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Nonlinear Unsteady Supersonic Flow Analysis for Slender Bodies of Revolution: Theory

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We construct analytical solutions for the problem of nonlinear supersonic flow past slender bodies of revolution due to small amplitude oscillations. The method employed is based on the splitting of the time dependent small perturbation equation to a nonlinear time independent partial differential equation (P.D.E.) concerning the steady flow, and a linear time dependent one, concerning the unsteady flow. Solutions in the form of three parameters family of surfaces for the first equation are constructed, while solutions including one arbitrary function for the second equation are extracted. As an application the evaluation of the small perturbation velocity resultants for a flow past a right circular cone is obtained making use of convenient boundary and initial conditions in accordance with the physical problem.

Keywords: Unsteady subersonic flow; monge equation; arbitrary function

Classification Categories: 76J99, 65M99, 65N20, 65N25

MAIN NOTATION

(x, r, θ) = nondimensional cylindrical co-ordinates, normalized by the true body length;
U = freestream velocity;
T = true time;
t = TU/L, nondimensional time;
ω = true angular frequency;

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k	=	$\omega L/U$, reduced frequency;
Ω, Φ	=	total and perturbed potentials, both normalized by $(UL)^{-1}$;
ϵ	=	maximum body radius/L, the body thickness ratio;
M	=	freestream Mach number;
γ	=	ratio of specific heats (1.40);
α	=	distance of the nonlifting position, normalized by L^{-1};
δ	=	oscillation amplitude.

1. INTRODUCTION

In a great number of aerodynamic problems one is interested in the perturbation of known fluid motion. The most common and obvious case is that of a uniform steady flow. Especially, the small perturbation theory is one of the most discussed items of aerodynamics. The major difficulty in obtaining analytical solutions of the general small perturbation steady or unsteady differential equations is the nonlinearity. In the past, the steady transonic flow problem for a two dimensional or an axisymmetric body has been treated by various analytical methods; see Oswatitsch and Keune [1955], Spreiter and Alksne [1959], Liepmann and Roshko [1957]. The method employed was based on the elimination of the nonlinear terms appearing in the nonlinear P.D.E. governing the steady perturbation velocity potential. More recent studies of this problem mostly adopted numerical techniques, notably the type sensitive difference sceme (Krupp and Murman [1972]), involving the flowfield calculations. However, for the unsteady flow problems, apart from the work by Stahara and Spreiter [1976], which is an extension of the earlier work by Liu, Platzer and Ruo [1970], efforts have been made in constructing analytical solutions making use of several approximate methods and techniques. We mention here the work by Liu, Platzer and Ruo [1977], which based on the linearized model according to Oswatitsch and Keune [1955] (the parabolic method) succeeded in obtaining approximate near-field solutions for slender bodies of revolution in unsteady transonic flow. Several other works which have used the same extensive concept of parabolic method for solving the sonic flow problem should be mentioned here (Kimble, Liu, Ruo and Wu [1977]; Ruo [1974], Platzer [1966] and Zierep [1965]).

In the present theoretical investigation, based on the work by Liu, Platzer and Ruo [1977], a successful attempt is made to solve analytically for supersonic flow the small perturbation unsteady transonic P.D.E. appearing in the harmonic small amplitude pitching oscillations of a body of revolution around its nonlifting position. Contrary to existing solution methodologies the only necessary approximation is the elimination of the nonlinear term appearing in the oscillatory perturbed potential. This approximation results from the application of the well known small perturbation procedure described by Liepmann and Roshko [1957]. The method employed is based on the splitting of the nonlinear time dependent transonic small perturbation equation to a nonlinear time independent P.D.E. concerning the steady axisymmetric potential, and a linear time dependent P.D.E. concerning the oscillatory potential.

Making use of a convenient ad hoc assumption a three parameters family of surfaces solution for the first equation is obtained, while an analytical solution including one arbitrary function for the second is constructed in case of supersonic flow. For both previous constructions, the Monge method was used (Ames [1965]), while for the second one the separation of variables technique was in addition used (Koshlyakov, Smirnov and Gliner [1964]). As an application of the developed analysis the evaluation in a closed-form of the perturbation velocity resultants for a right circular cone are obtained, using convenient boundary and initial conditions in accordance with the physical problem.

We must point out that the advantage of the proposed herein methodologies is the introduction of a time-distance dependent arbitrary function in the constructed solutions, by means of which one easily evaluates analytical expressions for arbitrary geometrical boundaries of the bodies under consideration.

2. MATHEMATICAL FORMULATION

Consider a rigid pointed body of revolution exposed to a steady uniform transonic flow U. The body performs harmonic, small amplitude pitching oscillations around its nonlifting position M, while it is assumed to be smooth and sufficiently slender so that the small disturbance concept can be applied. For the description of the problem we consider a space-fixed cartesian co-ordinate system $M(x_s, y_s, z)$, the axis x_s of which coincides with the central axis of the body in its steady position (Fig. 1). The small amplitude oscillations occur in the x_s, y_s-plane and so a body-fixed cartesian co-ordinate system M(x, y, z) is necessary to be introduced for our analysis. Finally, we consider a body-fixed cylindrical co-ordinate system (x, r, θ), where r is parallel to the yz-plane as Fig. 1 shows. The total velocity potential $\Omega(x, r, \theta, t)$ can be related to a perturbed velocity potential $\phi(x, r, \theta, t)$ by the equation (Liu, Platzer and Ruo [1977])

$$\Omega(x, r, \theta, t) = x\cos\delta + r\sin\delta\cos\theta + \Phi(x, r, \theta, t) \tag{2.1}$$

in which $\delta = \delta_0 \exp(ikt)$; δ_0 is the oscillation amplitude and k denotes the reduced frequency of the pitching motion. Using (2.1) one extracts the cylindrical velocity components

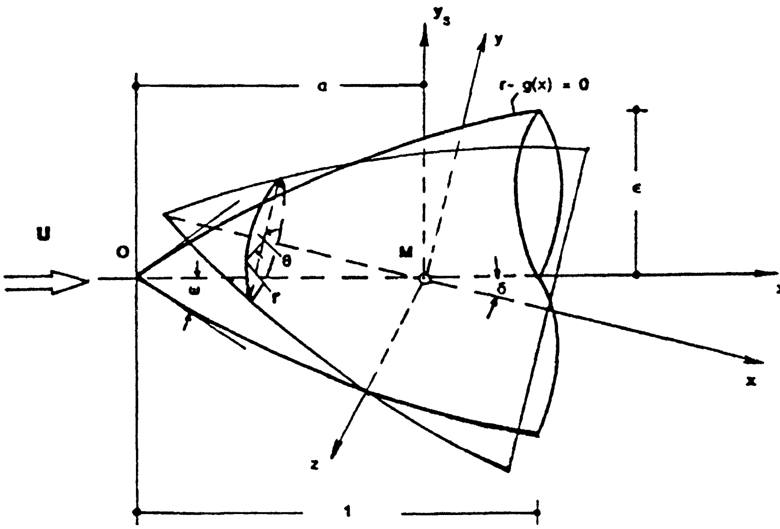

FIGURE 1 Geometry and sign convention of a rigid body of revolution under uniform flow.

$$V|_{\text{cylindrical}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \Omega_x \\ \Omega_r \\ \frac{1}{r}\Omega_\theta \end{bmatrix} = \begin{bmatrix} \cos\delta + \Phi_x \\ \sin\delta\cos\theta + \Phi_r \\ -\sin\delta\sin\theta + \frac{1}{r}\Phi_\theta \end{bmatrix}. \tag{2.2}$$

Introducing the expression for δ into (2.2), retaining only the real part and considering small-amplitude oscillations ($\delta_0 << 1$), we succeed in transforming relation (2.2) into the simpler one

$$V|_{\text{cylindrical}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \simeq \begin{bmatrix} 1 + \Phi_x \\ \delta_0 \cos kt \cos\theta + \Phi_r \\ -\delta_0 \cos kt \sin\theta + \frac{1}{r}\Phi_\theta \end{bmatrix}. \tag{2.3}$$

The time dependent transonic equation for small perturbations reads (Liu, Platzer and Ruo [1977])

$$\beta\,\Phi_{xx} + \Phi_{rr} + \frac{1}{r^2}\Phi_{\theta\theta} - M^2\Phi_{tt} - 2M^2\Phi_{xt} + \frac{1}{r}\Phi_r - (\gamma + 1)M^2\Phi_x\,\Phi_{xx} = 0 \tag{2.4}$$

where M is the freestream Mach number and $\beta = 1 - M^2$.

Note that the nonlinear term in (2.4) can be ignored only if the flow is sufficiently unsteady.

We assume now that the potential Φ splits into two terms; the first expresses the steady axisymmetric potential due to the geometry of the body, while the second represents the perturbed potential due to the oscillations of the body (Liepmann and Roshko [1957]; Liu, Platzer and Ruo [1977])

$$\Phi(x, r, \theta, t) = \phi(x, r) + \psi(x, r, \theta, t). \tag{2.5}$$

Differentiating (2.5) and inserting the results into (2.4) we deduce

$$\beta(\phi + \psi)_{xx} + (\phi + \psi)_{rr} + \frac{1}{r^2}\psi_{\theta\theta} - M^2\psi_{tt} - 2M^2\psi_{xt} + \frac{1}{r}(\phi + \psi)_r - $$
$$- (\gamma + 1)M^2\phi_x\phi_{xx} - (\gamma + 1)M^2(\phi_x\psi_{xx} + \phi_{xx}\psi_x) -$$
$$(\gamma + 1)M^2\psi_x\psi_{xx} = 0. \quad (2.6)$$

In this equation, for small-amplitude oscillations one can neglect the nonlinear term $(\gamma + 1)M^2\psi_x\psi_{xx}$ (Liu, Platzer and Ruo [1977]). Thus, we derive a simpler nonlinear P.D.E. which further can be splitted into one nonlinear P.D.E. concerning the steady flow, and one linear P.D.E. concerning the unsteady flow, namely

$$\beta\phi_{xx} + \phi_{rr} + \frac{1}{r}\phi_r = (\gamma + 1)M^2\phi_x\phi_{xx}; \quad (2.7a)$$

$$\beta\psi_{xx} + \psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} - M^2\psi_{tt} - 2M^2\psi_{xt} + \frac{1}{r}\psi_r = (\gamma + 1)M^2(\phi_{xx}\psi_x$$
$$+ \phi_x\psi_{xx}). \quad (2.7b)$$

In a large number of previous publications analytical solutions of both equations (2.7a, b) were obtained by means of approximate analyses and techniques. We mention here the papers by Oswatitsch and Keune [1955]; Spreiter and Alksne [1959]; Stahara and Spreiter [1976]; Liu [1968]; Liu and Platzer [1969]. In this paper, contrary to the above developed methodologies and techniques, we shall try to construct analytical solutions for both previous equations. Our attempt will be focussed on the formulation of solutions including one arbitrary function, which will permit us to define special solutions in accordance with the boundary and initial conditions of every physical problem under consideration.

3. ANALYTICAL SOLUTIONS OF EQNS (2.7 a,b)

3.1 Solutions of Equation (2.7a)

Since the function $\phi(x, r)$ resulting from the solution of Eq. (2.7a), must be used for the second equation (2.7b) a convenient solution of (2.7a) would be expressed in the form of a three parameters family of surfaces. Such a type of solution can be constructed as follows.

Using the well-known notation (Ames [1965])

$$p = \phi_x, q = \phi_r, \bar{r} = \phi_{xx}, s = \phi_{xr}, t = \phi_{rr}$$

one writes the nonlinear P.D.E. (2.7a) in the Monge form

$$R\bar{r} + Ss + Tt = V \tag{3.1}$$

where

$$R = \beta - (\gamma + 1)M^2 p, S = 0, T = 1, V = -q/r. \tag{3.2}$$

The corresponding to (3.1) Monge equations are

$$[\beta - (\gamma + 1)M^2 p]dpdr + dqdx + \frac{q}{r}dxdr = 0; \tag{3.3a}$$

$$[\beta - (\gamma + 1)M^2 p]dr^2 + dx^2 = 0; \tag{3.3b}$$

$$d\phi = pdx + qdr. \tag{3.3c}$$

Making use of the ad hoc assumption

$$p = \Delta = \text{constant} \tag{3.4}$$

we distinguish the following two cases concerning the solutions of the above system.

Case a

For

$$\beta - (\gamma + 1)M^2 \Delta < 0, \tag{3.5}$$

the equation (3.1) becomes of hyperbolic type valid only for supersonic flow and the solution of (3.3b) for dx/dr furnishes

$$dx/dr = \pm[(\gamma + 1)M^2 \Delta - \beta]^{\frac{1}{2}}.$$

The integration of this equation results in

$$x = \pm[(\gamma + 1)M^2 \Delta - \beta]^{\frac{1}{2}} r + a$$

where a is an integration constant. On the other hand, since p is considered to be constant, the integration of (3.3a) leads to the equation

$$rq = b,$$

where b is a new constant of integration. Noting that a and b are arbitrary, we derive

$$q = \frac{1}{r} H(x \mp [(\gamma + 1)M^2 \Delta - \beta]^{\frac{1}{2}} r) \tag{3.6}$$

in which H is an arbitrary function of its argument.

Therefore, the fundamental Monge equation (3.3c) becomes

$$d\phi = \Delta dx + \frac{1}{r} H(x \mp [(\gamma + 1)M^2 \Delta - \beta]^{\frac{1}{2}} r)dr.$$

This equation can be integrated in case when

$$H = A = \text{constant}$$

and consequently one obtains a solution of the nonlinear PDE under consideration in the form

$$\phi = \Delta x + A \ln r + B \tag{3.7}$$

where B is a third constant.

Case b

For

$$\beta - (\gamma + 1)M^2 \Delta > 0 \tag{3.8}$$

the equation (3.1) is of elliptic type valid only for subsonic flow and the previously developed procedure furnishes the following solution for the steady potential

$$\phi = \Delta x + \overset{*}{A} \ln r + \overset{*}{B} \tag{3.9}$$

where $\overset{*}{A}$ and $\overset{*}{B}$ are new constants with $\overset{*}{A}$ complex.

Both solutions (3.7) and (3.9) are of the same type and they can be rewritten in the following three parameters family of surfaces forms

$$F(x, r; \Delta, A, B) = \phi - \Delta x - A \ln r - B = 0; \overset{*}{F}(x, r; \Delta, A, B) = \phi - \Delta x - \overset{*}{A} \ln r - \overset{*}{B} = 0. \tag{3.10}$$

3.2 Solutions of Equation (2.7b)

For this equation we assume solutions of the separate form

$$\psi(x, r, \theta, t) = f(x, t)h(r)g(\theta), \tag{3.11}$$

where f, h and g are arbitrary smooth functions. By means of (3.11), as well as the steady solution (3.10), equation (2.7b) becomes

$$\frac{\ddot{h}}{h} + \frac{1}{r}\frac{\dot{h}}{h} + \frac{1}{r^2}\frac{\overset{**}{g}}{g} = [(\gamma + 1)M^2 \Delta - \beta]\frac{f_{xx}}{f} + 2M^2\frac{f_{xt}}{f} + M^2\frac{f_{tt}}{f}, \tag{3.12}$$

in which dot means differentiation with respect to r and asterisk with respect to θ. Since in (3.12) the left-hand side is a function of r and θ, and the right-hand side a function of x and t only, the equality is possible if both members are equal to the same constant $-\lambda^2$ ($\lambda > 0$) (separation constant). Consequently, (3.12) splits into the following two equations

$$\frac{\ddot{h}}{h} + \frac{1}{r}\frac{\dot{h}}{h} + \frac{1}{r^2}\frac{\overset{**}{g}}{g} + \lambda^2 = 0; \tag{3.13a}$$

$$[(\gamma + 1)M^2 \Delta - \beta]f_{xx} + 2M^2 f_{xt} + M^2 f_{tt} + \lambda^2 f = 0. \quad (3.13b)$$

Relation (3.13a) can be further written in the separate form

$$r^2 \frac{\ddot{h}}{h} + r\frac{\dot{h}}{h} + (\lambda r)^2 = -\frac{\overset{**}{g}}{g}, \quad (3.14)$$

which, taking into account that $g(\theta)$ reads as a periodic function of θ, splits into

$$\frac{\overset{**}{g}}{g} = -n^2; \quad (3.15a)$$

$$r^2 \ddot{h} + r\dot{h} + [(\lambda r)^2 - n^2]h = 0 \quad (3.15b)$$

where n is an integer.

The general solution of Eq. (3.15a) is given by

$$g_n(\theta) = c_{1n}\cos n\theta + c_{2n}\sin n\theta, \quad (3.16)$$

while the corresponding solution for Eq. (3.15b) reads

$$h(r) = c_{3n}J_n(\lambda r) + c_{4n}Y_n(\lambda r), \quad (3.17)$$

in which J_n and Y_n are the Bessel functions of integer order of the first and second kind respectively, while c_{in} ($i = 1,\ldots, 4$) are suitable constants of integration for each integer n.

Concerning now the linear P.D.E. (3.13b) the corresponding Monge equations are written as

$$[(\gamma + 1)M^2\Delta - \beta]\, dpdt + M^2dqdx + \lambda^2 fdxdt = 0; \quad (3.18a)$$

$$[(\gamma + 1)M^2\Delta - \beta]\, dt^2 - 2M^2dxdt + M^2dx^2 = 0; \quad (3.18b)$$

$$df = pdx + qdt, \quad (3.18c)$$

$$p = f_x,\ q = f_t.$$

Equation (3.18b), satisfying the characteristic directions, can be written under the form

$$M^2(dx/dt)^2 - 2M^2(dx/dt) + [(\gamma + 1)M^2\Delta - \beta] = 0 \qquad (3.19)$$

with discriminant 4D given by

$$D = M^2[1 - (\gamma + 1)M^2\Delta].$$

According to the sign of this discriminant we distinguish the following three cases.

Case a (Parabolic type)

In this case D = 0, that means

$$\Delta = 1/[(\gamma + 1)M^2]. \qquad (3.20)$$

Then, Eq. (3.19) furnishes

$$dx = dt \quad \text{or} \quad x = t + a \qquad (3.21)$$

where a is an integration constant.

Case b (Elliptic type)

In this case the inequality

$$D < 0 \quad \text{or} \quad \Delta > 1/(\gamma + 1)M^2 > 0 \qquad (3.22)$$

holds true, and Eq. (3.19) extracts the solutions

$$x = \frac{M \pm i[(\gamma + 1)M^2\Delta - 1]^{\frac{1}{2}}}{M} t + a. \qquad (3.23)$$

Case c (Hyperbolic type)

Here we have the inequality

$$D > 0 \quad \text{or} \quad \Delta < 1/(\gamma + 1)M^2 \tag{3.24}$$

and Eq. (3.19) gives

$$x = \frac{M \pm [1 - (\gamma + 1)M^2\Delta]^{\frac{1}{2}}}{M} t + a. \tag{3.25}$$

We shall prove now that in case (a), where equation (3.19) is of parabolic type, it is possible to obtain an analytical solution of (3.13b) including one arbitrary function. In fact, since (3.21) holds true, the Monge equation (3.18a), specifying the relations along the characteristics, leads to

$$dt = -M^2\, d(p + q)/\lambda^2 f. \tag{3.26}$$

On the other hand, the fundamental equation (3.18c) becomes

$$dt = df/(p + q). \tag{3.27}$$

Combination of (3.26) and (3.27) results in

$$M^2(p + q)d(p + q) = -\lambda^2 f\, df. \tag{3.28}$$

Since an integration constant a has been already introduced in (3.21), the integration of (3.28) results in

$$p + q = \pm i\lambda f/M \tag{3.29}$$

in which the integration constant has been depressed because the final solution must include only two constants of integration. Equation (3.29) is of quasi-linear form. Making use of the corresponding subsidiary Lagrange's equations

$$\frac{dx}{1} = \frac{dt}{1} = \mp i \frac{Mdf}{\lambda f} \tag{3.30}$$

and integrating, we deduce

$$x - t = a; \quad x \pm i \frac{M}{\lambda} \ln f = b \tag{3.31}$$

in which a and b are arbitrary constants. Combination of these equations results in

$$f(x, t) = \exp\{\pm i \frac{\lambda}{M} [x - F(x - t)]\} \tag{3.32}$$

where F is an arbitrary function.

In the Appendix I we verify that expression (3.32) constitutes a solution of the linear P.D.E. (3.13b).

By now, we are able to construct the solution for the unsteady potential ψ, including one arbitrary function. In fact, taking into account that there are no restrictions on the positive separation constant λ, we may combine expressions (3.16), (3.17) and (3.32), and take the sum of all particular solutions of the form (3.11). Thus, we deduce the solution for the unsteady potential ψ as follows

$$\psi(x, r, \theta, t) = \sum_{n=0}^{\infty} \Bigg[\Big(\int_0^{\infty} [A_n(\lambda) J_n(\lambda r) + B_n(\lambda) Y_n(\lambda r)] \exp \{\pm i \frac{\lambda}{M} [x - F(x - t)]\} \, d\lambda \Big) \cos n\theta + \Big(\int_0^{\infty} [\overline{A}_n(\lambda) J_n(\lambda r) + \overline{B}_n(\lambda) Y_n(\lambda r)] \exp\{+ i \frac{\lambda}{M} [x - F(x - t)]\} d\lambda \Big) \sin n\theta. \Bigg] \tag{3.33}$$

We note here that the solution (3.33) for the unsteady perturbation equation (2.7b) concerns only the parabolic case in which the parameter Δ is defined by the relation (3.20). This means that the steady nonlinear PDE (2.7a) becomes of hyperbolic type since for $\Delta = 1/(\gamma + 1)M^2$ only the inequality (3.5) holds true. Therefore, we conclude that the developed herein solutions of the problem under consideration concern only the case of supersonic flow.

In the next section, applying convenient boundary and initial conditions, we shall try to specify the analytical expressions of the flow velocity resultants given in (2.3) for a slender body of revolution with a given geometry of its boundary.

4. BOUNDARY AND INITIAL VALUE PROBLEM-SPECIAL SOLUTIONS

Let us consider the rigid, slender pointed body of revolution with configuration of a right circular cone shown in Fig. 2. The equation of the meridian of the body is given by

$$r = \epsilon(x + a) \tag{4.1}$$

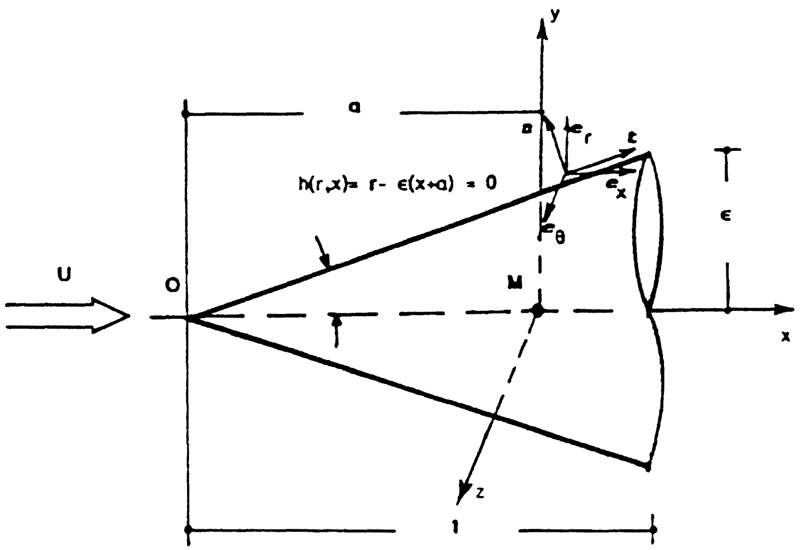

FIGURE 2 A right circular cone under uniform flow.

where ϵ denotes the body thickness ratio. We consider also at a generic point of the surface of the body the local right-handed Darboux co-ordinate system (t, $\boldsymbol{n}$, $\boldsymbol{e}_\theta$); where $\mathbf{t}$ is the tangent and $\boldsymbol{n}$ the normal to the surface unit vectors. The direction of the unit vector $\boldsymbol{e}_\theta$ coincides with the corresponding direction of the cylindrical co-ordinate system ($\boldsymbol{e}_x$, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$) (Fig. 2). For an arbitrary vector $\boldsymbol{G}$ expressed in both previous systems, we have the relation

$$\boldsymbol{G} = \begin{bmatrix} G_t \\ G_n \\ G_\theta \end{bmatrix} = \mathbf{A} \begin{bmatrix} G_x \\ G_r \\ G_\theta \end{bmatrix} \tag{4.2}$$

in which $\mathbf{A}$ is the 3×3 transformation matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.3}$$

In order to specify the already constructed solutions for the configuration of the circular cone shown in Fig. 2 we introduce the following boundary and initial conditions.

4.1 First Boundary Condition

The first set of boundary conditions concerns the velocity resultants, Eq. (2.1), at infinity. We read

$$\Omega_x = 1 + \phi_x + \psi_x; \ \Omega_r = \delta_0 \operatorname{cosktcos}\theta + \phi_r + \psi_r \text{ finite at infinity.} \tag{4.4}$$

These conditions are satisfied since the Bessel functions $J_n(\lambda r)$ and $Y_n(\lambda r)$, as well as the elementary function $\exp\{\pm i \frac{\lambda}{M} [x - F(x - t)]\}$ are bounded as $r \rightarrow \infty$ and $x \rightarrow \infty$ respectively.

4.2 Second Boundary and Initial Condition

We consider that for t = 0 the body, performing small amplitude oscillations around its nonlifting position M (Fig. 1), takes its maximum inclination δ_0. Also, in this position the flow is tangential to the solid surface. The above condition can be described by the well-known equation (Liepmann and Roshko [1957])

$$\text{For } t = 0, \quad \mathbf{u}\cdot \text{grad} h = 0 \tag{4.5}$$

where

$$\mathbf{u} = (u, v);\ h(x, r) = r - \epsilon(x + a) = 0. \tag{4.5a}$$

Combination of the above equations together with relations (2.3) and (2.5) leads to

$$\text{For } t = 0, \quad \epsilon(1 + \phi_x + \psi_x)|_{body} = \delta_0 \cos\theta + (\phi_r + \psi_r)|_{body}. \tag{4.6}$$

Furthermore, according to the formulation of the P.D.E.s (2.7a, b), relation (4.6) splits in the following two parts:

$$\text{For } t = 0,\ \epsilon(1 + \phi_x)|_{body} - \phi_r|_{body} = 0 \tag{4.7a}$$

and

$$\text{For } t = 0,\ \delta_0\cos\theta - \epsilon\psi_x|_{body} + \psi_r|_{body} = 0. \tag{4.7b}$$

By means of the solution (3.7) concerning the steady axisymmetric potential, as well as Eq. (3.20), the boundary condition (4.7a) reads

$$s = \frac{A}{\epsilon(1 + \Delta)}; \quad \Delta = \frac{1}{(\gamma + 1)M^2}, \tag{4.8}$$

where a parameter s has been already introduced such that

$$r = s, \quad x = \frac{s}{\epsilon} - a. \tag{4.9}$$

On the other hand, the solution (3.7) becomes

$$F(\phi, s; A, B, 1/(\gamma + 1)M^2) = \phi - \frac{1}{(\gamma + 1)M^2}(\frac{s}{\epsilon} - a) - A\ln s - B = 0. \tag{4.10}$$

Eliminating the parameter s between (4.9) and (4.10) we derive an equation of the form

$$\overset{*}{F}(\phi; A, B, 1/(\gamma + 1)M^2) = 0$$

which together with the equation

$$\partial\overset{*}{F}/\partial A = 0$$

results in the evaluation of the parameter **A**, namely

$$\ln\frac{A}{\epsilon(1 + \Delta)} + -1 - \frac{\Delta}{\epsilon^2(1 + \Delta)} \simeq - \frac{\Delta}{\epsilon^2(1 + \Delta)}$$

and hence

$$A \simeq \epsilon[1 + \frac{1}{(\gamma + 1)M^2}]\exp\{-\frac{1}{\epsilon^2[1 + (\gamma + 1)M^2]}\}. \tag{4.11}$$

Therefore, our solution concerning the steady potential ϕ can be written in the following one parameter family of surfaces form

$$\phi(x, r) = \frac{1}{(\gamma + 1)M^2}x + \epsilon[1 + \frac{1}{(\gamma + 1)M^2}]$$
$$\exp\{-\frac{1}{\epsilon^2[1 + (\gamma + 1)M^2]}\}\ln r + B. \tag{4.12}$$

Furthermore, the boundary condition (4.7b) can be approximated by noting that the product $\epsilon\psi_x$ is of lower order than the term ψ_r and may be neglected giving (Liepmann and Roshko [1957], p. 241)

$$\text{For } t = 0,\ \psi_r\big|_{body} + \delta_0 \cos\theta = 0. \tag{4.13}$$

Substituting the solution (3.33) into the last equation and taking into account the linear independence of 1; cosnθ; sinnθ (n = 0, 1,...) we deduce

$$\left.\begin{array}{l} A_n(\lambda)\dot{J}_n(x_1) + B_n(\lambda)\dot{Y}_n(x_1) = 0;\ n = 0, 2, 3,\ldots \\ \bar{A}_n(\lambda)\dot{J}_n(x_1) + \bar{B}_n(\lambda)\dot{Y}_n(x_1) = 0;\ n = 0, 1, 2,\ldots \end{array}\right\} \forall\, x_1 \tag{4.14}$$

and

$$\delta_0\cos\theta + \left(\int_0^\infty \lambda\, [A_1(\lambda)\overset{*}{J}_1(x_1) + B_1(\lambda)\overset{*}{Y}_1(x_1) \exp\{\pm i\frac{\lambda}{M}[x - F(x)]\}\, d\lambda\right) \cos\theta = 0 \quad \forall\, x_1 \tag{4.15}$$

in which

$$x_1 = \lambda\epsilon(x + a);\ (x + a)\epsilon(0, 1]$$

while dot means differentiation with respect to the argument x_1. Since J_n and Y_n are linearly independent, Eqs. (4.14) furnish

$$A_n = B_n = 0;\ n = 0, 2, 3,\ldots \tag{4.16}$$

$$\bar{A}_n = \bar{B}_n = 0;\ n = 0, 1, 2,\ldots$$

4.3 Third Boundary and Initial Condition

This condition concerns the singularity of the already constructed solutions appearing for $r \rightarrow 0$, namely near the axis of the slender body. In fact, for $r \rightarrow 0$, the Bessel function of the second kind $Y_1(\lambda r)$ introduced in the expression of the unsteady potential ψ, (Eq. (3.33)), as well as the logarithm function included in the expression of the steady potential ϕ, (Eq. (3.7)), become unbounded. In order to remove this peculiarity, we make use of the estimation of the radial velocity v near the axis of the body. According to Liepmann and Roshko [1957, p. 223] near the axis the resultant v is of the order 1/r. Thus, one writes

$$\text{For } t = 0, \lim_{r \to 0}(vr) = a_0 \tag{4.17}$$

where a_0 is a function of the variable x only, or a constant.

On the other hand, the expression for the velocity resultant v is given by (2.3), namely

$$\text{For} \quad t = 0, \quad v = \delta_0 \cos\theta + \phi_r + \psi_r. \tag{4.18}$$

Using (3.33), (4.12) and (4.14), as well as the last equation, we obtain

$$\text{For } t = 0, \lim_{r \to 0}(vr) = \lim_{r \to 0}(r\delta_0\cos\theta) + \lim_{r \to 0}(r\phi_r) + \lim_{r \to 0}(r\psi_r)$$

$$= \frac{\epsilon\left(1 + \dfrac{1}{(\gamma + 1)M^2}\right)}{\exp\left(\dfrac{1}{\epsilon^2[1 + (\gamma + 1)M^2]}\right)} - \lim_{r \to 0}\left(\int_0^\infty \{A_1(\lambda)[J_1(\lambda) - \lambda r J_0(\lambda r)] + \right.$$

$$\left. + B_1(\lambda)[Y_1(\lambda r) - \lambda r Y_0(\lambda r)]\{\exp \pm i\frac{\lambda}{M}[x - F(x)]\}\}d\lambda\right) \cos\theta.$$

Taking into account that for $r \to 0$, $J_1(0) = 0$; $J_0(0) = 1$; $Y_1(0) = Y_0(0) = -\infty$, the last equation is compatible with (4.17) if

$$B_1(\lambda) = 0, \tag{4.19}$$

giving

$$\lim_{r \to 0}(vr) = \epsilon\{1 + [1/(\gamma + 1)M^2]\} \exp\{-1/\epsilon^2[1 + (\gamma + 1)M^2]\} = \text{const.}$$

By now, relation (4.15) can be rewritten in the form

$$\left[\int_0^{\infty} \frac{\delta_0}{(1+\lambda)^2} + \lambda A_1(\lambda)\dot{J}_1(x_1)\exp\{\pm i\frac{\lambda}{M}[x - F(x)]\}d\lambda\right] \cos\theta = 0$$

which, since it must be valid for every θ, furnishes

$$\exp\{\pm i\frac{\lambda}{M}[x - F(x)]\} = -\frac{\delta_0}{\lambda(1+\lambda)^2 A_1(\lambda)\dot{J}_1(x_1)}. \tag{4.20}$$

Therefore, the arbitrary function included in the expression for the potential ψ, Eq. (3.33), is evaluated by the equation

$$\exp\{\pm i\frac{\lambda}{M}[x - F(x-t)]\} = -\frac{\delta_0}{\lambda(1+\lambda)^2 A_1(\lambda)\dot{J}_1(s)}\exp[\pm i\frac{\lambda}{M}t]; \tag{4.21}$$

$$s = \epsilon\lambda(x - t + \alpha),$$

where dot here means differentiation with respect to the argument $s = \epsilon\lambda(x - t + \alpha)$.

4.4 Fourth Boundary and Initial Condition

This condition is based on the fact that the flow has been already stabilized when it abandones the rigid cone. Considering the expression for the tangential velocity ν_t (Fig. 2)

$$\nu_t = u + \epsilon v \tag{4.22}$$

resulting by the transformation matrix (4.3), the above condition means that the tangential gradient of ν_t for $x = 1 - \alpha$; $r = \epsilon$; $t = 0$, is equal to zero, namely

$$\text{For } x = 1 - \alpha;\ r = \epsilon;\ t = 0,\ \left.\frac{\partial(u+\epsilon v)}{\partial x}\right|_{\text{body}} + \epsilon\left.\frac{\partial(u+\epsilon v)}{\partial r}\right|_{\text{body}} = 0. \tag{4.23}$$

Combining (4.23) with (2.3) and (2.5) we obtain

$$(\phi_{xx} + \psi_{xx} + \epsilon\phi_{xr} + \epsilon\psi_{xr})\Big|_{\text{body}} = (-\epsilon\phi_{xr} - \epsilon\psi_{xr} - \epsilon^2\phi_{rr} - \epsilon^2\psi_{rr})\Big|_{\text{body}}.$$

Observing that $\phi_{xx} = \phi_{xr} = 0$ and noting that $\epsilon << 1$ the above equation can be approximated in the form

$$\text{For } x = 1 = \alpha;\ r = \epsilon;\ t = 0,\ (\psi_{xx} + 2\epsilon\psi_{xr})\Big|_{\text{body}} = 0. \tag{4.24}$$

From now on, taking into account (4.21), (4.19) and (4.14), and retaining according to (2.3) only the real part in (4.21), we succeed in giving the final expression for the unsteady potential ψ as follows

$$\psi(x, r, \theta, t) = -\delta_0 \left[\int_0^\infty \frac{1}{\lambda(1+\lambda)^2} \frac{1}{\dot{J}_1(s)} J_1(\lambda r)\cos\frac{\lambda t}{M}\, d\lambda\right] \cos\theta;$$

$$s = \epsilon\lambda(x - t + \alpha). \tag{4.25}$$

Differentiating twice (4.25) with respect to x and r, introducing the results into (4.24) and using the well-known reccurence formulae for the derivatives of the Bessel function J_1, we deduce, after some algebra, an equation which, since it must be valid for every θ, results in the following transcendental equation

$$\left[\frac{3}{\mu^2} - 1\right]\left[1 + \frac{J_1(\mu)}{H(\mu)}\right] J_1(\mu) + 2\Lambda(\mu)\left[1 - \frac{\mu^2 J_1(\mu)\Lambda(\mu)}{H^2(\mu)}\right] = 0 \tag{4.26}$$

where

$$H(\mu) = J_1(\mu) - \mu J_0(\mu);\ \Lambda(\mu) = \left[\frac{2}{\mu^2} - 1\right] J_1(\mu) - \frac{1}{\mu} J_0(\mu);$$

$$\mu = \epsilon\lambda. \tag{4.26a}$$

By means of Eq. (4.26) we estimate an infinite number of positive eigenvalues μ_i ($i = 1, 2, \ldots$) to which correspond the values λ_i.

5. FINAL SOLUTIONS

Combination of equations (2.5), (4.12), (4.25) and (4.26, 26a) results in the following expression for the perturbed potential $\phi(x, r, \theta, t)$

$$\Phi(x, r, \theta, t) = \frac{1}{(\gamma + 1)M^2} x + \frac{\epsilon\left[1 + \frac{1}{(\gamma + 1)M^2}\right]}{\exp\left[\frac{1}{\epsilon^2[1 + (\gamma + 1)M^2]}\right]} \ln r + B -$$

$$- \delta_0 \epsilon \sum_{i=1}^{\infty} \left[\frac{1}{\mu_i\left[1 + \frac{\mu_i}{\epsilon}\right]^2} \frac{1}{\dot{J}_1(s_i)} J_1\left(\frac{\mu_i}{\epsilon} r\right) \cos\left(\frac{\mu_i}{\epsilon M} t\right)\right] \cos\theta; \qquad (5.1)$$

$$s_i = \mu_i(x - t + \alpha)$$

where dot means differentiation with respect to the argument s_i.

Introducing the solutions (5.1) into the matrix-equation (2.3) we derive, after some algebra, the following analytical expressions for the velocity resultants u, v and w

$$u = 1 + \frac{1}{(\gamma + 1)M^2} + \delta_0 \epsilon \sum_{i=1}^{\infty} \left[\frac{1}{\left[1 + \frac{\mu_i}{\epsilon}\right]^2} \frac{1}{H(s_i)} \left[1 + \frac{(1 - s_i^2)J_1(s_i)}{H(s_i)}\right] \times\right.$$

$$\left. \times J_i\left(\frac{\mu_i}{\epsilon} r\right)\cos\left(\frac{\mu_i t}{\epsilon M}\right)\right] \cos\theta;$$

$$v = \delta_0 \cos kt \cos\theta + \frac{\epsilon\left[1 + \frac{1}{(\gamma + 1)M^2}\right]}{\exp\frac{1}{\epsilon^2[1 + (\gamma + 1)M^2]}} \frac{1}{r} -$$

$$- \frac{\delta_0 \epsilon}{r} \sum_{i=1}^{\infty} \left[\frac{1}{\mu_i \left[1 + \frac{\mu_i}{\epsilon}\right]^2} \frac{s_i}{H(s_i)} H(\frac{\mu_i}{\epsilon} r) \cos(\frac{\mu_i t}{\epsilon M}) \right] \cos\theta; \qquad (5.2)$$

$$w = -\delta_0 \cos kt \sin\theta - \frac{\delta_0 \epsilon}{r} \sum_{i=1}^{\infty} \left[\frac{1}{\mu_i \left[1 + \frac{\mu_i}{\epsilon}\right]^2} \frac{s_i}{H(s_i)} J_1(\frac{\mu_i}{\epsilon} r) \cos(\frac{\mu_i t}{\epsilon M}) \right] \sin\theta,$$

in which H as in (4.26a) and s_i as in (5.1).

6. CONCLUSIONS

We have constructed analytical solutions for the problem of nonlinear supersonic flow analysis for slender bodies of revolution due to small amplitude oscillations. The above solutions refer to the nonlinear time independent P.D.E. concerning the steady flow potential, and the linear time dependent P.D.E. concerning the unsteady one. The total solution includes one arbitrary function, fact that permits us to investigate any boundary of the slender body under consideration. The evaluation of the flow velocity resultants is achived through a set of four boundary and initial conditions in accordance with the physical problem.

We underline that, contrary to the existing approximate and numerical techniques in obtaining solutions of the examined problem, the developed herein methodology succeeds in constructing analytical solutions in explicit form of the flow velocity resultants.

Finally, we believe that the suggested herein technique may proved powerful in the investigation of relative problems in fluid mechanics and gas dynamics, as well as in other domains of mechanics.

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APPENDIX I

We shall prove that the expression

$$f(x, t) = \exp\{\pm\, i\frac{\lambda}{M}[x - F(x - t)]\}$$

in which F is an arbitrary function, constitutes a solution of the linear P.D.E.

$$M^2 f_{xx} + 2M^2 f_{xt} + M^2 f_{tt} + \lambda^2 f = 0.$$

We compute the following partial derivatives

$$f_x = \pm i \frac{\lambda}{M} (1 - F')K(x, t);$$

$$f_t = \pm i \frac{\lambda}{M} F' K(x, t);$$

$$f_{xx} = [\mp i \frac{\lambda}{M} F'' - \frac{\lambda^2}{M^2} (1 - F')^2]K(x, t);$$

$$f_{xt} = [\pm i \frac{\lambda}{M} F'' - \frac{\lambda^2}{M^2} F' (1 - F')]K(x, t);$$

$$f_{tt} = [\mp i \frac{\lambda}{M} F'' - \frac{\lambda^2}{M^2} F'^2] K(x, t)$$

in which prime means differentiation with respect to the argument $(x - t)$, while

$$K(x, t) = \exp\{\pm i \frac{\lambda}{M} [x - F(x - t)]\}.$$

Introducing the above expressions into the P.D.E. under consideration we verify that the function $f(x, t)$ is a solution of this equation.