RATES OF CONVERGENCE FOR LINEAR ACTIONS OF COCOMPACT LATTICES ON THE COMPLEX PLANE

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Abstract

We consider the distribution of orbits of linear actions of certain discrete groups on the complex plane.

1. Introduction

In this note we will consider the distribution of orbits of linear actions of certain discrete groups on the complex plane. It is convenient to first formulate this problem in a slightly more general setting, so we first let $SL(2, K)$ denote the group of 2×2 matrices with unit determinant having entries in a field K (typically $\mathbb{R}$ or $\mathbb{C}$). Given a cocompact lattice $\Gamma < SL(2, K)$ (with $-I \in \Gamma$) we can consider the standard linear action

$$\begin{aligned} &\Gamma \times K^2 \to K^2 \\ &\gamma(x, y) = (ax + by, cx + dy), \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \end{aligned}$$

We define a norm by $||\gamma|| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}$ and given $X = (x_1, x_2) \in K^2$ we denote $||X|| = \sqrt{|x_1|^2 + |x_2|^2}$.

In [5], Ledrappier proved equidistribution results on the Γ-orbits when $K = \mathbb{R}$ (cf. also [9]). A decade later, Maucourant and Weiss showed in this setting a stronger result on the rates of convergence for the distribution of such orbits.

Theorem 1 (Maucourant-Weiss [7]). *Given a cocompact group $\Gamma < SL(2, \mathbb{R})$ with $-I \in \Gamma$ there exists $\rho > 0$ such that for any C^∞ compactly supported function F and any $X \in \mathbb{R}^2 - \{(0, 0)\}$ we can write*

$$\frac{1}{T} \sum_{\gamma \in \Gamma, \|\gamma\| \leq T} F(\gamma X) = \frac{1}{\|X\|} \int \frac{F(Y)}{\|Y\|} d\lambda(Y) + O\left(\frac{1}{T^\rho}\right) \qquad (1)$$

as $T \to +\infty$.

The approach in [7] uses estimates on horocycles due to Strombergsson, giving additional estimates on the value of ρ. Even sharper estimates in the case that $\Gamma = SL(2,\mathbb{Z})$ were proved by Laurent and Nogueira [4].

There are analogous equidistribution results on Γ-orbits to those in [5] with $K = \mathbb{C}$ in [6]. We begin by recalling a simple example due to Jorgensen [3].

Example 2. *Let us fix the values*

$$\beta = \frac{1}{4}(\sqrt{10} + 2\sqrt{17} + (-6 + 2\sqrt{17})) \text{ and } x = \frac{(\sqrt{10} - 2\sqrt{17} + i(6 + 2\sqrt{17}))}{(2(-6 + 2\sqrt{17})}.$$

We can then consider the matrices

$$A = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \text{ and } B = \begin{pmatrix} -(1+I)x/\sqrt{2} & -(1+x^2) \\ 1 & (1-i)x/\sqrt{2} \end{pmatrix}$$

and then let $\Gamma < SL(2,\mathbb{C})$ denote the group they generate. This is a discrete cocompact group and we can consider the the orbit of the $X = (1,0) \in \mathbb{C}^2$, say, under the linear action $\Gamma \times \mathbb{C}^2 \to \mathbb{C}^2$. In figure 1 we plot the real and imaginary coordinates of part of this orbit which lends credence to the relatively slow polynomial convergence.

We will show the following result, which is an analogue of Theorem 1 in the setting of $SL(2,\mathbb{C})$.

Theorem 3. *Given a cocompact group $\Gamma < SL(2,\mathbb{C})$ with $-I \in \Gamma$ there exists $\rho > 0$ such that if F is a C^∞ compactly supported function then for any $X \in \mathbb{C}^2 - \{0,0\}$:*

$$\frac{1}{T^2} \sum_{||\gamma|| \leq T} F(\gamma X) = \frac{1}{\|X\|} \int \frac{F(Y)}{\|Y\|} d\lambda(Y) + O\left(\frac{1}{T^\rho}\right) \tag{2}$$

as $T \to +\infty$. Moreover, we can choose ρ to be independent of Γ.

We establish Theorem 3 using a very straightforward approximation argument whose simplicity compensates for the lack of sharpness it gives on estimating ρ. The method we will use to prove Theorem 3 can also be used to recover Theorem 1, albeit with implicitly weaker estimates on the value of $\rho > 0$.

The main difference between the formulae (1) and (2) is that the normalization $1/T$ is replaced by $1/T^2$. In the proof this corresponds to the real dimension of the nilpotent subgroup of $SL(2,K)$ changing from 1 (when $K = \mathbb{R}$) to 2 (when $K = \mathbb{C}$).

Another interesting distinction between the cases $K = \mathbb{R}$ or $\mathbb{C}$ is the dependence of ρ on the group Γ. In Theorem 3, we observed that we can choose $\rho > 0$ independently of the choice of Γ. This is in complete contrast to the situation when $K = \mathbb{R}$ in Theorem 1, where for any $\epsilon > 0$ we can choose a cocompact group $\Gamma < SL(2,\mathbb{R})$ such that in (1) we require $\rho = \rho(\Gamma) < \epsilon$. To see that, we first observe that $\mathbb{H}^2/\Gamma$ is a compact surface and that the first eigenvalue of the Laplacian $\lambda_1(\Gamma)$ has the property that $C_1 \le \rho(\Gamma)/\lambda_1(\Gamma) \le C_2$, where $C_1, C_2 > 0$ are constants that depend only on the genus [10]. In particular, if the surface has a sufficiently small dividing closed geodesic then ρ can be made arbitrarily small.

Remark 4. The original version of this short note was written in 2003, but was unpublished at the time.

2. Case $K = \mathbb{C}$: Averages on Horospheres

We first need to establish a result on the distribution of stable manifolds for the flow $g_t : M \to M$ on the quotient space $M = PSL(2,\mathbb{C})/\Gamma$ of cosets defined by

$$g_t x = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} x$$

where $x = g\Gamma$. We denote $h_z^+ = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $h_z^- = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and then we can write

$$W^u(x) = \left\{ h_z^+ x : z \in \mathbb{C} \right\} \text{ and } W^s(x) = \left\{ h_z^- x : z \in \mathbb{C} \right\}.$$

Moreover, given $T > 0$ we let

$$W^u(x,T) = \left\{ h_z^+ x : |z| \le T \right\} \text{ and } W^s(x,T) = \left\{ h_z^- x : |z| \le T \right\}$$

denote the balls of radius T in $W^u(x)$ and in $W^s(x)$, respectively, in the induced metric. In particular, we see that each ball has volume πT^2.

Lemma 5. *For any $t \in \mathbb{R}$ and $T > 0$ we have that $g_t W^u(x,T) = W^u(g_t x, Te^t)$.*

Proof. This is immediate from the identity

$$g_t h_t^+ g_{-t} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & ze^t \\ 0 & 1 \end{pmatrix}.$$

□

Geometrically, g_t corresponds to the frame flow on the frame bundle over a three dimensional manifold V with sectional curvatures all equal to -1. The discrete group Γ corresponds to the fundamental group of V acting on the frame bundle over three dimensional hyperbolic space, identified with the upper half-space $\mathbb{H}^3 = \{z + jt : z \in \mathbb{C}, t > 0\}$ with the Poincaré metric. As is well-known, geodesics in $\mathbb{H}^3$ are semi-circular arcs which meet the boundary $\widehat{\mathbb{C}}$ perpendicularly and the horospheres are spheres which are tangent to the boundary $\widehat{\mathbb{C}}$. The frames are then translated on these horopheres [6].

Moreover, the frame flow is an extension of the geodesic flow $\psi_t : N \to N$ on the unit tangent bundle of V by the one parameter subgroup

$$K = \left\{ k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \le \theta < 1 \right\} = SO(2)$$

where $N = K \backslash PSL(2, \mathbb{C})/\Gamma$ and $\psi_t(Kg\Gamma) = Kg_t g\Gamma$ (since g_t commutes with $k_\theta \in K$).

The next result describes the equidistribution of horospheres.

Lemma 6. *There exists $\rho > 0$ such that for any C^∞ function $f : M \to \mathbb{R}$ and $x_0 \in M$ we have that*

$$\frac{1}{\pi T^2} \int_{W^u(x_0,T)} f(\xi) d\xi = \int f d\mu + O\left(\frac{1}{T^\rho}\right) \text{ as } T \to +\infty.$$

To prove Lemma 6, we first need the following classical estimate on the rate of mixing of the flow $g_t : M \to M$. Let μ be the normalized Haar measure on M.

Lemma 7. *There exists $\lambda > 0$ and $C > 0$ such that for any C^∞ functions $f_1, f_2 : M \to \mathbb{R}$ we have that*

$$\left| \int f_1(g_t x) f_2(x) d\mu(x) \right| \le C.e^{-\lambda t} ||f_1||.||f_2||, \text{ for } t > 0,$$

where $\|\cdot\|$ is a Sobolev norm. [1]

Proof. The first version of this result was proved by Fomin and Gelfand [2]. A more general result appears in [8]. In particular, $\lambda > 0$ is shown to be uniformly bounded away from zero. □

Remark 8. Unfortunately, we do not know if the analogue of this result holds when V have variable negative curvature, i.e., Do frame flows mix exponentially quickly with respect to the invariant measure equivalent to the volume if V has variable curvature?

We can now return to the proof of Lemma 6.

Proof of Lemma 6. Fix $\delta > 0$ and $x_0 \in M$. For any $t > 0$, we can use the scaling property of the frame flow in Lemma 1 to first write that

$$\frac{1}{\pi\delta^2} \int_{W^u(g_{-t}x_0,\delta)} f(\xi) d\mu^u(\xi) = \frac{1}{\pi\delta^2 e^{2t}} \int_{W^u(x_0,\delta e^t)} f(g_{-t}\xi) d\mu^u(\xi)$$

[1] i.e., for $l \ge 1$ we use the usual definition of $||\cdot|| = ||\cdot||_l$ in terms of the L^2-norms of the first l-derivatives.

where the integrals are with respect to the volume μ^u on the unstable manifolds induced from the Haar measure μ. We can choose a neighbourhood $U \supset W^u(g_{-t}x_0, \delta)$ defined by

$$U = \left\{k_\theta h_z^+ g_u x : x \in W^u(g_{-t}x_0, \delta), |\theta|, |z|, |u| < \epsilon(t)\right\}$$

where $\epsilon(t) = e^{-\alpha_1 t}$, where $\alpha_1 > 0$ will be chosen later. In particular, the image $g_t U$ of U under g_t will then be a neighbourhood of the set $g_t W^u(g_{-t}x_0, \delta) = W^u(x_0, e^t\delta)$. The effect of the flow is that:

1. g_t shrinks the set U exponentially in the strong stable directions, i.e., the h_t^+ orbit direction; and
2. g_t preserves distance in the neutral directions (corresponding to both the flow and frame directions, i.e., g_t and k_θ orbits) and thus it will at least stay within $\epsilon(t)$ of the image horosphere.

Since f is Lipschitz (for some $0 < \gamma \leq 1$) we can approximate

$$\left|\frac{1}{\pi\delta^2 e^{2t}} \int_{W^u(x_0, \delta e^t)} f(\xi) d\mu^+(\xi) - \frac{1}{\mu(U)} \int_{g_t U} f(\xi) d\mathrm{Vol}(\xi)\right| \leq \|f\|_{Lip} e^{-t\alpha_1} \qquad (3)$$

for $t > 0$, where $\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}$ and $d(\cdot, \cdot)$ denotes the Riemannian metric.

We can also L^1-approximate the indicator function χ_U by a smooth positive function h so that for any $\alpha_1 < \alpha_2$ and α_3, to be chosen later, so that:

(i) $|\int h d\mu - \mu(U)| < te^{-\alpha_2 t}$; and

(ii) $\|h\| < e^{\alpha_3 t}$.

The multiplicative term of t in (i) will be to accommodate the fact that the ball $W^u(x_0, e^t\delta)$ has a one-dimensional boundary. If $||h||$ is now taken to be a Sobolev l-norm, for some sufficiently large fixed $l \geq 1$, then one may choose $\alpha_3 := l\alpha_2$.

In particular, we can use (i) to approximate

$$\left|\int_{g_t U} f(\xi) d\mathrm{Vol}(\xi) - \int h(g_{-t}\xi) f(\xi) d\mathrm{Vol}(\xi)\right| \leq ||h||_\infty t e^{-\alpha_2 t} \qquad (4)$$

and we can use Lemma 7 and (ii) to write

$$\left|\int h(g_{-t}\xi) f(\xi) d\mu(\xi)\right| \leq \|h\|.\|f\| e^{-\lambda t} = O\left(e^{\alpha_3 t} e^{-\lambda t}\right). \qquad (5)$$

Comparing (3), (4) and (5) we get that

$$\frac{1}{2\delta e^t} \int_{-\delta e^t}^{\delta e^t} f(h_s x_0) ds = O\left(e^{-t\alpha_1 \gamma}, e^{-\alpha_2 t}, e^{-(\lambda - \alpha_3)t}\right) = O\left(e^{-t\alpha_1 \gamma}, e^{-\alpha_1 t}, e^{-(\lambda - l\alpha_1)t}\right).$$

The best bound on this last term comes from choosing $\gamma\chi_1 = \lambda - l\chi_1$. The result follows where the exponent can be taken to be $\rho = \frac{\gamma\lambda}{l+\gamma}$. □

Finally, we come to the proof of Theorem 3.

Proof of Theorem 3. This is an easy modification of the proof in [6], given the estimates in the preceding lemmas. One can define a map $\Psi : \mathbb{C}^2 - \{(0,0)\} \to SL(2,\mathbb{C})$, by associating to a point $X = (x,y) \in \mathbb{C}^2 - \{(0,0)\}$ an element $\gamma \in SL(2,\mathbb{C})$ such that $X = \gamma(1,0)$ and then defining

$$\Psi(X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \in SL(2,\mathbb{C})$$

where the matrices correspond to the compact and abelian components in the KAN decompositions (i.e., Ψ is a section for the horocycle foliation).

The geometric interpretation of the action of an element $\gamma \in SL(2,\mathbb{C})$ on $\mathbb{H}^3$ is that it carries a horosphere for the frame $\Psi(X)$ to a horosphere for the frame $\gamma\Psi(X)$. Moreover, the vectors are related by a horospherical translation, i.e., $\gamma\Psi(x)h_z^+ = \Psi(\gamma x)$, for some $z = z(\gamma, x)$. More precisely, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$ represents $X \in \mathbb{C}^2 - \{(0,0)\}$ and $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda' & 0 \\ 0 & 1/\lambda' \end{smallmatrix}\right)$ represents γX, and $z \in \mathbb{C}$ satisfies

$$\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \lambda' & 0 \\ 0 & 1/\lambda' \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

then $||\gamma||^2 = |X|^2|\gamma X|^2|z|^2 + \frac{|X|^2}{|\gamma X|^2} + \frac{|\gamma X|^2}{|X|^2}$.

For any small region $D \subset \mathbb{C}^2 - \{(0,0)\}$ we have $||\gamma|| \leq T$ and $\gamma X \in D$ if and only if $\Psi(X)h_z \in \gamma\Psi(D)$, for some $z \in \mathbb{C}$ with $|z| \leq \frac{T}{|X||\gamma X|}$. Assume that f is a smooth function whose support is in D. The above identity allows us to approximate

$$\sum_{\gamma\in\Gamma:||\gamma||\leq T} f(\gamma X) \quad \text{by} \quad \int_{|z|\leq \frac{T}{|X||Y|}} \tilde{f}(\Gamma X h_z) ds,$$

where h_z is the right multiplication by $h_z = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \in W$, and $\tilde{f}$ is a suitable function with small support on $\Gamma\backslash SL(2,\mathbb{C})$. More precisely, we first lift f to the sections, and then smooth it in the direction of the horospheres to obtain $\widetilde{f}$. The difference between these terms will then be $O(1)$, which is sufficient for our purposes. Theorem 3 now follows immediately from applying Lemma 6. □

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