsilon_{i,k+1} &= \frac{\partial v(x_i, t_{k+1})}{\partial x} - \widehat{v}_{\overline{x}} = \frac{\partial v(x_i, t_{k+1})}{\partial x} - \frac{\partial v(x_i^*, t_{k+1})}{\partial x} \\
&= u(x_i, t_{k+1}) - u(x_i^*, t_{k+1}) = \pi(u) \longrightarrow 0, \quad x_i^* \in (x_i - h, x_i), \\
\delta_{i,k+1} &= \frac{\partial v(x_i, t_{k+1})}{\partial t} - \widehat{v}_t = \frac{\partial v(x_i, t_{k+1})}{\partial t} - \frac{\partial v(x_i, t_{k+1}^*)}{\partial t} \\
&= \frac{\partial \varphi(u(x_i, t_{k+1}))}{\partial x} - \frac{\partial \varphi(u(x_i, t_{k+1}^*))}{\partial x} = \pi\left(\frac{\partial \varphi(u)}{\partial x}\right) \longrightarrow 0, \quad t_{k+1}^* \in (t_{k+1}, t_{k+1} + \tau).
\end{aligned} \tag{4.11}$$

Finally,

$$\eta_{i,k+1} = \frac{\partial \varphi(\widehat{u})}{\partial x} - \varphi_x(\widehat{u}) = \frac{\partial \varphi(u(x_i, t_{k+1}))}{\partial x} - \frac{\partial \varphi(u(x_i^{**}, t_{k+1}))}{\partial x} \longrightarrow 0, \quad x_i^{**} \in (x_i - h,\ x_i). \tag{4.12}$$

Therefore, $\gamma_{i,k+1} \to 0$, if $\tau \to 0$.

Theorem 4.1 (Maximum Principle). *The solution of problem* (4.2)–(4.4) *takes its maximum (or minimum) value on the boundary of the domain of definition of the solution, that is,*

$$0 \le V_{i,k+1} \le M = \max_{i,k}\{|u_0|,\ |u_1|\}. \tag{4.13}$$

Proof. At first, let us write (4.2) in the following form:

$$\begin{aligned}
\frac{V_{i,k+1} - V_{i,k}}{\tau} &= \frac{1}{h}\left[\left(\frac{V_{i+1,k+1} - V_{i,k+1}}{h}\right)^{\sigma} - \left(\frac{V_{i,k+1+1} - V_{i-1,k+1}}{h}\right)^{\sigma}\right] \\
&= \frac{1}{h}\left(\widehat{U}_i^{\sigma} - \widehat{U}_{i-1}^{\sigma}\right) = -w(x_i, t_k).
\end{aligned} \tag{4.14}$$

Assume that $V_{i,k+1}$ is not constant and $V_{i,k+1}$ takes the greatest value at some point of the grid ω_τ rather than at boundary nodes of γ_τ. Then, there is such a point $(x_1, t_1) \in \omega_\tau$ that $\widehat{V}$ takes the maximal value and even some neighborhood points $V(x_1, t_1)$ less than $\widehat{V}(x_1, t_1)$.

If $\widehat{V}_{x_1,t_1} > V_{x_1,t_1}$, because the function $\varphi(u) = u^\sigma$ is monotone, the left part of the relation (4.14) is positive, but the right part is negative. Hence we arrive to inconsistency. We arrive to the same inconsistency if $\widehat{V}_{\pm}(x_1, t_1) > \widehat{V}(x_1, t_1)$. Similarly, we can prove that $\widehat{V}$ does not take a minimal value at the inner nodes of the grid ω_τ. □

From Theorem 4.1, it follows that the solution of the problem (4.2)–(4.4) converges to the solution of problem (3.1)–(3.3) pointwise, that is,

$$\max_i |v_{i,k} - V_{i,k}| \longrightarrow 0. \tag{4.15}$$

It can be easily seen that the solution of the mentioned problem is continuously dependent on initial data.

Now, we will prove convergence of the $U_{i,k}$ to the solution of the main problem. To this end, by subtracting (4.2) from (4.9) and taking (4.5) into account, we have

$$R_t = \left(\varphi'(\tilde{u})(v_{\bar{x}} - V_{\bar{x}})\right)_x + \gamma_{i,k} \tag{4.16}$$

or

$$R_t = \left(\varphi'(\tilde{u})(u_{i,k} - U_{i,k})\right)_x + \gamma_{i,k}, \qquad \tilde{u} \in (u_{i,k}, U_{i,k}), \tag{4.17}$$

where $\widehat{R} = v_{i,k+1} - V_{i,k+1}$. By multiplying the last equation with $\widehat{R}$ and summing with respect to i and k, we get

$$\left(\widehat{R},\ R_t\right)_{L_2(\omega_\tau)} = \left(\widehat{R},\ \left(\varphi'(\tilde{u})(u_{i,k} - U_{i,k})\right)_x\right)_{L_2(\omega_\tau)} + \left(\widehat{R},\ v_{i,k}\right)_{L_2(\omega_\tau)}. \tag{4.18}$$

Here, the symbol $(f,\ g)$ denotes the differences analogy of the scalar product of the functions of f and g in $L_2(\omega_\tau)$ sense:

$$(f, g)_{L_2(\omega_\tau)} = \tau h \sum_{i=1} \sum_{k=0} f_{i,k}\ g_{i,k}. \tag{4.19}$$

With some algebra we have

$$\begin{aligned} \frac{1}{2}\sum_i R_i^2\Big|_{t_k} &= T + \left(u_{i,k} - U_{i,k},\ \varphi'(\tilde{u})(u_{i,k} - U_{i,k})\right)_{L_2(\omega_\tau)} \\ &\le \frac{1}{2}\sum_i R_i^2\Big|_{t_k=0} + T \max_i R_i^2 \left\|\gamma_{i,k}\right\|_{L_2(\omega_\tau)}. \end{aligned} \tag{4.20}$$

It follows that the numerical solution $U_{i,k}$ converges in mean to $u_{i,k}$.

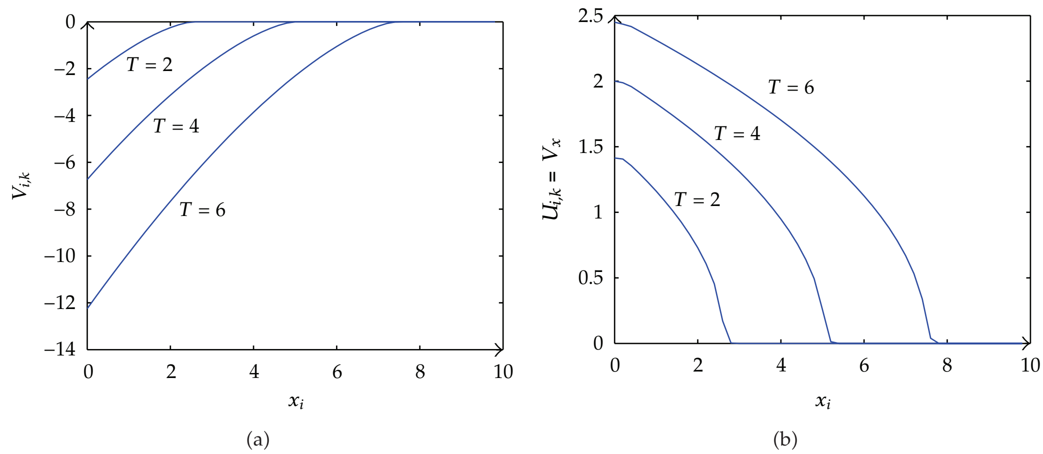


Figure 3: (a) Numerical solution of the problem (4.2)–(4.4). (b) Numerical solution of the main problem obtained by using the solution of the problem (4.2)–(4.4).

5. Numerical Experiments

In order to extract the significance of the suggested method, the numerical solution obtained using the proposed auxiliary problem is compared with the exact solution of the problem (2.1)–(2.3) on an equal footing. With this aim, firstly, computer experiments are performed on the algorithm (4.2)–(4.4) with the values $\sigma = 3$, $T = 2, 4$, and 6. The graphs of the obtained numerical solutions are presented in Figure 3(a). The numerical solutions of the main problem (2.1)–(2.3) obtained by using the algorithm (4.2)–(4.4) are shown in Figure 3(b). The numerical solution of the main problem obtained using the algorithm (4.6)–(4.7) is demonstrated in Figure 2(b).

Comparing Figures 1(a) and 3(b) shows that the numerical and the exact solutions of the main problem (2.1)–(2.3) coincide. Moreover, the graphs of the functions $v(x, t)$ and $V_{i,k}$ coincide, too.

Thus, the numerical experiments carried out show that the suggested numerical algorithms are efficient and economical from a computer point of view. The proposed algorithms permit us to develop the higher-resolution methods where the obtained solution correctly describes all physical features of the problem, even if the differentiability order of the solution of the problem is less than the order of differentiability which is required by the equation from the solution.

The finite differences scheme (4.2)–(4.4) has the first-order approximation with respect to t. But using the different versions of Runge-Kutta's methods we can increase the order of the algorithms mentioned previously.

6. Conclusions

The new method is suggested for obtaining the regular weak solution for the free boundary problem of the nonlinear parabolic type equation.

The auxiliary problem which has some advantages over the main problem is introduced and it permits us to find the exact solution with singular properties.

The auxiliary problem introduced previously allows us to develop the higher-resolution method where the obtained solution correctly describes all physical features of the problem, even if the differentiability order of the solution of the problem is less than the order of differentiability which is required by the equation from the solution.

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