Some New Exact van der Waerden numbers

Bruce Landman
Department of Mathematics
University of West Georgia, Carrollton, GA 30118
landman@westga.edu

Aaron Robertson
Department of Mathematics
Colgate University, Hamilton, NY
aaron@math.colgate.edu

Clay Culver
University of West Georgia, Carrollton, GA 30118

Received: 12/12/04, Accepted: 6/28/05, Published: 9/1/05

Abstract

For positive integers $r, k_0, k_1, ..., k_{r-1}$, the van der Waerden number $w(k_0, k_1, ..., k_{r-1})$ is the least positive integer n such that whenever $\{1, 2, \ldots, n\}$ is partitioned into r sets $S_0, S_1, ..., S_{r-1}$, there is some i so that S_i contains a k_i-term arithmetic progression. We find several new exact values of $w(k_0, k_1, ..., k_{r-1})$. In addition, for the situation in which only one value of k_i differs from 2, we give a precise formula for the van der Waerden function (provided this one value of k_i is not too small).

1 Introduction

A well-known theorem of van der Waerden [9] states that for any positive integers k and r, there exists a least positive integer, $w(k; r)$, such that any r-coloring of $[1, w(k; r)] = \{1, 2, \ldots, w(k; r)\}$ must contain a monochromatic k-term arithmetic progression $\{x, x+d, x+2d, \ldots, x+(k-1)d\}$. Some equivalent forms of van der Waerden's theorem may be found in in [7]. It is easy to see that the existence of the van der Waerden numbers $w(k; r)$ implies the existence of the (more general) van der Waerden numbers $w(k_0, k_1, ..., k_{r-1}; r)$ which are defined as follows.

Definition. *Let $k_0, k_1, ..., k_{r-1}$ be positive integers. The van der Waerden number $w(k_0, k_1, ..., k_{r-1}; r)$ is the least positive integer n such that every r-coloring $\chi : [1, n] \to \{0, 1, ..., r-1\}$ admits, for some i, $0 \le i \le r-1$, a k_i-term arithmetic progression of color i.*

If the value of r is clear from the context, we will denote the van der Waerden number $w(k_0, k_1, ..., k_{r-1}; r)$ more simply by $w(k_0, k_1, ..., k_{r-1})$.

For example, $w(4, 4) = w(4, 4; 2)$ has the same meaning as the classical van der Waerden number $w(4; 2)$; $w(3, 3, 3, 3)$ has the same meaning as $w(3; 4)$; and $w(5, 4, 7; 3) = w(5, 4, 7)$ represents the least positive n such that every (red,blue,green)-coloring of $[1, n]$ yields a red 5-term arithmetic progression, a blue 4-term arithmetic progression, or a green 7-term arithmetic progression.

The study of these "mixed" van der Waerden numbers apparently has received relatively little attention, especially when compared to, say, the classical (mixed) graph-theoretical Ramsey numbers $R(k_1, k_2, ..., k_r)$. It is easy to calculate by hand that $w(3, 3) = 9$. Other non-trivial values of the van der Waerden numbers were published by Chvátal [4], Brown [3], Stevens and Shantaram [8], Beeler and O'Neil [2], and Beeler [1], in 1970, 1974, 1978, 1979, and 1983, respectively.

The purpose of this note is to expand the table of known van der Waerden numbers. In Section 2, we present several new van der Waerden numbers. In Section 3, we give a formula for the van der Waerden numbers of the type $w(k, 2, 2, ..., 2; r)$, i.e., where all but one of the k_i's equal 2, provided k is large enough in relation to r.

2 Some New Values

Table 1 below gives all known non-trivial van der Waerden numbers, $w(k_0, k_1, ..., k_{r-1})$, where at least two of the k_i's exceed two (the cases where only one k_i exceeds two are handled, more generally, in the next section). The entries in the table due to Chvátal, Brown, Stevens and Shantaram, Beeler and O'Neil, or Beeler, are footnoted by a, b, c, d, or e, respectively. The entries that we are presenting here as previously unknown values are marked with the symbol *.

To calculate most of the new van der Waerden numbers, we used a slightly modified version of the "culprit" algorithm, introduced in [2]. In certain instances, we used a very simple backtracking algorithm, which can be found in [8]. These, and a third algorithm, are also described in [7].

r	k_0	k_1	k_2	k_3	k_4	w
2	3	3	-	-	-	9
2	4	3	-	-	-	18^a
2	4	4	-	-	-	35^a
2	5	3	-	-	-	22^a
2	5	4	-	-	-	55^a
2	5	5	-	-	-	178^c
2	6	3	-	-	-	32^a
2	6	4	-	-	-	73^d
2	7	3	-	-	-	46^a
2	7	4	-	-	-	109^e
2	8	3	-	-	-	58^d
2	9	3	-	-	-	77^d
2	10	3	-	-	-	97^d
2	11	3	-	-	-	114^*
2	12	3	-	-	-	135^*
2	13	3	-	-	-	160^*
3	3	3	2	-	-	14^b
3	3	3	3	-	-	27^a
3	4	3	2	-	-	21^b
3	4	3	3	-	-	51^d
3	4	4	2	-	-	40^b
3	4	4	3	-	-	89^*
3	5	3	2	-	-	32^b
3	5	3	3	-	-	80^*
3	5	4	2	-	-	71^b
3	6	3	2	-	-	40^b
3	6	4	2	-	-	83^*
3	7	3	2	-	-	55^*
4	3	3	2	2	-	17^b
4	3	3	3	2	-	40^b
4	3	3	3	3	-	76^d
4	4	3	2	2	-	25^b
4	4	3	3	2	-	60^*
4	4	4	2	2	-	53^b
4	5	3	2	2	-	43^b
4	6	3	2	2	-	48^*
4	7	3	2	2	-	65^*
5	3	3	2	2	2	20^*
5	3	3	3	2	2	41^*

TABLE 1. Van der Waerden Numbers

An r-coloring of an interval $[a,b]$ is said to be $(k_0,k_1,...,k_{r-1})$-*valid* (or simply *valid* if the specific r-tuple is clear from the context) if, for each $i = 0,1,...,r-1$, the coloring avoids k_i-term arithmetic progressions of color i.

For each of the new van der Waerden numbers $w = w(k_0,k_1,...,k_{r-1})$ we computed, the program also outputted all maximal length $(k_0,k_1,...,k_{r-1})$-valid r-colorings (i.e., all valid colorings of length $w-1$). We now describe all of these maximal-length valid colorings. For convenience, we will denote a coloring as a string of colors. For example, the coloring χ of $[1,5]$ such that $\chi(1) = \chi(2) = 0$ and $\chi(3) = \chi(4) = \chi(5) = 1$, will be denoted by the string 00111. Further, a string of $t \geq 2$ consecutive i's will be denoted by i^t. Thus, for example, $0^2 1^3$ represents 00111. In counting the number of valid colorings, we will consider two colorings to be the same if one can be obtained from the other by a renaming of the colors. For example, we consider 11001 and 00110 to be the same coloring of [1,5].

$\mathbf{w(11,3)}$: Corresponding to $w(11,3) = 114$, there are thirty different valid 2-colorings of [1,113]. These are:

$0^{10}10^{10}1^2 0^6 10^{10}1^2 01^2 0^9 1010^6 10^9 1^2 0^2 10^{10}10^6 1^2 0^4 10^7 a0^2 b$,

where at least one of a and b equals 1;

$0^7 c010^7 1^2 0^3 10^7 1^2 0^4 10^{10}1010^6 10^9 10^2 10^3 1010^2 d0^7 1010^3 1^2 0^{10}1e0^6 1$,

where the only restriction on c, d, and e is that c and e are not both 1;

$0^9 10^3 1010^2 f010^8 10^3 g1010^{10}10^8 10^9 1^2 0^5 1010^{10}101^2 010^{10}1010^3 h0^4$,

where the only restriction on f, g, and h is that f and g are not both 0;

and the fifteen colorings obtained by reversal of the fifteen colorings described above.

$\mathbf{w(12,3)}$: Corresponding to $w(12,3) = 135$, there is only one valid 2-coloring of $[1,134]$:

$$0^9 10^8 10^9 10^2 10^3 1010^7 10^2 1010^3 10^{11}1^2 0^{11}10^3 1010^2 10^7 1010^3 10^2 10^9 10^8 10^9$$

(it is its own reversal).

$\mathbf{w(13,3)}$: There are twenty-four different (13,3)-valid 2-colorings of [1,159]:

$0^{12}10^2 a0^7 101^2 0^{12}10^2 1^2 0^6 10^3 10^9 10^2 1^2 0^4 10^2 10^4 10^8 10^{10}10^4 10^7 1^2 0^2 10^9 10^3 1^2 0^9 10^{12}10^2$,

where a may be 0 or 1;

$0^2 10^4 10^5 10b0^{10}10^5 1^2 010^{10}10^7 10^3 10^2 10^9 10^2 c0^3 1010^{11}10^4 10^2 10^{10}10^6 1^2 0^{11}1^2 0^4 10^{12}10^6 10$,

with no restrictions on b or c;

$0^2 1010^4 d0^5 10^7 10^6 10^3 0^5 1^2 010^5 10^{10}e010^6 10^2 10^{12}101^2 0^3 10^7 10^5 10^4 10^5 1010^6 10^{12}0^5 10^2 10^7 f01$,

where the only restriction is that d and e are not both 1;

and the twelve others obtained from these twelve via reversal.

$\mathbf{w(4,4,3)}$: There are four (4,4,3)-valid 3-colorings of [1,88]:

$$0^2 a101^3 0^3 1^3 02121^2 2^2 010^2 202^2 01^2 2^2 0^3 101^3 0^3 1^3 02121^3 2010^2 202^2 0102^2 0^2 1^2 01^3 0^3 21^2 01^2 0$$

where a may be 0 or 1, along with the two reversals of these colorings.

$\mathbf{w(5,3,3)}$: There are forty-two different $(5,3,3)$-valid 3-colorings of [1,79]:

Twenty-one are of the form

$$0^2a102b0^2101^2210^310^420^3212^2020^21^2021^202^20^2101^2210^310^420^3212^2020^2c102d0^2,$$

where $ab \neq 00$, $cd \neq 00$, $b \neq 1$, $c \neq 2$, and $ad \neq 11$; by reversal of these colorings, the other twenty-one colorings are obtained (note: of course, since $k_2 = k_3$, interchanging of the colors 1 and 2 will not change the validity of any coloring, so we are not considering the forty-two valid colorings of [1,79] obtained in this manner, from those already listed, to be additional colorings).

$\mathbf{w(6,4,2)}$: There are twelve different $(6,4,2)$-valid 3-colorings of [1,82]:

$0^210101^20^310101^2010^31^20^310^51^201^221^20^410^210^31^2010^210^310101^20^510^31^20^2$;
$0^410^51^201010^4101^20^310^41^3010^410^3101^221^2010^410^21a0^31^3010^51b01c0$,
where either $a = 0$ or else $(a,b,c) = (1,0,0)$ (this gives five colorings);

and six more by reversal of the above colorings.

$\mathbf{w(7,3,2)}$: To describe the $(7,3,2)$-valid 3-colorings of $[1,54]$, we adopt the following notation. For $1 \leq i \leq 10$, denote by S_i the coloring

$$0^61^20^310^5101^20^31010^620^4a010^21^20^2b0cd0e,$$

where (a,b,c,d,e) is the ith element of the 10-element set $\{(0,0,0,1,1),(0,0,1,0,1),(0,0,1,1,1),$ $(0,1,0,0,0),$ $(0,1,0,0,1),$ $(0,1,1,0,0),(0,1,1,0,1),$ $(1,1,0,0,0),(1,1,1,0,0),$ $(0,1,0,1,1)\}$. Then the valid colorings of $[1,54]$ are:

$0S_i, \quad 1 \leq i \leq 9$;
$1S_i; \quad$ where $i \in \{1,4,5,8,10\}$;
$0^310^51^20^210^4101^20^520^61010^31^2010^510^3a$, where a may be 0 or 1;
$0^410^510^010^4101^20^520^51^20^51^201^20^610^2$;
$010^30^410^210^410^220^6101^20^31^20^610^31$;

and eighteen obtained by reversal of those already listed, giving a total of thirty-six valid colorings.

$\mathbf{w(4,3,3,2)}$: There are eight different $(4,3,3,2)$-valid 4-colorings of [1,59]:

$$1012^201020^2a01^22021^20^3320^320^21012^20121^20120^22^20^3b021^2,$$

where $a, b \in \{1,2\}$, and four others obtained by reversal.

$\mathbf{w(6,3,2,2)}$: There are twenty-eight different $(6,3,2,2;4)$-valid colorings of [1,47]:

$$
\begin{array}{l}
00001000211011000011010000300001000110000010011 \\
0000101001000001021100000100100031001000a011010,\ a \in \{0,1\} \\
0000101001000001021100100000100131001000001001a,\ a \in \{0,1\} \\
00010000112110000110100003000010001100000100110 \\
0100001001000001121100000100100031001000001101a,\ a \in \{0,1\} \\
0100101001000001021100000100100031001000a011010,\ a \in \{0,1\} \\
0100101001000001021100100000100131001000001001a,\ a \in \{0,1\} \\
1a001010010000210001001000000130101001000001001 1,\ a \in \{0,1\}
\end{array}
$$

plus fourteen more colorings obtained by reversing the above fourteen colorings.

$\mathbf{w(7,3,2,2)}$: There are five different $(7,3,2,2)$-valid colorings of [1,64]:

$$0^3 10^6 1^2 0a10^6 10^2 10^5 12310^6 1^2 01^2 0^6 10^2 10^5 10^3$$

where $a \in \{0,1\}$, along with the reversals of these; and

$$0^2 10^3 10^6 1^2 01^2 0^2 20^6 1010^4 1010^6 30^2 1^2 01^2 0^6 10^3 10^2,$$

which is its own reversal.

$\mathbf{w(3,3,2,2,2)}$: There are five different $(3,3,2,2,2;5)$-valid colorings of [1,19]:

$$
\begin{array}{l}
0^2 10^2 1^2 231410^2 1^2 0^2 1,\ \text{and its reversal;} \\
0^2 1^2 0^2 1^2 2340^2 1^2 0^2 1^2; \\
0^2 1^2 21^2 0^2 30^2 1^2 41^2 0^2; \\
0101^2 0102340101^2 010.
\end{array}
$$

$\mathbf{w(3,3,3,2,2)}$: There are forty-two different $(3,3,3,2,2)$-valid 5-colorings of [1,40]:

$0xy^2 2y2x^2 2^2 x^2 2y2y^2 x3xy^2 2y2x^2 2^2 x^2 2y2y^2 xb4,$
where $xy = 01$ or $xy = 10$, and $b \in \{0,1,2\}$;
$010x^2 yxy0^2 y^2 0^2 yxyx^2 ab0x^2 yxy0^2 y^2 0^2 yxyx^2 cd,$
where $xy = 12$ or $xy = 21$, $abc = 034$ or $abc = 340$, and $d \in \{0,1,2\}$;
$012^2 0201^2 0^2 1^2 0202^2 1312^2 0201^2 0^2 1^2 0202^2 1a4,$ where $a \in \{0,1,2\}$;

along with the reversals of the (twenty-one) colorings above.

3 When All But One k_i Equals 2

Consider $w(k, 2, 2, ..., 2; r)$. For ease of notation, we will denote this function more simply as $w_2(k; r)$. We give an explicit formula for $w_2(k; r)$ provided k is large enough in relation to r.

We adopt the following notation. Let $p_1 < p_2 < \cdots$ be the sequence of primes. For $r \geq 2$, denote by $\pi(r)$ the number of primes not exceeding r, and denote by $\#r$ the product $p_1p_2 \cdots p_{\pi(r)}$. For $k, r \geq 2$, let $j_{k,r} = \min\{j \geq 0 : \gcd(k - j, \#r) = 1\}$, and $\ell_{k,r} = \min\{\ell \geq 0 : \gcd(k - \ell, \#r) = r\}$.

Theorem Let $k > r \geq 2$. Let $j = j_{k,r}$, $\ell = \ell_{k,r}$, and $m = \min\{j, \ell\}$.

I. $w_2(k; r) = rk$ if $j = 0$.

II. $w_2(k; r) = rk - r + 1$ if either (i) $j = 1$; or (ii) r is prime and $\ell = 0$.

III. If r is composite and $j \geq 2$, then $w_2(k; r) \geq rk - j(r - 2)$, with equality provided either (i) $j = 2$ and $k \geq 2r - 3$; or (ii) $j \geq 3$ and $k \geq (\pi(r))^3(r - 2)$.

IV. If r is prime, $j \geq 2$, and $\ell \geq 1$, then $w_2(k; r) \geq rk - m(r - 2)$, with equality provided either (i) $\ell = 1$, (ii) $m = 2$ and $k \geq 2r - 3$, or (iii) $m \geq 3$ and $k \geq (\pi(r))^3(r - 2)$.

Proof. (I) To show $w_2(k; r) \leq rk$, let $g : [1, rk] \to \{0, 1, 2, ..., r - 1\}$ be any r-coloring. If g is valid, then no k consecutive elements of $[1, rk]$ have color 0, and for each i, $1 \leq i \leq r - 1$, there is at most one element with color i. This is not possible, since these last two conditions would imply that no more than $r - 1 + r(k - 1) = rk - 1$ integers have been colored. Hence, g is not valid.

To show that rk is also a lower bound, we show that the following coloring of $[1, rk - 1]$: is valid:

$$0^{k-1}10^{k-1}20^{k-1}3 \ldots 0^{k-1}(r - 1)0^{k-1}.$$

Obviously, this coloring admits no 2-term monochromatic arithmetic progression having color other than 0. For a contradiction, assume there is a k-term arithmetic progression $A = \{a + id : 0 \leq i \leq k - 1\}$ of color 0. Since $k > r$, we have $1 < d \leq r$. Hence, by hypothesis, $(d, k) = 1$, so A is a complete residue system modulo k. Therefore there is some $x \in A$ with $x \equiv 0 (\text{mod } k)$. Then x does not have color 0, a contradiction.

(II) (i) We first note that k must be even, for otherwise $(k - 1, \#r) \neq 1$. We prove that $r(k - 1) + 1$ is an upper bound by induction on r. First, let $r = 2$. Let g be an arbitrary (0,1)-coloring of $[1, 2k - 1]$. If g is $(k, 2)$-valid and if there is at most one integer having color 1, then g must be the coloring $0^{k-1}10^{k-1}$ (since there cannot be k consecutive integers with color 0). This contradicts the validity of g since $g(\{2i - 1 : 1 \leq i \leq k\}) = 0$. This establishes the desired upper bound for $r = 2$.

Now let $r \geq 3$ and assume that for all s, $2 \leq s < r$, we have $w_2(k; s) \leq s(k - 1) + 1$ whenever $j_{k,s} = 1$. Let h be any r-coloring of $[1, r(k - 1) + 1]$, with $j_{k,r} = 1$. Without loss of generality, we may assume $h = B_1 1 B_2 2 B_3 3 ... B_{r-1}(r-1)B_r$, where each B_i is a (possibly empty) string of 0's. For each i, $1 \le i \le r-1$, let x_i be the integer for which $h(x_i) = i$.

For a contradiction, assume h is $(k, 2, 2, ..., 2; r)$-valid. Note that $|B_i| \le k-1$ for $1 \le i \le r-2$, so that $|B_r| \ge 1$. Now, for each t, $1 \le t \le r-1$, since all primes $p \le t$ do not divide $k-1$ and since k is even, $j_{k,t} = 1$. By the induction hypothesis, $w_2(k,t) \le t(k-1)+1$, and therefore we may assume that h admits at least $t+1$ colors within $[1, t(k-1)+1]$ (or else h would not be valid). Thus, $x_t \le t(k-1)+1$ for $1 \le t \le r-1$. By a symmetric argument (considering intervals of the form $[(r-t)(k-1)+1, r(k-1)+1]$), it follows that $x_t \ge t(k-1)+1$ for all t, $1 \le t \le r-1$. Hence for each t, $1 \le t \le r-1$, we have

$$x_t = t(k-1)+1.$$

Consider $S = \{1 + ir : 0 \le i \le (k-1)\}$. We claim that S is monochromatic of color 0, the truth of which contradicts the assumption that h is (k, r)-valid. Let $1 + cr \in S$. If $h(1+cr) \ne 0$, then $cr = n(k-1)$ for some n, $1 \le n \le r-1$. However, $(k-1)$ cannot divide cr since $\gcd(k-1, r) = 1$ and $c \le k-2$ (since $|B_r| \ge 1$). Hence, $h(S) = 0$, which completes the proof that $r(k-1)+1$ is an upper bound.

The coloring $0^{k-1}10^{k-2}20^{k-2}\ldots(r-2)0^{k-2}(r-1)0^{k-2}$ is $(k, 2, 2, ..., 2; r)$-valid, which establishes that $r(k-1)+1$ is also a lower bound. To see that this coloring is valid, note that if $S = \{a + id : 0 \le i \le k-1\} \subseteq [1, r(k-1)]$ is an arithmetic progression, then since $d \le r-1$, we have $\gcd(k-1, d) = 1$. Thus, S represents all residue classes modulo $k-1$, in particular the class $1(\text{mod } (k-1))$, with a and $a+(k-1)d$ in the same class. Hence, since no member of $\{1 + i(k-1) : 1 \le i \le r-1\}$ has color 0, S cannot be monochromatic.

(II) To show $r(k-1)+1$ is a lower bound, we show that the coloring

$$0^{k-1}10^{k-1}2...(r-2)0^{k-1}(r-1)0^{k-r}$$

is valid. Let A be a k-term arithmetic progression. Obviously, there is no k-term arithmetic progression having color 0 and gap $d = 1$. So we may assume $2 \le d < r$. Since A forms a complete residue system modulo k, some member of A is a multiple of k. Hence A is not monochromatic, so the coloring is valid.

To complete the proof, let f be any r-coloring of $[1, r(k-1)+1]$, and assume (for a contradiction) that f is valid. By the proof of Part I, we may assume that for $1 \le s \le r-1$, the interval $[1, ks]$ assumes at least $s+1$ colors, and the interval $[k-r+2, r(k-1)+1]$ assumes all r colors. Since all non-zero colors occur at most once, $[ks+1, r(k-1)+1]$ assumes at most $r-s$ colors for $1 \le s \le r-1$. Similarly, $[1, k(r-s)-r+1]$ assumes at most $r-s$ colors. From this we may conclude that $f([sk+1, ks+k-r+1]) = 0$ for each s, $0 \le s \le r-1$. Since $r|k$, we obtain $f(1+jr) = 0$ for $j = 0, 1, ..., k-1$, a contradiction.

(III) Note that the coloring

$$0^{k-1}10^{k-j-1}20^{k-j-1}3\ldots0^{k-j-1}(r-1)0^{k-1}$$

is a valid coloring of $[1, n] = [1, kr - j(r-2) + 1]$. (The argument is very similar to that used to establish the lower bounds in Parts I and II of the theorem.)

To complete the proof, we will show that, subject to the stated restrictions on k, the above coloring is the *only* valid r-coloring of $[1, n]$. The desired result then follows since the coloring clearly cannot be extended to a valid r-coloring of $[1, n+1]$.

Let $\tau : [1, n] \to \{0, 1, \ldots, r-1\}$ be valid. We may assume that

$$\tau = 0^{\alpha_1} 1 0^{\alpha_2} 2 \ldots 0^{\alpha_{r-1}} (r-1) 0^{\alpha_r},$$

with $0 \leq \alpha_i \leq k-1$ for $i = 1, 2, \ldots, r-1$. For $i = 1, 2, \ldots, r-1$, let $\tau(y_i) = i$. Let $y_0 = 0$ and $y_r = n+1$. Define

$$B_i = \{x : y_{i-1} < x < y_i\}$$

for $i = 1, 2, \ldots, r$, so that $|B_i| = \alpha_i$.

Let $b \geq 1$, $t \geq 2$, and assume that $b + t \leq r$. We show that there do not exist i_1 and i_2 such that

$$b \leq i_1 < i_2 \leq b + t - 1 \text{ and } y_{i_1} \equiv y_{i_2} \pmod{t}. \tag{1}$$

Let

$$a = \sum_{i=b}^{b+t} \alpha_i.$$

We know that $\sum_{i=1}^{r} \alpha_i = kr - j(r-2) - 1 - (r-1) = r(k-1) - j(r-2)$. Since $\alpha_i \leq k-1$ for all $i \in [1, r]$, we have

$$a \leq r(k-1) - j(r-2) - (k-1)(r-t-1) = (k-1)(t+1) - j(r-2). \tag{2}$$

Now consider the interval

$$I = \left(\bigcup_{i=b}^{b+t} B_i \right) \bigcup \{y_b, y_{b+1}, \ldots, y_{b+t-1}\}.$$

We claim that $|I| \geq kt$ (for both Cases (i) and (ii)). Using (2), we see that $|I| \geq (k-1)(t+1) - 2(r-2) + t = kt + k - 1 - 2(r-2)$. Since $k \geq 2r - 3$, the claim holds in Case (i) .

For Case (ii) we turn to a result of Jurkat and Richert [6]. Let $C(r)$ be the length of the longest string of consecutive integers, each divisible by one of the first $\pi(r)$ primes (this is a particular case of what is known as Jacobsthal's function). Jurkat and Richert showed that $C(r) < (\pi(r))^2 \exp(\log(\pi(r))^{13/14})$. (As an aside, better asymptotic bounds have been proved: Iwaniec [5] showed that $C(r) \ll \pi(r)^2 \log^2(\pi(r))$.) Using the slightly weaker bound $C(r) < \pi(r)^3$, we have that $j < \pi(r)^3$. Hence, from (2) we see that $|I| > kt + k - 1 - \pi(r)^3(r-2)$. Since $k \geq \pi(r)^3(r-2)$, we have $|I| \geq kt$.

Having established that $|I| \geq kt$, now assume, for a contradiction, that there exist i_1 and i_2 satisfying (1). Then there is some c, $0 \leq c \leq t-1$, so that no member of

$\{y_m, y_{m+1}, \ldots, y_{m+t-1}\}$ is congruent to c modulo t. Since $\{y_m, y_{m+1}, \ldots, y_{m+t-1}\}$ are the only members of I *not* of color 0, the set $\{x \in I : x \equiv c \,(\mathrm{mod}\ t)\}$ is an arithmetic progression of length at least k and of color 0, a contradiction. Hence, (1) is true.

By (1) we have, in particular, that for all $i \in [1, r-2]$, $y_{i+1} - y_i \not\equiv 0 \,(\mathrm{mod}\ t)$ for each $t = 2, 3, \ldots, r-1$. By assumption, for all $u \in [k-j+1, k]$, there exists $t \in [2, r-1]$ that divides u. Hence, for each $i \in [1, r-2]$ we must have $y_{i+1} - y_i \leq k - j$. Thus,

$$\alpha_i \leq k - j - 1 \quad \text{for } 2 \leq i \leq r-1.$$

Since

$$\sum_{i=2}^{r-1} \alpha_i \geq (r-2)(k-j-1)$$

we must have $\alpha_i = k - j - 1$ for $i \in [2, r-1]$, and hence $\alpha_1 = \alpha_r = k - 1$.

(IV) The proof is essentially the same as that for Part III, where we use the r-coloring

$$0^{k-1}10^{k-m-1}20^{k-m-1}3 \ldots 0^{k-m-1}(r-1)0^{k-1},$$

which is a valid coloring of $[1, n] = [1, kr - m(r-2) + 1]$. For this reason, we include the details for only Case (i), where we need only the restriction that $k > r$.

To prove the result for Case (i), note first that since $\gcd(k-1, \#(r-1)) = 1$, by II.ii of this theorem, we know that $w_2(k; r-1) = k(r-1) - r + 2$. Using the notation from the proof of Part III , we thus may assume that $y_1 \geq k$ and $y_{r-1} \leq k(r-1) - r + 2$. This implies that

$$\sum_{i=2}^{r-1} \alpha_i = (r-2)(k-2),$$

and therefore, for any u, $1 \leq u \leq r-2$, we have

$$\sum_{i=u}^{u+2} \alpha_i \geq 3(k-1) - r + 2.$$

Now if some α_i is odd, where $2 \leq i \leq r-1$, then there is a 0-colored arithmetic progression with gap 2 and length at least

$$\frac{3k - r - 1}{2} + \frac{1}{2} = \frac{3k - r}{2} \geq k.$$

Otherwise, if each α_i, $2 \leq i \leq r-1$, is even, then since k is even (by hypothesis) and $\alpha_i \leq k-1$, we must have $\alpha_i = k-2$ for $i = 2, \ldots, r-1$. $\square$

Remarks. It is easy to see that the converse of Part I is also true. It is also clear that, for any fixed r, the values of $w_2(k; r)$ given by the theorem may be expressed in terms of the residue classes of k modulo $\#r$. This is convenient for small values of r. As examples, for $k \geq 4$, we have $w_2(k; 3) = 3k$ if $k \equiv \pm 1 (\mathrm{mod}\ 6)$, $w_2(k; 3) = 3k - 1$ if $k \equiv 3 (\mathrm{mod}\ 6)$, and $w_2(k; 3) = 3k - 2$ otherwise; and, for $k \geq 5$, we have $w_2(k; 4) = 4k - 3$ if $k \equiv 0$ or 2 (mod

6), $w_2(k;4) = 4k-4$ if $k \equiv 3 (\text{mod } 6)$, $w_2(k;4) = 4k-6$ if $k \equiv 4 (\text{mod } 6)$ (the theorem gives this last equality only for $k \geq 16$, but it is easy to show that it holds provided $k \geq 5$), and $w_2(k;4) = 4k$ otherwise. We believe that the restrictions on the magnitude of k (in relation to r) in Parts III.ii and IV.iii can be weakened substantially. We also would like to obtain precise formulas in the other cases, but with weaker restrictions on the magnitude of k.

Acknowledgement

We thank Carl Pomerance for pointing us to some relevant literature.

References

[1] M. Beeler, A new van der Waerden number, *Discrete Applied Math.* **6** (1983), 207.

[2] M. Beeler and P. O'Neil, Some new van der Waerden numbers, *Discrete Math.* **28** (1979), 135-146.

[3] T.C. Brown, Some new van der Waerden numbers (preliminary report), *Notices American Math. Society* **21** (1974), A-432.

[4] V. Chvátal, Some unknown van der Waerden numbers, in *Combinatorial Structures and their Applications*, 31-33, Gordon and Breach, New York, 1970.

[5] H. Iwaniec, On the error term in the linear sieve, *Acta Arith.* **19** (1971), 1-30.

[6] W. B. Jurkat and H. E. Richert, An improvement of Selberg's sieve method I, *Acta Arith.* **11** (1965), 217-240.

[7] B. Landman and A. Robertson, Ramsey Theory on the Integers, *American Math. Society*, Providence, RI, 317pp., 2004.

[8] R. Stevens and R. Shantaram, Computer-generated van der Waerden partitions, *Math. Computation* **32** (1978), 635-636.

[9] B.L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk.* **15** (1927), 212-216.