A NEW ALGORITHM OF CONTINUED FRACTIONS RELATED TO REAL ALGEBRAIC NUMBER FIELDS OF DEGREE ≤ 5

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Abstract

In an earlier paper, we introduced a new algorithm which is something like the modified Jacobi-Perron algorithm, and gave some computer experiments by which we can expect that the expansion obtained by our algorithm for $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s) \in K^s$ (with some natural conditions on $\underline{\alpha}$) becomes periodic for any real number field K as far as $s + 1 = deg_{\mathbb{Q}}(K) \leq 4$. But, it seems very likely that the algorithm will not work well if $deg_{\mathbb{Q}}(K) = 5$. In this paper we give a new algorithm and discuss some properties and experimental results by which we can expect that the expansion of $\underline{\alpha}$ by the new algorithm always becomes periodic for any real number field K with $deg_{\mathbb{Q}}(K) \leq 5$.

1. Introduction

Among problems related to higher dimensional continued fractions, a central problem has been to find a higher-dimensional generalization of Lagrange's theorem concerning the periodic continued fractions. Related to this problem the following conjecture has been believed.

Conjecture. Let $1, \alpha_1, \ldots, \alpha_s$ be a $\mathbb{Q}-$basis of real number field K with $[K : \mathbb{Q}] = s + 1$. Then, the expansion of $(\alpha_1, \ldots, \alpha_s)$ by the Jacobi-Perron algorithm is eventually periodic, cf. [7].

But this conjecture may be false even in the case where $s = 2$, cf. [8]. We also presented an algorithm in [8](say, the algebraic Jacobi-Perron Algorithm) which is something like the Jacobi-Perron algorithm and discussed some experiments by which we can expect that the expansion of $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$ with some natural conditions by the algorithm always becomes periodic for any real field K with $s + 1 = deg_{\mathbb{Q}}(K) \leq 4$.

In this paper, we have two main objectives:

(1) To give computer experiments of expansions of $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) \in K^s$ (with some natural conditions on $\underline{\alpha}$) obtained by the algorithm [8] by which we can expect that the expansions are not always periodic, cf. Tables E.

(2) To introduce a new algorithm and give experimental results according to the new algorithm by which we can expect that the resulting expansion of $\underline{\alpha}$ always becomes periodic for any $\underline{\alpha} \in K^s$(with some natural conditions on $\underline{\alpha}$) and any real number field K with $deg_{\mathbb{Q}}(K) = s+1 \le 5$, cf. Tables A-E. We also give experiments for $\underline{\alpha} \in K^5$ with $deg_{\mathbb{Q}}(K) = 6$ such that the experiments for some $\underline{\alpha}$ obtained by our new algorithm are expected to be non periodic except for some accidental cases. (cf. Tables F, and a problem given in Section 6.)

2. Cubic Case

In this section, we shall give two algorithms AJPA1 and AJPA2. Throughout this section K denotes a real cubic field (including totally real cases and not totally real cases). First, we define Algebraic Jacobi-Perron Algorithm(AJPA) presented in [8], which will be referred to as AJPA1. We denote by X_K the set defined by

$$X_K := \{(\alpha, \beta) \in K^2 | 1, \alpha, \beta \text{ are linearly independent over } \mathbb{Q}\} \cap [0,1)^2.$$

We define the transformation $T_K^{(1)}$ on X_K by:

$$T_K^{(1)}(\alpha,\beta) := \begin{cases} \left(\dfrac{1}{\alpha} - \left\lfloor \dfrac{1}{\alpha} \right\rfloor, \dfrac{\beta}{\alpha} - \left\lfloor \dfrac{\beta}{\alpha} \right\rfloor\right) & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ \left(\dfrac{\alpha}{\beta} - \left\lfloor \dfrac{\alpha}{\beta} \right\rfloor, \dfrac{1}{\beta} - \left\lfloor \dfrac{1}{\beta} \right\rfloor\right) & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}} \end{cases}$$

for $(\alpha, \beta) \in X_K$, where $\lfloor x \rfloor$ is the floor function of x and $N(x)$ is the norm of $x \in K$ over $\mathbb{Q}$.

We define the integer-valued functions $a^{(1)}, b^{(1)}$ and $e^{(1)}$ for $(\alpha, \beta) \in X_K$ by:

$$a^{(1)}(\alpha,\beta) := \begin{cases} \left\lfloor \frac{1}{\alpha} \right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ \left\lfloor \frac{\alpha}{\beta} \right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}}, \end{cases}$$

$$b^{(1)}(\alpha,\beta) := \begin{cases} \left\lfloor \frac{\beta}{\alpha} \right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ \left\lfloor \frac{1}{\beta} \right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}}, \end{cases}$$

$$e^{(1)}(\alpha,\beta) := \begin{cases} 0 & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ 1 & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}}. \end{cases}$$

We put (for $n \in \mathbb{Z}_{>0}$):

$$\begin{aligned}(a_n^{(1)}, b_n^{(1)}, e_n^{(1)}) &= (a_n^{(1)}(\alpha,\beta), b_n^{(1)}(\alpha,\beta), e_n^{(1)}(\alpha,\beta)) \\ &:= (a^{(1)}(T_K^{(1)n-1}(\alpha,\beta)), b^{(1)}(T_K^{(1)n-1}(\alpha,\beta)), e^{(1)}(T_K^{(1)n-1}(\alpha,\beta)), \\ \mathcal{S}^{(1)}(\alpha,\beta) &:= \{(a_n^{(1)}(\alpha,\beta), b_n^{(1)}(\alpha,\beta), e_n^{(1)}(\alpha,\beta))\}_{n=1}^{\infty}.\end{aligned}$$

The sequence $\mathcal{S}^{(1)}(\alpha,\beta)$ will be referred to as *the expansion of* $(\alpha,\beta) \in X_K$ *by* $T_K^{(1)}$; $T_K^{(1)}$ gives rise to a 2-dimensional continued fraction expansion algorithm, which will be called AJPA1.

Secondly, we define an algorithm, which will be referred to as AJPA2. For an algebraic number θ, we mean by $\phi_\theta \in \mathbb{Q}[x]$ the monic minimal polynomial of θ. We define $\nu(\theta) := \frac{|\theta|}{\sqrt{|D(\theta)|}}$, where $D(\theta) := \left[\frac{d}{dx}\phi_\theta(x)\right]_{x=\theta}$ is the differential coefficient of $\phi_\theta(x)$ at $x = \theta$.

We define the transformation $T_K^{(2)}$ on X_K by:

$$T_K^{(2)}(\alpha,\beta) := \begin{cases} \left(\frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor, \frac{\beta}{\alpha} - \left\lfloor \frac{\beta}{\alpha} \right\rfloor\right) & \text{if } \ \nu(\alpha) > \nu(\beta) \ , \\ & \text{or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha > \beta, \\ \left(\frac{\alpha}{\beta} - \left\lfloor \frac{\alpha}{\beta} \right\rfloor, \frac{1}{\beta} - \left\lfloor \frac{1}{\beta} \right\rfloor\right) & \text{if } \ \nu(\alpha) < \nu(\beta) \ , \\ & \text{or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha < \beta. \end{cases}$$

We define the integer valued functions $a^{(2)}, b^{(2)}$ and $e^{(2)}$ on X_K as follows:

$$a^{(2)}(\alpha,\beta) := \begin{cases} \left\lfloor \frac{1}{\alpha} \right\rfloor & \text{if } \ \nu(\alpha) > \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha > \beta, \\ \left\lfloor \frac{\alpha}{\beta} \right\rfloor & \text{if } \ \nu(\alpha) < \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha < \beta, \end{cases}$$

$$b^{(2)}(\alpha,\beta) := \begin{cases} \left\lfloor \frac{\beta}{\alpha} \right\rfloor & \text{if } \nu(\alpha) > \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha > \beta, \\ \left\lfloor \frac{1}{\beta} \right\rfloor & \text{if } \nu(\alpha) < \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha < \beta, \end{cases}$$

$$e^{(2)}(\alpha,\beta) := \begin{cases} 0 & \text{if } \nu(\alpha) > \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha > \beta, \\ 1 & \text{if } \nu(\alpha) < \nu(\beta) \text{ or } \ \nu(\alpha) = \nu(\beta) \text{ with } \ \alpha < \beta \end{cases}$$

for $(\alpha,\beta) \in X_K$.

We put (for $n \in \mathbb{Z}_{>0}$)

$$\begin{aligned}(a_n^{(2)}, b_n^{(2)}, e_n^{(2)}) &= (a_n^{(2)}(\alpha,\beta), b_n^{(2)}(\alpha,\beta), e_n^{(2)}(\alpha,\beta)) \\ &:= (a^{(2)}(T_K^{(2)n-1}(\alpha,\beta), b^{(2)}(T_K^{(2)n-1}(\alpha,\beta)), e^{(2)}(T_K^{(2)n-1}(\alpha,\beta)), \\ \mathcal{S}^{(2)}(\alpha,\beta) &:= \{(a_n^{(2)}(\alpha,\beta), b_n^{(2)}(\alpha,\beta), e_n^{(2)}(\alpha,\beta))\}_{n=1}^{\infty}.\end{aligned}$$

The sequence $\mathcal{S}^{(2)}(\alpha,\beta)$ will be referred to as the expansion of $(\alpha,\beta)\in X_K$ by $T_K^{(2)}$; T_K gives rise to a 2-dimensional continued fraction expansion, which will be called as AJPA2. Throughout our paper α_n,β_n are numbers defined by

$$(\alpha_n,\beta_n):=T_K^{(2)n}(\alpha,\beta),\quad (n\in\mathbb{Z}_{\geq 0}).$$

Notice that $(a_n^{(2)},b_n^{(2)},e_n^{(2)})\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\times\{0,1\},\quad (n\in\mathbb{Z}_{\geq 0})$.

For each $(a,b,e)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geq 0}\times\{0,1\}$, we put

$$A_{(a,b,e)}:=\begin{cases}\begin{pmatrix}0&0&1\\0&1&b\\1&0&a\end{pmatrix} & \text{if } e=0,\\ \\ \begin{pmatrix}1&0&a\\0&0&1\\0&1&b\end{pmatrix} & \text{if } e=1,\end{cases} \tag{1}$$

$$M_n(\alpha,\beta)=\begin{pmatrix}p_n''(\alpha,\beta)&p_n'(\alpha,\beta)&p_n(\alpha,\beta)\\q_n''(\alpha,\beta)&q_n'(\alpha,\beta)&q_n(\alpha,\beta)\\r_n''(\alpha,\beta)&r_n'(\alpha,\beta)&r_n(\alpha,\beta)\end{pmatrix}:=A_{(a_1^{(2)},b_1^{(2)},e_1^{(2)})}\cdots A_{(a_n^{(2)},b_n^{(2)},e_n^{(2)})}. \tag{2}$$

Definition (*The set of periodic points.*) We define $\mathcal{P}_K^{AJPA1}$ and $\mathcal{P}_K^{AJPA2}$ as follows:

$$\begin{aligned}\mathcal{P}_K^{AJPA1}&=\{(\alpha,\beta)\in X_K|\text{ there exist } m,n\in\mathbb{Z}_{>0}\text{ such that } m\neq n\text{ and}\\&\quad T_K^{(1)m}(\alpha,\beta)=T_K^{(1)n}(\alpha,\beta)\},\\\mathcal{P}_K^{AJPA2}&=\{(\alpha,\beta)\in X_K|\text{ there exist } m,n\in\mathbb{Z}_{>0}\text{ such that } m\neq n\text{ and}\\&\quad T_K^{(2)m}(\alpha,\beta)=T_K^{(2)n}(\alpha,\beta)\}.\end{aligned}$$

If $(\alpha,\beta)\in\mathcal{P}_K^{AJPAj}$, the expansion $\mathcal{S}^{(j)}(\alpha,\beta)$ by $T_K^{(j)}$ becomes periodic, and vice versa for each $j=1,2$. In what follows we mean by "the period" the period obtained by choosing the shortest period and preperiod. For the periodic continued fraction obtained by AJPA2, we have the following Proposition 1 ,which can be shown in a way similar to Perron [5].

Proposition 1. *Let $(\alpha,\beta)\in\mathcal{P}_K^{AJPA2}$. Then, there exists a constant $c(\alpha,\beta)>0$ and $\eta(\alpha,\beta)>0$ such that $\eta(\alpha,\beta)\leq\frac{3}{2}$ and both*

$$|\alpha-\frac{p_n}{r_n}|\leq\frac{c(\alpha,\beta)}{r_n^{\eta(\alpha,\beta)}}\quad\textit{and}\quad|\beta-\frac{q_n}{r_n}|\leq\frac{c(\alpha,\beta)}{r_n^{\eta(\alpha,\beta)}}$$

hold. Further, $\eta(\alpha,\beta)=\frac{3}{2}$ holds if and only if K is not a totally real cubic field.

Remarks.

1. Based on our many experiments (cf. Tables A, B, C, and D), we can hope that $X_K = \mathcal{P}_K^{AJPA2}$.

2. In view of Proposition 1, we see that $(\alpha, \beta) \neq (\alpha', \beta')$ if and only if $\mathcal{S}^{(2)}(\alpha, \beta) \neq \mathcal{S}^{(2)}(\alpha', \beta')$, for $(\alpha, \beta), (\alpha', \beta') \in \mathcal{P}_K^{AJPA2}$.

Bernstein [1] gave some classes of periodic continued fractions obtained by the Jacobi-Perron algorithm. We can also give some examples of periodic expansions obtained by AJPA2 in the similar manner as in [8].

Theorem 2. *Let* $K = \mathbb{Q}(\sqrt[3]{m^3+1})$ *with* $m \in \mathbb{Z}_{>0}$*. Let* $(\alpha, \beta) = (\sqrt[3]{m^3+1} - m, \sqrt[3]{(m^3+1)^2} - m^2)$*. Then,* $(\alpha, \beta) \in \mathcal{P}_K^{AJPA2}$*.*

Proof. Let $m \geq 4$. We put $u = \sqrt[3]{m^3+1}$. We get $\mathcal{S}^{(2)}(\alpha, \beta)$ as in the following tables. We prove the cases where $n = 1$ in the tables. We can prove the other cases in the similar manners.

We see easily that $|D(\alpha)| = 3u^2$ and $|D(\beta)| = 3u(m^3+1)$.

We have

$$\begin{aligned} &|D(\alpha)D(\beta)| \left(\frac{\beta^2}{|D(\beta)|} - \frac{\alpha^2}{|D(\alpha)|} \right) \\ &= (-2m^2u^2 + (m^3+1)u + m^4)(3u^2) - 3u(1+m^3)(u^2 - 2mu + m^2). \end{aligned} \tag{3}$$

Putting $u = m + \epsilon$, we have

$$\begin{aligned} &|D(\alpha)D(\beta)| \left(\frac{\beta^2}{|D(\beta)|} - \frac{\alpha^2}{|D(\alpha)|} \right) \\ &= -9\epsilon m^5 - 27\epsilon^2 m^4 - 24\epsilon^3 m^3 + 3y^3 - 6\epsilon^4 m^2 + 9\epsilon m^2 + 6\epsilon^2 m. \end{aligned}$$

Since

$$(m+\epsilon)^3 = m^3 + 1, \tag{4}$$

we have $3m^3 = 9\epsilon m^5 + 9\epsilon^2 m^4 + 3\epsilon^3 m^3$ and $9\epsilon m^2 = 27\epsilon^2 m^4 + 27\epsilon^3 m^3 + 9\epsilon^4 m^2$, which implies

$$\begin{aligned} &- 9\epsilon m^5 - 27\epsilon^2 m^4 - 24\epsilon^3 m^3 + 3m^3 - 6\epsilon^4 m^2 + 9\epsilon m^2 + 6\epsilon^2 m \\ &= -18\epsilon^2 m^4 - 21\epsilon^3 m^3 + 9\epsilon^2 m^3 - 6\epsilon^4 m^2 + 9\epsilon m^2 + 6\epsilon^2 m \\ &= 9\epsilon^2 m^4 + 6\epsilon^3 m^3 + 9\epsilon^2 m^3 + 3\epsilon^4 m^2 + 6\epsilon^2 m > 0. \end{aligned}$$

Thus, we have $\frac{\beta^2}{|D(\beta)|} - \frac{\alpha^2}{|D(\alpha)|} > 0$. Therefore, we have $e_1^{(2)} = 1$. Then, we see that $\frac{\alpha_0}{\beta_0} = \frac{u-m}{u^2-m^2} = \frac{1}{u+m} < 1$, which implies $a_1 = 0$. We have $\frac{1}{\beta_0} = \frac{m^2u^2+(m^3+1)u+m^4}{2y^3+1}$. We are going to prove the following inequalities:

$$\frac{3m}{2} + \frac{1}{2} > \frac{m^2u^2 + (m^3+1)u + m^4}{2m^3+1} > \frac{3m}{2}. \tag{5}$$

First, we see that

$$\frac{m^2u^2+(m^3+1)u+m^4}{2m^3+1}-\frac{3m}{2}=\frac{6\epsilon m^3+2\epsilon^2m^2-m+2\epsilon}{4m^3+2}.$$

On the other hand, by virtue of $3\epsilon m^2+3\epsilon^2m+\epsilon^3=1$ we see easily that $\frac{1}{6m^2}<\epsilon<\frac{1}{3m^2}$. Hence, we get $\frac{6\epsilon m^3+2\epsilon^2m^2-m+2\epsilon}{4m^3+2}>0$. Therefore, we have $\frac{m^2u^2+(m^3+1)u+m^4}{2m^3+1}>\frac{3m}{2}$. Secondly, by (5) we see that

$$\begin{aligned}\frac{3m}{2}+\frac{1}{2}-\frac{m^2u^2+(m^3+1)u+m^4}{2m^3+1} &= \frac{-6\epsilon m^3+2m^3-2\epsilon^2m^2+m-2\epsilon+1}{4m^3+2}\\ &= \frac{(-3\epsilon+1)m^3+(1-2\epsilon^2m)m+(-2\epsilon+1)}{4m^3+2}\\ &> 0.\end{aligned}$$

Hence we get that $b_1^{(2)}=\frac{3m}{2}$ for even m, and $b_1^{(2)}=\frac{3m}{2}-\frac{1}{2}$ for odd m.

Case 1. $m=1$.

n	1	2
$a_n^{(2)}$	0	2
$b_n^{(2)}$	1	1
$e_n^{(2)}$	1	0

For $n\geq 3$, $a_n^{(2)}=a_{n-2}^{(2)}$, $b_n^{(2)}=b_{n-2}^{(2)}$ and $e_n^{(2)}=e_{n-2}^{(2)}$.

n	α_n	β_n
0	$u-1$	u^2-1
1	$\frac{1+u^2-u}{3}$	$\frac{-2+u^2+2u}{3}$

For $n\geq 2$, $\alpha_n=\alpha_{n-2}$ and $\beta_n=\beta_{n-2}$.

Case 2. $m=3$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$a_n^{(2)}$	0	6	0	5	0	4	0	3	27	3	0	4	3	5
$b_n^{(2)}$	4	3	5	2	6	3	9	0	0	0	9	0	6	0
$e_n^{(2)}$	1	0	1	0	1	0	1	0	0	0	1	0	1	0

n	15	16	17	18	19	20	21	22
$a_n^{(2)}$	2	6	3	9	0	0	0	9
$b_n^{(2)}$	5	0	4	0	3	27	3	0
$e_n^{(2)}$	1	0	1	0	1	1	1	0

For $n\geq 23$, $a_n^{(2)}=a_{n-22}^{(2)}$, $b_n^{(2)}=b_{n-22}^{(2)}$ and $e_n^{(2)}=e_{n-22}^{(2)}$.

n	α_n	β_n
0	$-3+u$	$-9+u^2$
1	$\frac{9}{55}+\frac{1}{55}u^2-\frac{3}{55}u$	$-\frac{139}{55}+\frac{9}{55}u^2+\frac{28}{55}u$
2	$-3+u$	$-6+u^2-u$
3	$\frac{1}{9}+\frac{1}{36}u^2-\frac{1}{18}u$	$-\frac{29}{9}+\frac{7}{36}u^2+\frac{11}{18}u$
4	$-3+u$	$-3+u^2-2u$
5	$\frac{1}{29}+\frac{1}{29}u^2-\frac{1}{29}u$	$\frac{-109}{29}+\frac{7}{29}u^2+\frac{22}{29}u$
6	$-3+u$	u^2-3u
7	$\frac{1}{28}u^2$	$-6+\frac{9}{28}u^2+u$
8	$-3+u$	$9+u^2-6u$
9	$-18+u^2+3u$	$-3+u$
10	$\frac{-123}{61}+\frac{27}{244}u^2+\frac{41}{122}u$	$\frac{9}{61}+\frac{1}{244}u^2-\frac{3}{122}u$
11	$-9+u^2$	$-3+u$
12	$\frac{-139}{55}+\frac{9}{55}u^2+\frac{28}{55}u$	$\frac{9}{55}+\frac{1}{55}u^2-\frac{3}{55}u$
13	$-6+u^2-u$	$-3+u$
14	$\frac{-29}{9}+\frac{7}{36}u^2+\frac{11}{18}u$	$\frac{1}{9}+\frac{1}{36}u^2-\frac{1}{18}u$
15	$-3+u^2-2u$	$-3+u$
16	$\frac{-109}{29}+\frac{7}{29}u^2+\frac{22}{29}u$	$\frac{1}{29}+\frac{1}{29}u^2-\frac{1}{29}u$
17	u^2-3u	$-3+u$
18	$-6+\frac{9}{28}u^2+u$	$\frac{1}{28}u^2$
19	$9+u^2-6u$	$-3+u$
20	$-3+u$	$-18+u^2+3u$
21	$\frac{9}{61}+\frac{1}{244}u^2-\frac{3}{122}u$	$\frac{-123}{61}+\frac{27}{244}u^2+\frac{41}{122}u$

For $n \geq 22$, $\alpha_n = \alpha_{n-22}$ and $\beta_n = \beta_{n-22}$.

`Case 3.` m is even.

n	1	2	3	4	5	6	7	8	9	10
$a_n^{(2)}$	0	$2m$	0	$\frac{3m}{2}$	0	m	$3m^2$	m	0	$\frac{3m}{2}$
$b_n^{(2)}$	$\frac{3m}{2}$	0	$2m$	0	$3m$	0	0	0	$3m$	0
$e_n^{(2)}$	1	0	1	0	1	0	0	0	1	0

n	11	12	13	14	15	16	17	18
$a_n^{(2)}$	0	$2m$	0	$3m$	0	0	0	$3m$
$b_n^{(2)}$	$2m$	0	$\frac{3m}{2}$	0	m	$3m^2$	m	0
$e_n^{(2)}$	1	0	1	0	1	1	1	0

For $n \geq 19$, $a_n^{(2)} = a_{n-18}^{(2)}$, $b_n^{(2)} = b_{n-18}^{(2)}$ and $e_n^{(2)} = e_{n-18}^{(2)}$.

n	α_n	β_n
0	$u-m$	u^2-m^2
1	$\frac{u^2-mu+m^2}{2m^3+1}$	$\frac{2m^2u^2-(-2m^3-2)u-4m^4-3m}{4m^3+2}$
2	$u-m$	$\frac{2u^2-mu-m^2}{2}$
3	$\frac{8u^2-4mu+2m^2}{9m^3+8}$	$\frac{m^2u^2+(6m^3+8)u-12m^4-12m}{9m^3+8}$
4	$u-m$	u^2-mu
5	$\frac{u^2}{m^3+1}$	$\frac{m^2u^2+(m^3+1)u-2m^4-2m}{m^3+1}$
6	$u-m$	$u^2-2um+m^2$
7	$u^2+mu-2m^2$	$u-m$
8	$\frac{3m^2u^2+(3m^3+1)u-6m^4-2m}{9m^3+1}$	$\frac{u^2-2mu+4m^2}{9m^3+1}$
9	u^2-m^2	$u-m$
10	$\frac{2m^2u^2-(-2m^3-2)u-4m^4-3m}{4m^3+2}$	$\frac{u^2-mu+m^2}{2m^3+1}$
11	$\frac{2u^2-mu-m^2}{2}$	$u-m$
12	$\frac{m^2u^2+(6m^3+8)u-12m^4-12m}{9m^3+8}$	$\frac{8u^2-4mu+2m^2}{9m^3+8}$
13	u^2-mu	$u-m$
14	$\frac{m^2u^2+(m^3+1)u-2m^4-2m}{m^3+1}$	$\frac{u^2}{m^3+1}$
15	$u^2-2um+m^2$	$u-m$
16	$u-m$	$u^2+mu-2m^2$
17	$\frac{u^2-2mu+4m^2}{9m^3+1}$	$\frac{3m^2u^2+(3m^3+1)u-6m^4-2m}{9m^3+1}$

For $n \geq 18$, $\alpha_n = \alpha_{n-18}$ and $\beta_n = \beta_{n-18}$.

`Case 4.` m is odd.

n	1	2	3	4	5	6	7	8	9
$a_n^{(2)}$	0	$2m$	0	$\frac{1}{2}(3m+1)$	0	$m+1$	$3m^2$	m	0
$b_n^{(2)}$	$\frac{1}{2}(3m-1)$	m	$2m-1$	$\frac{1}{2}(m+1)$	$3m-3$	3	3	0	$3m$
$e_n^{(2)}$	1	0	1	0	1	0	0	0	1

n	10	11	12	13	14	15	16	17	18
$a_n^{(2)}$	$\frac{1}{2}(3m-1)$	m	$2m-1$	$\frac{1}{2}(m+1)$	$3m-3$	3	3	0	$3m$
$b_n^{(2)}$	0	$2m$	0	$\frac{1}{2}(3m+1)$	0	$m+1$	$3m^2$	m	0
$e_n^{(2)}$	0	1	0	1	0	1	1	1	0

For $n \geq 19$, $a_n^{(2)} = a_{n-18}^{(2)}$, $b_n^{(2)} = b_{n-18}^{(2)}$ and $e_n^{(2)} = e_{n-18}^{(2)}$.

n	α_n	β_n
0	$u-m$	u^2-m^2
1	$\frac{u^2-um+m^2}{2m^3+1}$	$\frac{2m^2u^2+(2m^3+2)u-4m^4+2m^3}{4m^3+2}$ $+\frac{-3m+1}{4m^3+2}$
2	$u-m$	$\frac{2u^2+(1-m)u-m^2-m}{2}$
3	$\frac{8u^2+(-4m-4)u+2m^2+4m+2}{9m^3+3m^2+3m+9}$	$\frac{(6m^2+2)u^2+(6m^3+2m+8)u}{9m^3+3m^2+3m+9}$ $+\frac{-12m^4+3m^3-m^2-11m+5}{9m^3+3m^2+3m+9}$
4	$u-m$	$u^2+(1-m)u-m$
5	$\frac{u^2-u+1}{m^3+2}$	$\frac{(m^2-m+1)u^2+(m^3-m^2+m+1)u}{m^3+2}$ $+\frac{-2m^4+2m^3+m^2-5m+5}{m^3+2}$
6	$u-m$	$m^2+(-2u-3)m+u^2+3u$
7	$u^2+mu-2m^2$	$u-m$
8	$\frac{3m^2u^2+(3m^3+1)u-6m^4-2m}{9m^3+1}$	$\frac{u^2-2mu+4m^2}{9m^3+1}$
9	u^2-m^2	$u-m$
10	$\frac{2m^2u^2+(2m^3+2)u-4m^4+2m^3}{4m^3+2}$ $+\frac{-3m+1}{4m^3+2}$	$\frac{u^2-um+m^2}{2m^3+1}$
11	$\frac{2u^2+(1-m)u-m^2-m}{2}$	$u-m$
12	$\frac{(6m^2+2)u^2+(6m^3+2m+8)u}{9m^3+3m^2+3m+9}$ $+\frac{-12m^4+3m^3-m^2-11m+5}{9m^3+3m^2+3m+9}$	$\frac{8u^2+(-4m-4)u+2m^2+4m+2}{9m^3+3m^2+3m+9}$
13	$u^2+(1-m)u-m$	$u-m$
14	$\frac{(m^2-m+1)u^2+(m^3-m^2+m+1)u}{m^3+2}$ $+\frac{-2m^4+2m^3+m^2-5m+5}{m^3+2}$	$\frac{u^2-u+1}{m^3+2}$
15	$u^2+(1-m)u-m$	$u-m$
16	$u-m$	$u^2+mu-2m^2$
17	$\frac{u^2-2mu+4m^2}{9m^3+1}$	$\frac{3m^2u^2+(3m^3+1)u-6m^4-2m}{9m^3+1}$

For $n \geq 18$, $\alpha_n = \alpha_{n-18}$ and $\beta_n = \beta_{n-18}$. □

Theorem 3. *Let δ_m be the root of $x^3 - mx + 1 = 0$ ($m \in \mathbb{Z}, m \geq 3$) determined by $0 < \delta_m < 1$. Then, $K = \mathbb{Q}(\delta_m)$ is a totally real cubic number field and $(\delta_m, \delta_m^2) \in \mathcal{P}_K^{AJPA2}$.*

Proof. We see easily that $K = \mathbb{Q}(\delta_m)$ is a totally real cubic number field. We put $(\alpha, \beta) = (\delta_m, \delta_m^2)$. We get $\mathcal{S}^{(2)}(\alpha, \beta)$ as in the following tables. We prove the cases where $n = 1$ in the tables. We see easily that $|D(\alpha)| = m - 3\delta_m^2$ and $|D(\beta)| = m^2 - m\delta_m^2 - 3\delta_m$.

We have

$$|D(\alpha)D(\beta)|\left(\frac{\alpha^2}{|D(\alpha)|}-\frac{\beta^2}{|D(\beta)|}\right) \tag{6}$$
$$=\delta_m^2(m^2-m\delta_m^2-3\delta_m)-(m-3\delta_m^2)(m\delta_m^2-\delta_m) \tag{7}$$
$$=2\delta_m^4 m+\delta_m m-6\delta_m^3.$$

On the other hand, by $\delta_m^3-m\delta_m+1=0$ we have

$$\delta_m=\frac{\delta_m^3+1}{m}, \tag{8}$$

which implies $\frac{1}{m}<\delta_m<\frac{2}{m}$. Therefore, we get

$$\begin{aligned}2\delta_m^4 m+\delta_m m-6\delta_m^3 &> 2\delta_m^3+\delta_m m-6\delta_m^3\\ &=\delta_m m-4\delta_m^3>\delta_m(m-\frac{24}{m^2})\\ &>0.\end{aligned}$$

Thus, we have $e_1=0$. Then, we see that $\frac{\beta_0}{\alpha_0}=\delta_m$, which implies $b_1=0$. We have $\frac{1}{\beta_0}=m-\delta_m^2$, which implies $a_1=m-1$. Hence, we have $\alpha_1=1-\delta_m^2$ and $\beta_1=\delta_m$.

n	$a_n^{(2)}$	$b_n^{(2)}$	$e_n^{(2)}$	α_{n-1}	β_{n-1}
1	$m-1$	0	0	δ_m	δ_m^2
2	1	0	0	$1-\delta_m^2$	δ_m
3	0	$m-1$	1	$\frac{-(m-1)\delta_m^2-\delta_m+1}{m(m-2)}$	$\frac{-\delta_m^2-(m-1)\delta_m+m-1}{m(m-2)}$
4	0	1	1	δ_m	$-\delta_m^2-\delta_m+1$
5	$m-2$	1	0	$\frac{-\delta_m+1}{m-2}$	$\frac{-\delta_m^2+1}{m-2}$
6	1	0	0	$-\delta_m^2-\delta_m+1$	δ_m
7	1	$m-2$	1	$\frac{-\delta_m^2+1}{m-2}$	$\frac{-\delta_m+1}{m-2}$

For $n\geq 8$, $a_n^{(2)}=a_{n-4}^{(2)}$, $b_n^{(2)}=b_{n-4}^{(2)}$, $e_n^{(2)}=e_{n-4}^{(2)}$, $\alpha_{n-1}=\alpha_{n-5}$ and $\beta_{n-1}=\beta_{n-5}$.
□

3. General Definition

In Section 2, we defined AJPA2 for cubic cases, which can be generalized to any real number field K with $d=deg_{\mathbb{Q}}(K)\geq 3$. In this section, we mean by X_K and

X'_K the set defined by

$$X_K := \{(\alpha_1, \ldots, \alpha_{d-1}) \in (K \cap I)^{d-1} | \text{ there exists a integer } i \text{ with } 1 \leq i \leq d-1 \\ \text{such that } K = \mathbb{Q}(\alpha_i) \text{ and } 1, \alpha_1, \ldots, \alpha_{d-1} \text{ are linearly independent over } \mathbb{Q}\}.$$

The function $\nu(\theta)$ defined in Section 2 can be extended to $\theta \in K$ by

$$\nu(\theta) := \begin{cases} \frac{\theta}{|D(\theta)|^{\frac{1}{d-1}}} & \text{if } K = \mathbb{Q}(\theta), \\ -1 & \text{if } K \neq \mathbb{Q}(\theta), \end{cases} \tag{9}$$

where $\left[\frac{d}{dx}\phi_\theta(x)\right]_{x=\theta}$ which is the differential coefficient of $\phi_\theta(x)$ at $x = \theta$ as in Section 2.

For $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in X_K$, we define $\rho(\alpha) = \max\{\nu(\alpha_i) | 1 \leq i \leq d-1\}$. We denote by $\omega(\alpha)$ the number $i \in \{1, \ldots, d-1\}$ uniquely determined by

$$\alpha_i = \max\{\alpha_j | \rho(\alpha) = \nu(\alpha_j)\}.$$

We define a transformation $T_K^{(2)}$ on X_K as follows:
For $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in X_K$, $T_K(\alpha) = (\beta_1, \ldots, \beta_{d-1})$ with

$$\beta_i := \begin{cases} \dfrac{1}{\alpha_{\omega(\alpha)}} - \left\lfloor \dfrac{1}{\alpha_{\omega(\alpha)}} \right\rfloor & \text{if } i = \omega(\alpha), \\ \dfrac{\alpha_i}{\alpha_{\omega(\alpha)}} - \left\lfloor \dfrac{\alpha_i}{\alpha_{\omega(\alpha)}} \right\rfloor & \text{if } i \neq \omega(\alpha) \qquad (i = 1, \ldots, d-1). \end{cases}$$

We easily see that $T_K^{(2)}$ is well-defined on X_K. The transformation $T_K^{(2)}$ gives rise to an algorithm of continued fraction of dimension $d-1$, which will be also referred to AJPA2. We define $\mathcal{P}_K^{AJPA2}$, in a similar fashion to Section 2, that is,

$$\mathcal{P}_K^{AJPA2} : = \{\alpha \in X_K | \text{there exist } m_1, m_2 \in \mathbb{Z}_{>0} \text{ such that } m_1 \neq m_2 \text{ and} \\ T_K^{(2)m_1}(\alpha) = T_K^{(2)m_2}(\alpha)\}.$$

We put, for $\frac{p}{q}$ $(p, q \in \mathbb{Z}$ are coprime)

$$\mathrm{dh}(\frac{p}{q}) := \max\{\lfloor \log_{10}|p| + 1\rfloor, \lfloor \log_{10}|q| + 1\rfloor\}, \quad \mathrm{dh}(0) := 0.$$

The function dh can be extended to $\mathbb{Q}[x]$: for $g(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Q}[x]$, put

$$\mathrm{dh}(g) := \max_{0 \leq i \leq n}\{\mathrm{dh}(a_i)\}.$$

We define $\overline{\mathrm{dh}}$, $\mathrm{dh}_{\mathrm{AJPA2}}$ and $\mathrm{rdh}_{\mathrm{AJPA2}}$, namely, for $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in X_K$ and $n \in \mathbb{Z}_{\geq 0}$

$$\overline{\mathrm{dh}}(\alpha) := \max_{i \in \{1,\ldots,d-1\}} \{\mathrm{dh}(p_i^{d/deg(p_i)})\},$$
$$\mathrm{dh}_{\mathrm{AJPA2}}(n;\alpha) := \overline{\mathrm{dh}}(T_K^{(2)n}(\alpha)),$$
$$\mathrm{rdh}_{\mathrm{AJPA2}}(n;\alpha) := \frac{\overline{\mathrm{dh}}(T_K^{(2)n}(\alpha))}{\overline{\mathrm{dh}}(\alpha)},$$

where $p_i = \phi_{\alpha_i} \in \mathbb{Q}[x]$ $(i \in \{1, \ldots, d-1\})$ is the monic minimal polynomial of α_i.

We define $\overline{\mathrm{dh}}_{\mathrm{AJPA2}}$ and $\overline{\mathrm{rdh}}_{\mathrm{AJPA2}}$for $\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \in X_K$ as follows:

$$\overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\alpha) := \sup_{n \in \mathbb{Z}_{\geq 0}} \mathrm{dh}_{\mathrm{AJPA2}}(n;\alpha),$$
$$\overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\alpha) := \frac{\overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\alpha)}{\overline{\mathrm{dh}}(\alpha)}.$$

The function $\mathrm{dh}_{\mathrm{AJPA2}}(n;\alpha)$ (resp., $\mathrm{rdh}_{\mathrm{AJPA2}}(n;\alpha)$) is referred to as *the nth decimal height of* α (resp., *the nth relative decimal height of* α) with respect to the AJPA2. The function $\overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\alpha)$ (resp., $\overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\alpha)$) is referred to as *the decimal height of* α (resp.,*the relative decimal height of* α) with respect to the AJPA2.

For an algorithm $\mathcal{A}$ like as AJPA2 on X_K, we can define $\mathrm{dh}_{\mathcal{A}}(n;\alpha)$, $\mathrm{rdh}_{\mathcal{A}}(n;\alpha)$, $\overline{\mathrm{dh}}_{\mathcal{A}}(\alpha)$, $\overline{\mathrm{rdh}}_{\mathcal{A}}(\alpha)$ and $\mathcal{P}_K^{\mathcal{A}}$ analogously.

4. Numerical Experiments

In this section, we compare the expansion obtained by AJPA1, AJPA2, and modified Jacobi-Perron algorithm (abbr. MJPA). The modified Jacobi-Perron algorithm is a classical one, which is defined as follows: For $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1)^n$ $(1, x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$), we define $\psi(\mathbf{x}) \in \{1, \ldots, n\}$ and a transformation $\bar{T}$ by

$$x_{\psi(\mathbf{x})} = \max\{x_1, \ldots, x_n\},$$
$$\bar{T}(x_1, \ldots, x_n) = (u_1, \ldots, u_n),$$

where

$$u_i := \begin{cases} \dfrac{y}{x_{\psi(\mathbf{x})}}, & \text{if } i \neq \psi(\mathbf{x}), \\ \dfrac{1}{x_{\psi(\mathbf{x})}} - \left\lfloor \dfrac{1}{x_{\psi(\mathbf{x})}} \right\rfloor, & \text{if } i = \psi(\mathbf{x}). \end{cases}$$

This algorithm has connection with accelerated Brun's algorithm [7],[3] and the modified Jacobi-Perron algorithm by Podsypanin [6]. It is not difficult to see that the expansion obtained by the modified Jacobi-Perron algorithm is eventually periodic if and only if it is eventually periodic by accelerated Brun's algorithm. We computed the length of the periods of the expansion $\mathcal{S}^{(2)}(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle)$ for all $m \in \mathbb{Z}$ with $2 \leq m \leq 5000$ ($\sqrt[3]{m} \notin \mathbb{Q}$) and these decimal heights, cf. Table A. For the calculation of the tables, we used a computer equipped with GiNaC [4] on GNU C++. We confirmed that $(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle) \in \mathcal{P}_K^{AJPA2}$ for all m with $2 \leq m \leq 5000(\sqrt[3]{m} \notin \mathbb{Q})$. There are no reports on the periodicity of such pairs numbers obtained by Jacobi-Perron algorithm or the modified Jacobi-Perron algorithms. In [2] Elsner and Hasse gave numerical results for 36 pairs of cubic numbers with respect to the Jacobi-Perron algorithm. Among the pairs, they found 14 periodic cases, while the other 22 cases are suspicious to be periodic.

We confirmed that $(\langle\tau_m\rangle, \langle\tau_m^2\rangle) \in \mathcal{P}_K^{AJPA2}$ for all integers m with $3 \leq m \leq 5000$, where τ_m is the maximal root of $x^3 - mx + 1$, cf. Table B. Notice that $\mathbb{Q}(\tau_m)$ is a totally real cubic field for any integer $m \geq 3$.

We computed the length of the period of the expansion of $(\langle\sqrt[4]{m}\rangle, \langle\sqrt[4]{m^2}\rangle, \langle\sqrt[4]{m^3}\rangle)$ with $\sqrt{m} \notin \mathbb{Q}$) obtained by AJPA2 and relative decimal heights for m all $2 \leq m \leq 5000$, cf. Table C.

We confirmed that $(\langle\sqrt[4]{m}\rangle, \langle\sqrt[4]{m^2}\rangle, \langle\sqrt[4]{m^3}\rangle) \in \mathcal{P}_K^{AJPA2}$ for all $2 \leq m \leq 5000(\sqrt{m} \notin \mathbb{Q})$.

We computed the length of the period of the expansion of $(\langle\sqrt[5]{m}\rangle, \langle\sqrt[5]{m^2}\rangle, \langle\sqrt[5]{m^3}\rangle, \langle\sqrt[5]{m^4}\rangle)$ with $\sqrt[5]{m} \notin \mathbb{Q}$) obtained by T_K for m all $2 \leq m \leq 2000$ and relative decimal heights given above, cf. Table D. We confirmed that $(\langle\sqrt[5]{m}\rangle, \langle\sqrt[5]{m^2}\rangle, \langle\sqrt[5]{m^3}\rangle, \langle\sqrt[5]{m^4}\rangle) \in \mathcal{P}_K^{AJPA2}$ for all $2 \leq m \leq 2000(\sqrt[5]{m} \notin \mathbb{Q})$.

We compare the expansions of $(\langle\sqrt[5]{m}\rangle, \langle\sqrt[5]{m^2}\rangle, \langle\sqrt[5]{m^3}\rangle, \langle\sqrt[5]{m^4}\rangle)$ with $\sqrt[5]{m} \notin \mathbb{Q}$) obtained by AJPA1, AJPA2 and the modified Jacobi-Perron algorithm for $2 \leq m \leq 30$, cf. Table E. We seek the periods of the cases related to each algorithms in so far as the decimal heights are less than 20 and we terminate seeking the periods if these are greater than 20 or equal to 20. We find all cases of periodicity related to AJPA2, 8 cases of periodicity related to AJPA1 and no case of periodicity related to MJPA. We also try to seek the lengths of the periods of the expansion of $(\langle\sqrt[6]{m}\rangle, \langle\sqrt[6]{m^2}\rangle, \langle\sqrt[6]{m^3}\rangle, \langle\sqrt[6]{m^4}\rangle, \langle\sqrt[6]{m^5}\rangle)$ with $\sqrt[6]{m^2}, \sqrt[6]{m^3} \notin \mathbb{Q}$ obtained by AJPA2, AJPA1 and MJPA for $2 \leq m \leq 30$ in so far as the decimal heights are less than 100 and we terminate seeking the periods if these are greater than 100 or equal to 100; cf. Table F. We find 2 cases of periodicity related to AJPA2, 2 cases of periodicity related to AJPA1 and no case of periodicity related to MJPA.

5. A Conjecture

Conjecture

(1) Let K be a real number field with $deg_{\mathbb{Q}}(K) \leq 5$. Then, $X_K = \mathcal{P}_K^{AJPA2}$.

(2) Let K be a real number field with $deg_{\mathbb{Q}}(K) \leq 5$. Then, there exists an absolute constant c independent of K and $\alpha \in X_K$ such that $\overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\alpha) \leq c$ holds.

(3) For some real number fields K with $deg_{\mathbb{Q}}(K) = 5$, $X_K \neq \mathcal{P}_K^{AJPA1}$ holds.

(4) For some real number fields K with $deg_{\mathbb{Q}}(K) = 6$, $X_K \neq \mathcal{P}_K^{AJPA2}$ holds.

6. Problem

The algebraic quantity $\nu(\theta)$ given by (9) in Section 3 plays an important role in the definition of our Algorithm AJPA2. Is it possible to find a suitable modification of the value $\nu(\theta)$ such that the resulting algorithm $\mathcal{A}$ has as a property $X_K = \mathcal{P}_K^{\mathcal{A}}$ for any real algebraic number field K with $deg_{\mathbb{Q}}(K) = 6$?

7. Tables

In Table A, below, $L_A(m_1, m_2)$, $H_A(m_1, m_2)$ and $R_A(m_1, m_2)$ are numbers defined by

$L_A(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle \sqrt[3]{m} \rangle, \langle \sqrt[3]{m^2} \rangle)$ by AJPA2 for $m_1 \leq m \leq m_2$ with $\sqrt[3]{m} \notin \mathbb{Q}$,

$$H_A(m_1, m_2) := \max_{m_1 \leq m \leq m_2,\ \sqrt[3]{m} \notin \mathbb{Q}} \overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\langle \sqrt[3]{m} \rangle, \langle \sqrt[3]{m^2} \rangle),$$

$$R_A(m_1, m_2) := \max_{m_1 \leq m \leq m_2,\ \sqrt[3]{m} \notin \mathbb{Q},} \overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\langle \sqrt[3]{m} \rangle, \langle \sqrt[3]{m^2} \rangle),$$

which are well-defined by the periodicity.

TABLE A

range of m $m_1 \leq m \leq m_2$	$L_A(m_1, m_2)$	$H_A(m_1, m_2)$	$R_A(m_1, m_2)$
$2 \leq m \leq 200$	474	5	$5/3$
$201 \leq m \leq 400$	1062	6	$3/2$
$401 \leq m \leq 600$	1806	6	$3/2$
$601 \leq m \leq 800$	1586	6	$6/5$
$801 \leq m \leq 1000$	3338	6	$6/5$
$1001 \leq m \leq 1200$	2910	6	$6/5$
$1201 \leq m \leq 1400$	3762	7	$7/5$
$1401 \leq m \leq 1600$	5642	7	$7/5$
$1601 \leq m \leq 1800$	8694	7	$7/5$
$1801 \leq m \leq 2000$	8586	7	$7/5$
$2001 \leq m \leq 2200$	5846	7	$7/5$
$2201 \leq m \leq 2400$	6958	7	$7/5$
$2401 \leq m \leq 2600$	8706	7	$7/5$
$2601 \leq m \leq 2800$	6966	7	$7/6$
$2801 \leq m \leq 3000$	5522	7	$7/6$
$3001 \leq m \leq 3200$	12790	7	$7/6$
$3201 \leq m \leq 3400$	7286	7	$7/6$
$3401 \leq m \leq 3600$	10818	7	$7/6$
$3601 \leq m \leq 3800$	8210	8	$4/3$
$3801 \leq m \leq 4000$	17014	8	$4/3$
$4001 \leq m \leq 4200$	14574	7	$7/6$
$4201 \leq m \leq 4400$	17818	8	$4/3$
$4401 \leq m \leq 4600$	16214	8	$4/3$
$4601 \leq m \leq 4800$	16402	8	$4/3$
$4801 \leq m \leq 5000$	11574	8	$4/3$

TABLE B

range of m $m_1 \leq m \leq m_2$	$L_B(m_1, m_2)$	$H_B(m_1, m_2)$	$R_B(m_1, m_2)$
$3 \leq m \leq 200$	1038	6	2
$201 \leq m \leq 400$	4588	7	7/3
$401 \leq m \leq 600$	5916	8	7/3
$601 \leq m \leq 800$	11352	7	7/4
$801 \leq m \leq 1000$	13020	8	2
$1001 \leq m \leq 1200$	15666	8	2
$1201 \leq m \leq 1400$	18946	8	2
$1401 \leq m \leq 1600$	22396	8	2
$1601 \leq m \leq 1800$	30256	8	2
$1801 \leq m \leq 2000$	39788	9	9/4
$2001 \leq m \leq 2200$	28572	9	9/4
$2201 \leq m \leq 2400$	68866	9	9/4
$2401 \leq m \leq 2600$	54294	9	9/4
$2601 \leq m \leq 2800$	54198	9	9/4
$2801 \leq m \leq 3000$	56542	9	9/4
$3001 \leq m \leq 3200$	50394	9	9/4
$3201 \leq m \leq 3400$	69318	9	9/4
$3401 \leq m \leq 3600$	60490	9	9/4
$3601 \leq m \leq 3800$	81362	9	9/4
$3801 \leq m \leq 4000$	94904	9	9/4
$4001 \leq m \leq 4200$	102666	10	5/2
$4201 \leq m \leq 4400$	128206	10	5/2
$4401 \leq m \leq 4600$	119286	10	5/2
$4601 \leq m \leq 4800$	124616	10	5/2
$4801 \leq m \leq 5000$	109448	10	5/2

In Table B, $L_B(m_1, m_2)$, $H_B(m_1, m_2)$ and $R_B(m_1, m_2)$ are numbers defined by

$L_B(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle\tau_m\rangle, \langle\tau_m^2\rangle)$ by AJPA2 for $m_1 \leq m \leq m_2$, where τ_m is a maximal root of $x^3 - mx + 1$,

$$H_B(m_1, m_2) := \max_{m_1 \leq m \leq m_2} \overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\langle\tau_m\rangle, \langle\tau_m^2\rangle),$$

$$R_B(m_1, m_2) := \max_{m_1 \leq m \leq m_2} \overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\langle\tau_m\rangle, \langle\tau_m^2\rangle),$$

which are well-defined by the periodicity.

TABLE C

range of m $m_1 \leq m \leq m_2$	$L_C(m_1, m_2)$	$H_C(m_1, m_2)$	$R_C(m_1, m_2)$
$2 \leq m \leq 200$	2316	8	2
$201 \leq m \leq 400$	9822	9	3/2
$401 \leq m \leq 600$	14182	10	10/7
$601 \leq m \leq 800$	16770	10	10/7
$801 \leq m \leq 1000$	12802	11	11/8
$1001 \leq m \leq 1200$	19116	11	11/8
$1201 \leq m \leq 1400$	27178	11	11/8
$1401 \leq m \leq 1600$	26578	11	11/8
$1601 \leq m \leq 1800$	39660	11	11/8
$1801 \leq m \leq 2000$	27726	13	13/8
$2001 \leq m \leq 2200$	32892	11	11/9
$2201 \leq m \leq 2400$	42564	12	4/3
$2401 \leq m \leq 2600$	57774	12	4/3
$2601 \leq m \leq 2800$	118830	12	4/3
$2801 \leq m \leq 3000$	41802	12	4/3
$3001 \leq m \leq 3200$	34758	13	13/9
$3201 \leq m \leq 3400$	72366	12	4/3
$3401 \leq m \leq 3600$	98418	12	4/3
$3601 \leq m \leq 3800$	74874	13	13/9
$3801 \leq m \leq 4000$	47918	13	13/9
$4001 \leq m \leq 4200$	99462	12	4/3
$4201 \leq m \leq 4400$	44928	12	4/3
$4401 \leq m \leq 4600$	43934	13	13/9
$4601 \leq m \leq 4800$	79794	12	4/3
$4801 \leq m \leq 5000$	78162	13	13/9

In Table C, $L_C(m_1, m_2)$, $H_C(m_1, m_2)$ and $R_C(m_1, m_2)$ are numbers defined by

$L_C(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle\sqrt[4]{m}\rangle, \langle\sqrt[4]{m^2}\rangle, \langle\sqrt[4]{m^3}\rangle)$ by AJPA2 for $m_1 \leq m \leq m_2$ with $\sqrt{m} \notin \mathbb{Q}$,

$$H_C(m_1, m_2) := \max_{m_1 \leq m \leq m_2, \sqrt{m} \notin \mathbb{Q}} \overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\langle\sqrt[4]{m}\rangle, \langle\sqrt[4]{m^2}\rangle, \langle\sqrt[4]{m^3}\rangle),$$

$$R_C(m_1, m_2) := \max_{m_1 \leq m \leq m_2, \sqrt{m} \notin \mathbb{Q}} \overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\langle\sqrt[4]{m}\rangle, \langle\sqrt[4]{m^2}\rangle, \langle\sqrt[4]{m^3}\rangle),$$

which are well-defined by the periodicity.

TABLE D

range of m $m_1 \leq m \leq m_2$	$L_D(m_1, m_2)$	$H_D(m_1, m_2)$	$R_D(m_1, m_2)$
$2 \leq m \leq 200$	94548	17	8/3
$201 \leq m \leq 400$	235884	19	19/9
$401 \leq m \leq 600$	308576	21	21/10
$601 \leq m \leq 800$	406580	18	9/5
$801 \leq m \leq 1000$	745932	19	19/11
$1001 \leq m \leq 1200$	897654	20	20/11
$1201 \leq m \leq 1400$	1176156	14	14/11
$1401 \leq m \leq 1600$	1073388	19	19/11
$1601 \leq m \leq 1800$	1000436	19	19/11
$1801 \leq m \leq 2000$	1528364	19	19/12

In Table D, $L_D(m_1, m_2)$, $H_D(m_1, m_2)$ and $R_D(m_1, m_2)$ are numbers defined by

$L_D(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle \sqrt[5]{m} \rangle, \langle \sqrt[5]{m^2} \rangle, \langle \sqrt[5]{m^3} \rangle, \langle \sqrt[5]{m^4} \rangle)$ by AJPA2 for $m_1 \leq m \leq m_2$ with $\sqrt[5]{m} \notin \mathbb{Q}$,

$$H_D(m_1, m_2) := \max_{m_1 \leq m \leq m_2,\, \sqrt[5]{m} \notin \mathbb{Q}} \overline{\mathrm{dh}}_{\mathrm{AJPA2}}(\langle \sqrt[5]{m} \rangle, \langle \sqrt[5]{m^2} \rangle, \langle \sqrt[5]{m^3} \rangle, \langle \sqrt[5]{m^4} \rangle),$$

$$R_D(m_1, m_2) := \max_{m_1 \leq m \leq m_2,\, \sqrt[5]{m} \notin \mathbb{Q}} \overline{\mathrm{rdh}}_{\mathrm{AJPA2}}(\langle \sqrt[5]{m} \rangle, \langle \sqrt[5]{m^2} \rangle, \langle \sqrt[5]{m^3} \rangle, \langle \sqrt[5]{m^4} \rangle),$$

which are well-defined by the periodicity.

TABLE E

	AJPA2		AJPA1		MJPA	
m	length of period	$\overline{\mathrm{dh}}_{\mathrm{AJPA2}}$	length of period	$\overline{\mathrm{dh}}_{\mathrm{AJPA1}}$	length of period	$\overline{\mathrm{dh}}_{\mathrm{MJPA}}$
2	4	2	4	2		(148)
3	366	5		(4552)		(196)
4	2584	8		(2654)		(218)
5	2618	7		(5861)		(245)
6	136	7		(6359)		(192)
7	40	4	76	4		(163)
8	1194	6	294	6		(172)
9	1016	7		(2000)		(111)
10	9536	8		(2686)		(211)
11	42	6	24	6		(222)
12	1284	12		(2707)		(230)
13	204	7	90	(166)		(166)
14	1748	9		(3772)		(222)
15	2500	8		(4190)		(141)
16	2058	7		(6745)		(194)
17	3756	9		(4248)		(123)
18	516	8		(5175)		(165)
19	138	6	156	7		(157)
20	1950	11		(2365)		(110)
21	18556	10		(3946)		(104)
22	1160	8	408	7		(159)
23	2260	14		(2708)		(250)
24	4884	9		(2445)		(170)
25	216	7		(4139)		(141)
26	2390	8		(2271)		(303)
27	3906	9		(1667)		(117)
28	450	7		(3853)		(116)
29	126	8	34	8		(158)
30	9486	10		(4648)		(278)

For $\alpha_m = (\langle\sqrt[5]{m}\rangle, \langle\sqrt[5]{m^2}\rangle, \langle\sqrt[5]{m^3}\rangle, \langle\sqrt[5]{m^4}\rangle)$ with $1 < m \leq 30$ and $\sqrt[5]{m} \notin \mathbb{Q}$, we seek to find a period of the expansions related to each algorithms during the decimal heights $\mathrm{dh}_*(\mathrm{n}; \alpha_\mathrm{m})$ keeps being less than 20 and we terminate seeking to find a period if it exceeds 20 or is equal to 20. The number n with parentheses in the columns $\overline{\mathrm{dh}}_*$ denotes the minimum number n such that $\mathrm{dh}_*(n; \alpha_m)$ exceeds 20 or is equal to 20.

TABLE F

m	AJPA2		AJPA		MJPA	
	length of period	$\overline{\mathrm{dh}}_{\mathrm{AJPA2}}$	length of period	$\overline{\mathrm{dh}}_{\mathrm{AJPA1}}$	length of period	$\overline{\mathrm{dh}}_{\mathrm{MJPA}}$
2	5	3	8	3		(1285)
3		(20935)		(7427)		(1226)
5		(17777)		(6966)		(1115)
6		(17543)		(6601)		(1047)
7		(18516)		(8014)		(1193)
10		(18080)		(7158)		(1055)
11		(17286)		(6517)		(1108)
12		(20031)		(6836)		(1023)
13		(16304)		(6666)		(1155)
14		(16774)		(6607)		(1183)
15		(17406)		(6197)		(1166)
17		(15241)		(6955)		(1014)
18	140	7	54	7		(1131)
19		(15681)		(7534)		(1268)
20		(19786)		(7184)		(1098)
21		(15430)		(6268)		(1030)
22		(14375)		(7043)		(1140)
23		(19745)		(7200)		(1106)
24		(18464)		(6490)		(1170)
26		(16857)		(6983)		(1126)
28		(15433)		(7385)		(1161)
29		(15742)		(6986)		(1260)
30		(17328)		(5962)		(1270)

For $\alpha'_m = (\langle \sqrt[6]{m} \rangle, \langle \sqrt[6]{m^2} \rangle, \langle \sqrt[6]{m^3} \rangle, \langle \sqrt[6]{m^4} \rangle, \langle \sqrt[6]{m^5} \rangle)$ with $1 < m \leq 30$ and $\sqrt[6]{m^2}$, $\sqrt[6]{m^3} \notin \mathbb{Q}$), we seek to find a period of the expansions related to each algorithms during the decimal heights $\mathrm{dh}_*(\mathrm{n}; \alpha'_{\mathrm{m}})$ keeps being less than 100 and we terminate seeking the periods if it exceeds 100 or is equal to 100. The number n with parentheses in the columns $\overline{\mathrm{dh}}_*$ denotes the minimum number n such that $\mathrm{dh}_*(n; \alpha_m)$ exceeds 100 or is equal to 100.

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