CRITICAL EXPONENT OF INFINITE WORDS CODING BETA-INTEGERS ASSOCIATED WITH NON-SIMPLE PARRY NUMBERS

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Abstract

In this paper, we study the critical exponent of infinite words $\mathbf{u}_\beta$ coding β-integers for β being a non-simple Parry number. In other words, we investigate the maximal consecutive repetitions of factors that occur in the infinite word in question. We calculate also the ultimate critical exponent that expresses how long repetitions occur in the infinite word $\mathbf{u}_\beta$ when the factors of length growing ad infinitum are considered. The basic ingredients of our method are the description of all bispecial factors of $\mathbf{u}_\beta$ and the notion of return words. This method can be applied to any fixed point of any primitive substitution.

1. Introduction

In this paper the infinite words associated with non-simple Parry numbers β are studied. These words, denoted by $\mathbf{u}_\beta$, have two equivalent definitions, they are the words coding the gaps between consecutive β-integers and, at the same time, they are fixed points of the substitutions φ_β canonically assigned to β. Our aim is to find the maximal repetitions of motifs occurring in $\mathbf{u}_\beta$, more precisely, to compute the critical exponent and the ultimate critical exponent of these words (for definition see (1) and (2)).

The β-integers proved to be a convenient discrete set for the description of positions of atoms in the materials with long range order, so-called quasicrystals [4].

Physical properties of these materials are determined by the spectrum of the discrete Schrödinger operator assigned to this aperiodic structure. Damanik showed in [6] that there is a strong connection between the properties of the spectrum and the value of the critical exponent of the word $\mathbf{u}_\beta$.

In 1912, A. Thue studied words with minimal repetitions; he discovered a word with the critical exponent equal to two – the lowest possible critical exponent for binary words – which is now known as Thue-Morse word [15]. A great effort was made to compute the critical exponent of Sturmian words. For the most prominent Sturmian word, namely the Fibonacci word, the critical exponent was calculated by Mignosi and Pirillo in 1992 in [11]. The general result for all Sturmian words was provided independently by Carpi and de Luca [5] and by Damanik and Lenz [7]; the formula comprises the coefficients of the continued fraction of the slope of a given Sturmian word.

The major contribution to the problem of computing the critical exponent of fixed points of substitution is due to D. Krieger [10]. She proved that the critical exponent of such words is either infinite or belongs to the algebraic field generated by the eigenvalues of the incidence matrix of the respective substitution.

In the present paper we provide the formula for computing the critical exponent of the words $\mathbf{u}_\beta$ associated to non-simple Parry numbers. The basic ingredients of our method are the description of all bispecial factors of $\mathbf{u}_\beta$ and the notion of return words. This method can be applied to any fixed point of any primitive substitution.

The paper is organized as follows. In Section 2 we recall basic notions of combinatorics on words and we introduce a connection of the studied words $\mathbf{u}_\beta$ with some numeration systems. Section 3 shows that description of bispecial factors and return words is crucial for evaluation of the critical exponent of any infinite word. Therefore, Section 4 is focused on these objects in the word $\mathbf{u}_\beta$. In Section 5 the main theorem is stated. Its proof is contained in Section 6. The last section is devoted to derivation of a simple form of the ultimate critical exponent.

2. Preliminaries

2.1. Combinatorics on Words

A *finite word* w over a finite alphabet $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$ is a string of letters from $\mathcal{A}$, i.e., $w = w_1 w_2 \dots w_n$, where $w_i \in \mathcal{A}$ for all $i = 1, 2, \dots, n$. The *length* of the word $w = w_1 w_2 \dots w_n$ will be denoted $|w| = n$, by $|w|_a$ we denote the number of occurrences of the letter a in w. The *Parikh vector* of a finite word w is the row vector $\Psi(w) = (|w|_{a_1}, |w|_{a_2}, \dots, |w|_{a_m}) \in \mathbb{N}^m$. Clearly, $|w| = \Psi(w)\vec{e}$, where $\vec{e}$ is the column vector from $\mathbb{R}^m$ whose entries are all equal to 1.

For the set of finite words over the alphabet $\mathcal{A}$, the notation $\mathcal{A}^*$ is used. An *infinite word* $\mathbf{u}$ over the alphabet $\mathcal{A}$ is a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. The set of such sequences is denoted $\mathcal{A}^{\mathbb{N}}$. The set $\mathcal{A}^*$ together with concatenation forms a monoid, with the empty word ϵ as its neutral element. The notation w^k for $w \in \mathcal{A}^*$ and $k \in \mathbb{N}$ stands for concatenation of k copies of the word w; the symbol w^ω means the infinite repetition of w.

If a word w arises by concatenation of x and y, i.e., $w = xy$, then x is called a *prefix* of w and y is a *suffix* of w. The prefix x can be obtained from w by the "inverse" procedure to concatenation, namely by erasing the suffix y, therefore we will also use $x = wy^{-1}$ and, analogously, $y = x^{-1}w$. The *cyclic shift* on $\mathcal{A}^*$ is the mapping $w \to S(w) = awa^{-1}$, where a is the last letter of w. Any iteration $S^k(w)$ of the cyclic shift for $k \in \mathbb{N}$ is called a *conjugate* of w. We say that a word $w \in \mathcal{A}^*$ is *primitive* if it has $|w|$ conjugates.

A word $w \in \mathcal{A}^*$ is said to be a *factor* of an infinite word $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ if there exists an index $i \in \mathbb{N}$ such that w is a prefix of $u_i u_{i+1} \cdots$. The index i is an *occurrence* of w in $\mathbf{u}$. The set of all factors of $\mathbf{u}$ is denoted $\mathcal{L}(\mathbf{u})$.

An infinite word $\mathbf{u}$ is *recurrent*, if each of its factors has at least two occurrences in $\mathbf{u}$. If, moreover, the gaps between neighboring occurrences of a given factor are bounded for any factor, $\mathbf{u}$ is *uniformly recurrent.*

A word v is called a *power* of w if v is a prefix of w^ω. If v is not a power of any word w' shorter than w, then w is the *root* of v. The *index* of a finite word $w \neq \epsilon$ in an infinite word $\mathbf{u}$ is

$$\operatorname{ind}(w) = \sup \left\{ \tfrac{|v|}{|w|} | v \in \mathcal{L}(\mathbf{u}) \text{ and } v \text{ is a power of } w \right\} .$$

Let us limit our considerations to uniformly recurrent aperiodic infinite words. Under this assumption, any factor $w \in \mathcal{L}(\mathbf{u})$ has a finite index. A power v of w for which the supremum is attained is called the *maximal power* of w in $\mathbf{u}$. The *critical exponent* of an infinite word $\mathbf{u}$ is defined as

$$\mathrm{E}(\mathbf{u}) = \sup\{\operatorname{ind}(w) \mid w \in \mathcal{L}(\mathbf{u})\} . \tag{1}$$

In [3], the authors introduce $\mathrm{E}^*(\mathbf{u})$ which is closely related to $\mathrm{E}(\mathbf{u})$. The characteristics $\mathrm{E}^*(\mathbf{u})$ expresses how long repetitions occur in the infinite word $\mathbf{u}$ when the factors of length growing ad infinitum are considered. In order to provide an exact definition of $\mathrm{E}^*(\mathbf{u})$, let us denote by $\operatorname{ind}_n(\mathbf{u}) = \max\{\operatorname{ind}(w) \bigm| w \in \mathcal{L}(\mathbf{u}),\ |w| = n\}$. The *ultimate critical exponent* of an infinite word $\mathbf{u}$ is defined as

$$\mathrm{E}^*(\mathbf{u}) = \limsup_{n\to\infty} \operatorname{ind}_n(\mathbf{u}). \tag{2}$$

Clearly, $\mathrm{E}(\mathbf{u}) \geq \mathrm{E}^*(\mathbf{u})$. In case $\mathrm{E}(\mathbf{u}) \notin \mathbb{Q}$, then $\mathrm{E}^*(\mathbf{u}) = \mathrm{E}(\mathbf{u})$.

Let us recall that a *morphism* on $\mathcal{A}^*$ is a mapping $\varphi : \mathcal{A}^* \to \mathcal{A}^*$ such that $\varphi(wv) = \varphi(w)\varphi(v)$ for all $w, v \in \mathcal{A}^*$. To any morphism φ, one can assign its *incidence matrix* M_φ by the prescription $(M_\varphi)_{a,b} = |\varphi(a)|_b$. The incidence matrix enables us to express the Parikh vector of the image of w by φ. One has

$$\Psi(\varphi(w)) = \Psi(w)M_\varphi\,. \tag{3}$$

We say that a morphism φ is *primitive* if there exists an exponent $k \in \mathbb{N}$ such that all entries of M_φ^k are positive.

The image of an infinite word $\mathbf{u}$ by φ is naturally defined as $\varphi(\mathbf{u}) = \varphi(u_0u_1u_2\ldots) = \varphi(u_0)\varphi(u_1)\varphi(u_0)\ldots$. The word $\mathbf{u} \in \mathcal{A}^\mathbb{N}$ is a *fixed point* of a morphism φ if $\varphi(\mathbf{u}) = \mathbf{u}$. If $\varphi(b) \neq \epsilon$ for every letter $b \in \mathcal{A}$ and if there exists a letter $a \in \mathcal{A}$ and $w \in \mathcal{A}^* - \{\epsilon\}$ such that $\varphi(a) = aw$, then φ is called a *substitution*. Any substitution has at least one fixed point, namely $\lim\limits_{n\to\infty} \varphi^n(a)$ (taken in the product topology). A substitution φ in general may have more fixed points. If φ is primitive, then any of its fixed points is uniformly recurrent and the languages of all fixed points of φ coincide.

Variability in an infinite word $\mathbf{u}$ is measured by the *factor complexity* $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ defined for every $n \in \mathbb{N}$ by

$$\mathcal{C}(n) = \#\{w \mid w \in \mathcal{L}(\mathbf{u}) \text{ and } |w| = n\}\,.$$

It is known [13] that the factor complexity of a fixed point of a primitive substitution is sublinear, i.e., there exist constants $a, b \in \mathbb{R}$ such that $\mathcal{C}(n) \leq an+b$ for all $n \in \mathbb{N}$.

For evaluation of the complexity of an infinite word $\mathbf{u}$, the special factors play an important role. Let us denote by $\mathrm{Rext}(w) = \{a \in \mathcal{A} \mid wa \in \mathcal{L}(\mathbf{u})\}$ and $\mathrm{Lext}(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(\mathbf{u})\}$ the set of all possible right and left extensions of a factor w, respectively. Clearly $\#\mathrm{Rext}(w) \geq 1$ for any factor w. If $\mathbf{u}$ is recurrent, then also $\#\mathrm{Lext}(w) \geq 1$. A factor $w \in \mathcal{L}(\mathbf{u})$ is said to be *right special* (RS) if $\#\mathrm{Rext}(w) \geq 2$ and *left special* (LS) if $\#\mathrm{Lext}(w) \geq 2$. We say that a factor w is *bispecial* (BS) if it is at once right and left special.

2.2. β-Integers

In 1957, A. Rényi introduced the *β-expansions* of positive numbers [14]. Consider a base $\beta > 1$. Then any $x \in [0, \infty)$ can be uniquely expressed in the form

$$x = \sum_{i=-\infty}^{N} x_i\beta^i, \tag{4}$$

where $x_i \in \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ and

$$0 \leq x - \sum_{i=n}^{N} x_i\beta^i < \beta^n \quad \text{for all } n \leq N+1, n \in \mathbb{Z}.$$

As it is usual in the everyday used cases of $\beta = 10$ and $\beta = 2$, we write

$$(x)_\beta = x_Nx_{N-1}\cdots x_1x_0\,\bullet\,x_{-1}x_{-2}x_{-3}\cdots$$

and we call this infinite word the β-expansion of x.

A number $x \in [0,\infty)$ is a *β-integer* if $x_i = 0$ for all negative indices i, i.e., $(x)_\beta = x_N x_{N-1} \cdots x_1 x_0\bullet$. All β-integers distributed on the positive real line form a discrete set and the distances between two neighboring β-integers are always ≤ 1. The set of all these distances can be described precisely using the *Rényi expansion of unity* $\mathrm{d}_\beta(1) = t_1 t_2 t_3 \cdots$, where $t_1 = \lfloor \beta \rfloor$ and $0 t_2 t_3 t_4 \cdots$ is the β-expansion of $1 - t_1/\beta$. Parry [12] proved that an infinite sequence $t_1 t_2 t_3 \cdots$ of nonnegative integers is the Rényi expansion of unity for some $\beta > 1$ if, and only if, the following so-called *Parry condition* is satisfied:

$$t_i t_{i+1} t_{i+2} \cdots \prec t_1 t_2 t_3 \cdots \qquad \text{for all } i = 2, 3, 4, \dots\ . \tag{5}$$

Thurston [16] proved that the distances between neighboring β-integers take values in the set $\{\triangle_k \mid k = 0, 1, 2, \ldots\}$ with

$$\triangle_k = \sum_{i=1}^{\infty} \frac{t_{i+k}}{\beta^i}. \tag{6}$$

A number $\beta > 1$ is said to be a *Parry number* if its set of the distances defined in (6) is finite. In such cases the distribution of distances between β-integers can be coded as an infinite word over a finite alphabet, we denote this word by $\mathbf{u}_\beta$. It is easy to see that β is a Parry number if, and only if, the Rényi expansion of unity is eventually periodic. In particular, we distinguish *simple Parry numbers* for which

$$\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m 0^\omega$$

and *non-simple Parry numbers* for which

$$\mathrm{d}_\beta(1) = t_1 t_2 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega.$$

The positive integers m, p are taken the least possible. This implies that $t_m \neq 0$ in the case of a simple Parry number and $t_m \neq t_{m+p}$ for the non-simple case. As shown by S. Fabre [8], the word $\mathbf{u}_\beta$ is also the unique fixed point of the canonical substitution φ_β associated with a Parry number β. For a simple Parry number β, the substitution φ_β acts on the alphabet $\mathcal{A} = \{0, 1, \ldots, m-1\}$ and is given by

$$\begin{array}{rcl}
\varphi_\beta(0) & = & 0^{t_1}1 \\
\varphi_\beta(1) & = & 0^{t_2}2 \\
 & \vdots & \\
\varphi_\beta(m-2) & = & 0^{t_{m-1}}(m-1) \\
\varphi_\beta(m-1) & = & 0^{t_m}
\end{array}$$

For a non-simple Parry number β, the alphabet is $\mathcal{A} = \{0, 1, \dots, m+p-1\}$ and

$$\begin{array}{rcl} \varphi_\beta(0) & = & 0^{t_1}1 \\ \varphi_\beta(1) & = & 0^{t_2}2 \\ & \vdots & \\ \varphi_\beta(m{-}1) & = & 0^{t_m}m \\ \varphi_\beta(m) & = & 0^{t_{m+1}}(m{+}1) \\ & \vdots & \\ \varphi_\beta(m{+}p{-}2) & = & 0^{t_{m+p-1}}(m{+}p{-}1) \\ \varphi_\beta(m{+}p{-}1) & = & 0^{t_{m+p}}m. \end{array} \tag{7}$$

In both cases the substitution is primitive.

As we said, in this paper we focus on the non-simple Parry numbers β. For them the incidence matrix of φ_β reads

$$M = \begin{array}{c} \begin{pmatrix} t_1 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ t_2 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{m+p-2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ t_{m+p-1} & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \end{pmatrix} \\ \begin{array}{cccccccc} \phantom{t_{m+p-1}} & & & & \uparrow & & & \\ & & & & m\text{-th} & & & \end{array} \end{array} \tag{8}$$

Since the substitution φ_β is primitive, the dominant eigenvalue of M is simple. It is not difficult to prove that this dominant eigenvalue is equal to β and that the vector $(\triangle_0, \triangle_1, \dots, \triangle_{m+p-1})^T$, with $\triangle_i$ defined in (6), is a right eigenvector corresponding to it.

For description of the bispecial factors of $\mathbf{u}_\beta$ it will be important to track the last letters of words $\varphi_\beta^n(a), a \in \mathcal{A}, n = 0, 1, \cdots$. Therefore, we introduce the following notation.

Definition 1. For all $k, \ell \in \mathbb{N}$ we define the addition $\oplus : \mathbb{N} \times \mathbb{N} \to \mathcal{A}$ as follows.

$$k \oplus \ell = \begin{cases} k + \ell & \text{if } k + \ell < m + p, \\ m + (k + \ell - m \mathrm{mod} p) & \text{otherwise.} \end{cases}$$

Similarly, if used with parameters t_i, we define for all $k, \ell \in \mathbb{N}, k + \ell > 0$,

$$t_{k \oplus \ell} = \begin{cases} t_{k+\ell} & \text{if } 0 < k + \ell < m + p + 1, \\ t_{m+1+(k+\ell-m-1 \bmod p)} & \text{otherwise.} \end{cases}$$

For example, employing this notation one can show that the word $\varphi_\beta^n(a), a \in \mathcal{A}$, has the suffix $0^{t_{a \oplus n}}(a \oplus n)$.

3. Maximal Powers and Bispecial Factors

The critical exponent $\mathrm{E}(\mathbf{u})$ is defined as the supremum of the set of indices $\mathrm{ind}(w)$ of all factors $w \in \mathcal{L}(\mathbf{u})$. We will show that the set of factors important for evaluation of $\mathrm{E}(\mathbf{u})$ can be significantly reduced.

Lemma 2. *Let $w \in \mathcal{A}^*$ have the maximal index in a recurrent infinite word $\mathbf{u}$ between all its conjugates and let this index be strictly greater than one. Let $w^\ell w'$ be the maximal power of w in u, where $\ell \geq 1$ and w' is a proper prefix of w. Further, let b be the last letter of w and let a be the letter such that $w'a$ is a prefix of w. Then*

(i) $b \notin \mathrm{Lext}(w^\ell w')$ *and* $a \notin \mathrm{Rext}(w^\ell w')$,

(ii) *for* $k = 0, 1, \ldots, \ell - 1$, $w^k w'$ *is a BS factor such that* $b \in \mathrm{Lext}(w^k w')$ *and* $a \in \mathrm{Rext}(w^k w')$.

Proof. (*i*) If $b \in \mathrm{Lext}(w^\ell w')$, then the index of a conjugate of w, namely bwb^{-1}, is greater than the index of w. If $a \in \mathrm{Rext}(w^\ell w')$, then $w^\ell w'$ is not the maximal power of w.

(*ii*) By (*i*), there exists at least one letter $x \neq b$ such that $x \in \mathrm{Lext}(w^\ell w')$; hence, $\{b, x\} \subset \mathrm{Lext}(w^k w')$. Analogously for the case of right extensions. □

Definition 3. Denote by $\mathcal{B}(\mathbf{u})$ the set of (ordered) pairs (v, w) of factors of an infinite word $\mathbf{u}$ satisfying the following conditions:

(B1) v is a BS factor,

(B2) wv is a power of w in $\mathbf{u}$.

Having this set defined, we can propose the following straightforward corollary of Lemma 2.

Corollary 4. *Given a uniformly recurrent infinite word $\mathbf{u}$, we have:*

$$\mathrm{E}(\mathbf{u}) = \sup\{\mathrm{ind}(w) \mid (v, w) \in \mathcal{B}(\mathbf{u}) \textit{ for some } v\}.$$

Of course, the equality is true even if we consider for a given BS factor v only the shortest w such that $(v, w) \in \mathcal{B}(\mathbf{u})$. And this will be our strategy: we will first find all BS factors v in $\mathbf{u}_\beta$ and then the corresponding shortest w. Usually, for a given BS factor v, it is not difficult to find w such that v is a power of w and to verify that wv is a factor of $\mathbf{u}$. What may be a problem is to prove that this w is the shortest such factor. Sometimes it is convenient to use the notion of return words.

Definition 5. Let $w \in \mathcal{L}(\mathbf{u})$. If v_L and v_R satisfy

(i) $wv_R = v_L w \in \mathcal{L}(\mathbf{u})$,

(ii) there are exactly two occurrences of w in $wv_R = v_L w$,

then v_L is a *left return word (LRW)* of w, v_R is a *right return word (RRW)* of w, and $wv_L = v_R w$ is a *complete return word (CRW)* of w in $\mathbf{u}$.

For example, if

$$\mathbf{u} = 00\underline{0}010\underline{0}0100\underline{0}01\underline{0}01000\cdots,$$

then all LRWs of 0010 visible in this prefix are 0010, 00100, 001. Thus, a LRW of w may be shorter than w itself!

Lemma 6. *Let v be a power of w and $\tilde{w}$ a prefix of v. If w is a LRW of $\tilde{w}$, then w is the root factor of v.*

This simple observation turns out to be very useful. In the case of $\mathbf{u}_\beta$, there exists a simple tool for generating all BS factors. For any BS factor v we will find easily a factor w such that $(v, w) \in \mathcal{B}(\mathbf{u}_\beta)$. Then, by a good choice of the prefix $\tilde{w}$ from the previous lemma, we will prove that this w is the shortest possible.

4. Bispecial Factors in $\mathbf{u}_\beta$

Throughout the following text, the coefficient t_1 from (7) will be greater than 1. Corollary 4 claims that to get the critical exponent of $\mathbf{u}_\beta$, it suffices to go through all BS factors v and corresponding w (if it exists) such that $(v, w) \in \mathcal{B}(\mathbf{u}_\beta)$. In what follows, we will take advantage of having described all BS factors of $\mathbf{u}_\beta$ in [9]. In order to present the necessary results we need some more sophisticated notation for BS factors.

Definition 7. Let $a, b, c, d \in \mathcal{A}$ such that $a \neq b$ and $c \neq d$. A factor $v \in \mathcal{A}^*$ is an $(a - c, b - d)$-*bispecial factor* of an infinite word $\mathbf{u}$ if both avc and bvd are factors of $\mathbf{u}$.

In the sequel, the aim is to introduce a mapping (Definition 11) which will help us to describe all BS factors of $\mathbf{u}_\beta$ as sequences of words generated by this mapping from a finite number of short BS factors. We start with some technical results.

Lemma 8. *Let $a \in \mathcal{A} \setminus \{0\}$. Then the letter*

$$b = \max\{j \mid 0^j \textit{ is a suffix of } t_1 t_2 \cdots t_a\}$$

is a left extension of the factor a. Another possible left extension of a is c, where

$$\begin{aligned} c &= \max\{j \mid 0^j \textit{ is a suffix of } t_{m+1} \cdots t_{m+p}\} && \textit{for } a = m, \\ c &= \max\{j \mid 0^j \textit{ is a suffix of } t_{m+1} \cdots t_{m+p} t_{m+1} \cdots t_a\} && \textit{for } a > m. \end{aligned}$$

There are no other left extensions.

Proof. The statement is a direct consequence of this simple fact: if $t_a > 0$, then 0 is a left extension of a; if $t_a = 0$ and $t_{a-1} > 0$, then 1 is a left extension. Continuing in this manner we get that b defined as above is always a left extension of a. In fact, b is the next to last letter of $\varphi_\beta^a(0)$.

Since $a \geq m$ can appear not only as the last letter of $\varphi_\beta^a(0)$ but also as the last letter of $\varphi_\beta^{p+a-m}(m)$, the letter c can be the other left extension. □

Note that due to the assumption $t_1 > 1$ we must have $\mathrm{Lext}(0) = \mathcal{A}$. If $t_1 = 1$, then clearly 00 cannot be a factor.

Let us denote throughout the following text

$$t = \min\{t_m, t_{m+p}\} \quad \text{and} \quad \mathrm{Lext}(0^t m) = \{0, z\}. \tag{9}$$

Corollary 9. *The nonzero left extension z of $0^t m$ is given by*

$$z = \begin{cases} 1 + \max\{j \mid 0^j \text{ is a suffix of } t_{m+1} \cdots t_{m+p} t_{m+1} \cdots t_{m+p-1}\} & t_m > t_{m+p}, \\ 1 + \max\{j \mid 0^j \text{ is a suffix of } t_1 \cdots t_{m-1}\} & t_{m+p} > t_m. \end{cases} \tag{10}$$

Proof. Since $t_m = t_{m+p}$ is not admissible, 0 must be a left extension of $0^t m$. The other left extension z is then determined by the (unique) left extension of $m-1$, if $t = t_m$, or of $m+p-1$, otherwise. □

Since $\mathrm{Lext}(0) = \mathcal{A}$, the factor 0 is LS and, clearly, even its image $0^{t_1}1$ is LS. Other short LS factors are described in the following Corollary which is consequence of Lemma 8.

Corollary 10. *If v is a LS factor of $\mathbf{u}_\beta$ containing at least one nonzero letter, then one of the following factors is a prefix of v:*

(i) $0^{t_1}1$,

(ii) $0^t m$,

(iii) $0^{t_k}k$, *if* $k > m$ *and* $t = t_{m+1} = t_{m+2} = \cdots = t_{k-1} = 0$.

Now, let us introduce the announced mapping that, when iterated, produces all BS factors from a finite number of some short ones.

Definition 11. Let $\{a, b\}$ be a set of two distinct letters of $\mathcal{A}$. We define:

$$f_L(b, a) = f_L(a, b) = \text{the longest common suffix of } \varphi_\beta(a) \text{ and } \varphi_\beta(b)$$

and

$$f_R(b, a) = f_R(a, b) = \text{the longest common prefix of } \varphi_\beta(a) \text{ and } \varphi_\beta(b).$$

If v is an $(a-c, b-d)$-bispecial factor of $\mathbf{u}_\beta$, then the *f-image* of v is the factor

$$f(v) = f_L(a, b)\varphi_\beta(v) f_R(c, d).$$

The f-image is defined so that it maps any BS factor to another one.

The form of the substitution φ_β means that, for distinct letters a and b, the factors $\varphi_\beta(a)$ and $\varphi_\beta(b)$ ends in letters $a \oplus 1$ and $b \oplus 1$. These letters are again distinct if and only if $\{a, b\} \neq \{m-1, m+p-1\}$: in such a case $f_L(a, b) = \epsilon$ and the left extension of the f-image are $a \oplus 1$ and $b \oplus 1$. If $\{a, b\} = \{m-1, m+p-1\}$, we get $f_L(a, b) = 0^t m$ and the left extensions are 0 and z by (9). The case of right extensions is even more straightforward.

Lemma 12. *Let v be an $(a-c, b-d)$-bispecial factor of $\mathbf{u}_\beta$. Then we have*

$$f_L(a,b) = \begin{cases} 0^t m & \text{if } \{a,b\} = \{m-1, m+p-1\}, \\ \epsilon & \text{otherwise}, \end{cases}$$

and

$$f_R(c,d) = 0^{\min\{t_{c\oplus 1}, t_{d\oplus 1}\}}.$$

The f-image of v is then an $(a'-c', b'-d')$-bispecial factor, where c' and d' are the first letters of factors $0^{t_{c\oplus 1} - \min\{t_{c\oplus 1}, t_{d\oplus 1}\}}(c \oplus 1)$ and $0^{t_{d\oplus 1} - \min\{t_{c\oplus 1}, t_{d\oplus 1}\}}(d \oplus 1)$, respectively, and a' and b' are either 0 and z, if $\{a, b\} = \{m-1, m+p-1\}$, or $a \oplus 1$ and $b \oplus 1$, otherwise.

Since the f-image is again a BS factor, we can construct a sequence of f^n-images of some starting BS factor. It is easy to see that any $(a-c, b-d)$-bispecial factor containing at least two nonzero letters has a unique f-preimage, i.e., it is equal to $f(v)$ for a unique v from Definition 11. This, together with Corollary 10 (note that $0^{t_k}k$ from (iii) are just φ_β^{k-m}-images of m), implies that any BS factor is an f^n-image of one of these BS factors:

(I) $0^k, 0 < k \leq t_1 - 1$,

(II) $0^t m 0^\ell, 0 \leq \ell \leq t_1$.

In fact, as we shall see, even $0^t m 0^\ell$ is an f^n-image of an $(a-c, b-d)$-bispecial factor ϵ where p divides $(a-b)$.

Lemma 13. *Let v be an $(a-c, b-d)$-bispecial factor of $\mathbf{u}_\beta$. Then there exist $n \in \mathbb{N}$ and $a', b', c', d' \in \mathcal{A}$ such that v is an f^n-image of the $(a'-c', b'-d')$-bispecial factor 0^k with $0 \leq k \leq t_1 - 1$.*

Proof. The only thing to show is that if the factor $0^t m 0^\ell, 0 \leq \ell \leq t_1$ is BS, then it is an f^n-image of ϵ. We first find a condition under which $0^t m 0^\ell$ is BS. Since $0^t m$ occurs only as a suffix of $\varphi_\beta^{m+pk}(0), k \geq 0$, the factor $0^t m$ is always followed in $\mathbf{u}_\beta$ either by $\varphi_\beta^m(x), x \in \mathcal{A}$, or by $\varphi_\beta^{m+pk}(y), k \geq 1, y \in \mathcal{A}$. Since $\varphi_\beta^{m+pk}(y)$ always begins in $0^{t_1}1$, in order for $0^t m 0^\ell$ to be BS, we need $\varphi_\beta^m(x)$ does not begin in $0^{t_1}1$ for some $x \in \mathcal{A}$. This implies that $t_{x \oplus k} = 0$ for $k \in \{1, \ldots, m-1\}$. Thus, we have the condition $\varphi_\beta^m(x) = 0^{t_{x\oplus m}}(x \oplus m)$. But in such a case, v is the f^m-image of $(a'-x, b'-0)$-bispecial factor ϵ with $a' \in \mathrm{Lext}(x)$ and $b' = a' + p$. $\square$

Definition 14. The $(a-c, b-d)$-bispecial factors $0^k, 0 \le k < t_1$, will be called *initial*.

Thus, all BS factors can be generated from a few short initial factors applying a very simple rule repetitively. This rule can be even more simplified.

Definition 15. Let $n \in \mathbb{N}$ and $n = \ell m + k, 0 \le k < m$. Then we put

$$z^{(n)} = \begin{cases} \epsilon & \text{if } n < m, \\ \varphi_\beta^k(0^t m)\varphi_\beta^{k+m}(0^t m)\cdots\varphi_\beta^{(\ell-1)m+k}(0^t m) & \text{if } n \ge m \text{ and } z \text{ is a multiple of } p, \\ \varphi_\beta^{n-m}(0^t m) & \text{otherwise.} \end{cases}$$

Lemma 16. *Let v be an $(a-c, b-d)$-bispecial factor such that $t_{c\oplus 1}t_{c\oplus 2}\cdots \preceq t_{d\oplus 1}t_{d\oplus 2}\cdots$. The f^n-image of v is equal to $u_1\varphi_\beta^n(v)u_2$, where:*

$$u_2 = \operatorname{lcp}(\varphi_\beta^n(c), \varphi_\beta^n(d)) = \varphi_\beta^n(c)(c\oplus n)^{-1}$$

and

$$u_1 = \begin{cases} \epsilon & \textit{if } p \textit{ does not divide } a-b, \\ z^{(n+\min\{a,b\})} & \textit{otherwise.} \end{cases}$$

Proof. The fact that $u_2 = \varphi_\beta^n(c)(c\oplus n)^{-1}$ is proved in [9, Lemma 45,46]. As for the form of u_1, if $p \nmid a-b$, then $f_L(a,b) = f_L(a\oplus 1, b\oplus 1) = \cdots = f_L(a\oplus n, b\oplus n) = \epsilon$ by Lemma 12, therefore $u_1 = \epsilon$. If $p \mid a-b$ (assume $a < b$), then $a < m$. We must have $a \oplus (m-a) = b \oplus (m-a) = m$ and so the f^{m-a}-image of v begins in $z^{(m)} = 0^t m$. The rest is obvious. $\square$

5. Main Theorem

Having the simple tool for description of all BS factors, it remains to find the shortest factors which form with them a pair from $\mathcal{B}(\mathbf{u}_\beta)$.

Imagine that we are given a BS factor v which arises from a nonempty initial factor $0^s, s > 0$. According to Lemma 16, v can have only one of the following two forms:

(*a*) $v = \varphi_\beta^n(0^s)\varphi_\beta^n(c)(c\oplus n)^{-1}$ for some $c \in \mathcal{A} \setminus \{0\}$. In this case it is obvious that $(v, \varphi_\beta^n(0))$ is a pair from $\mathcal{B}(\mathbf{u}_\beta)$. Moreover, it is not difficult to prove by induction on n that $\varphi_\beta^n(0)$ is the shortest such factor (see Lemma 24).

(*b*) $v = z^{(n+r)}\varphi_\beta^n(0^s)\varphi_\beta^n(c)(c\oplus n)^{-1}$ for some $s > 0$, $c \in \mathcal{A}$, and $r \in \mathbb{N}$. The cases when $r > 0$ will be studied later on. For now assume that $v = z^{(n)}\varphi_\beta^n(0^s)\varphi_\beta^n(c)(c\oplus n)^{-1}$. As a direct consequence of the definitions of φ_β and $z^{(n)}$, we get that $z^{(n)}$ is a suffix of $\varphi_\beta^n(0)$. This yields that $(v, z^{(n)}\varphi_\beta^n(0)(z^{(n)})^{-1})$ is a good candidate for being an element of $\mathcal{B}(\mathbf{u}_\beta)$.

To prove that $w = z^{(n)}\varphi_\beta^n(0)(z^{(n)})^{-1}$ is the shortest possible choice is a bit more problematic than in case (a). In order to do so we will use Lemma 6 with $\tilde{w} = z^{(n)}$.

Let us take as an example β such that $d_\beta(1) = 33(02)^\omega$. Then f^2-image of $(0-1, 2-0)$-bispecial factor 00 equals to $(0-3, 2-0)$-bispecial factor

$$v = \overbrace{002}^{z^{(2)}}\underbrace{0001000100010002}_{\varphi_\beta^2(0)}\underbrace{0001000100010002}_{\varphi_\beta^2(0)}\overbrace{000100010001}^{\varphi_\beta^2(1)(3)^{-1}}.$$

Clearly, the factor $w = 0020001000100010$ is the LRW of 002 and so w is the root factor of v. It turns out that this argument can be used in general if we replace 002 with $z^{(n)}$ and as the factor v we take $v^{(n)}$ defined as follows:
Denote by $v^{(0)}$ a $(0-c, b-d)$-bispecial factor 0^s, $s > 0$, and by $v^{(n)}$ its f^n-image. By Lemma 16 we have

$$v^{(n)} = z^{(n)}\varphi_\beta^n(0^s c)(c \oplus n)^{-1}. \tag{11}$$

Using these techniques, we will prove the following theorem.

Theorem 17. *Let z be defined as in (9). If $t_1 \geq 4$ or if $t_1 = 3$ and p does not divide z, then the critical exponent satisfies*

$$\mathrm{E}(\mathbf{u}_\beta) = \sup_{n\in\mathbb{N}}\left\{t_1 + \frac{|z^{(n)}| + |\varphi_\beta^n(1)| - 1}{|\varphi_\beta^n(0)|}\right\}$$

and the ultimate critical exponent equals

$$\mathrm{E}^*(\mathbf{u}_\beta) = \begin{cases}\beta + \frac{1}{\beta^m - 1}(t + \triangle_m) & \text{if } p \text{ divides } z,\\ \beta + \frac{1}{\beta^m}(t + \triangle_m) & \text{otherwise,}\end{cases}$$

where Δ_m is defined in (6).

6. Proof of the Main Theorem

First let us describe all return words of $z^{(n)}$ since they are playing a crucial role in the following sections.

6.1. Return Words of $z^{(n)}$

We can distinguish three types of return words of $0^t m$. Denote by $X, Y \in \mathcal{A}$, nonzero letters such that $X0^{t_m}m$ and $Y0^{t_{m+p}}m$ are factors of $\mathbf{u}_\beta$; Y is always unique and X is unique if $m > 1$. Let v be a CRW of $0^t m$; then v is a suffix of exactly one of the following factors of $\mathbf{u}_\beta$:

(A) $w_1 X 0^{t_m} m$,

(B) $X 0^{t_m} m w_2 Y 0^{t_{m+p}} m$,

(C) $Y 0^{t_{m+p}} m w_3 Y 0^{t_{m+p}} m$,

where w_1 is long enough and w_2 and w_3 do not contain $0^t m$ as a factor. The following lemma is based on this observation.

Lemma 18. *Let v be a CRW of $0^t m$, then v satisfies exactly one of the following conditions:*

(A) $X0^{t_m}m$ is a suffix of v; in this case $\varphi_\beta^m(0)$ must be also a suffix of v,

(B) v is a suffix of the φ_β^m-image of $0wy$, where y is a letter of the form $sp, s \geq 1$, and $w \in \mathcal{A}^*$ does not contain 0 or any multiple of p,

(C) v is a suffix of φ_β^p-image of $mw'm$, where $w' \in \mathcal{A}^*$ and $\varphi_\beta^p(w')$ does not contain the factor $0^t m$ and $0^t m w' m$ is a return word of $0^t m$, which is not of type (A).

Proof. If $X0^{t_m}m$ is a suffix of v, then $\varphi_\beta^m(0)$ must be a suffix of v as well since $X0^{t_m}m$ appears in $\mathbf{u}_\beta$ only as a suffix of $\varphi_\beta^m(0)$ and $\varphi_\beta^m(0)$ contains $0^t m$ only at the end.

Let v be of type (B). Then since $X0^{t_m}m$ occurs only as a suffix of $\varphi_\beta^m(0)$ and $Y0^{t_{m+p}}m$ occurs only as a suffix of $\varphi_\beta^m(sp)$ with $s \geq 1$, the factor v must be a suffix of $\varphi_\beta^m(0w(sp))$ for some $w \in \mathcal{A}^*$. Since v has to be a return word of $0^t m$, the factor w can contain neither 0 nor any multiple of p.

Let v be of type (C). Then since $Y0^{t_{m+p}}m$ occurs only as a suffix of $\varphi_\beta^p(m)$, the factor v must be a suffix of $\varphi_\beta^p(mw'm)$. Since v has to be a return word of $0^t m$, the factor $\varphi_\beta^p(w')$ cannot contain $0^t m$.

The factor $0^t m w' m$ is obviously a complete return word of $0^t m$. Assume it is of type (A). Then $\varphi_\beta^m(0)$ is a suffix of $w'm$ and so w' contains all letters $\leq m-1$. Consequently, $\varphi_\beta^p(w')$ contains all letters of $\mathcal{A}$ and also the factor $0^t m$ - a contradiction. ☐

Corollary 19. *Let v be a CRW of $z^{(n)}, n \geq m$. Then v is a suffix of φ_β^{n-m}-image of $0v'$ or $w'zv'$, where v' is a CRW of $0^t m$ satisfying exactly one of the conditions (A), (B), and (C), and $w' \in \mathcal{A}^*$ is long enough so that $z^{(n)}$ is a suffix of $\varphi_\beta^{n-m}(w'z0^t m)$.*

Proof. If z is not a multiple of p, then the CRWs of $z^{(n)}, n > m$, are exactly the φ_β^{n-m}-images of the CRWs of $0^t m$. Otherwise, we need to extend these φ_β^{n-m}-images to the left so that they contain the complete factor $z^{(n)}$. Due to the properties of φ_β, $z^{(n)}$ is always a proper suffix of $\varphi_\beta^{n-m}(00^t m)$. But this might not be true for $\varphi_\beta^{n-m}(z0^t m)$, and therefore we must consider a long enough prolongation by a factor w'. ☐

Example 20. Let us illustrate the situation where a CRW of $z^{(n)}$ is a suffix of $\varphi_\beta^{n-m}(w'z0^t m)$, however $z^{(n)}$ is not a suffix of $\varphi_\beta^{n-m}(z0^t m)$. In fact, such a situation occurs whenever $\varphi_\beta^m(z) = 0^{t_{m+p}}m$, where $t_{m+p} < t_m$ and z is a multiple of p. Let $d_\beta(1) = 21^\omega$. The substitution φ_β then reads: $0 \mapsto 001, 1 \mapsto 01$ and we have $m = 1, p = 1, t = 1$, and $0^t m = 01$. It is easy to calculate $z^{(3)} = 01\varphi_\beta(01)\varphi_\beta^2(01)$ and $\varphi_\beta^{3-m}(z0^t m) = \varphi_\beta^2(101) = \varphi_\beta(01)\varphi_\beta^2(01)$.

Definition 21. In the terms of the previous lemma and its corollary, we distinguish three types of LRWs, RRWs, and CRWs of $z^{(n)}, n \geq m$: type (A), type (B), and type (C).

Example 22. Let $d_\beta(1) = 221(12)^\omega$. The substitution φ_β then reads: $0 \mapsto 001, 1 \mapsto 002, 2 \mapsto 03, 3 \mapsto 04, 4 \mapsto 003$ and we have $m = 3, p = 2, t = 1$, and $0^t m = 03$.

Any CRW of 03 of type (A) ends in $\varphi_\beta^3(0) = 001001002001001002001001003$, there are three such CRWs:

$$03001001002001001002001001003,$$
$$03001001002001001004001001002001001002001001003,$$
$$0300100100200100100200104001001002001001002001001003.$$

There are two multiples of $p = 2$ less than $m + p = 5$; we have to consider $y = sp, s = 1, 2$, to get all CRWs of type (B). Since $t_{2p} = t_4 > 0$ and $t_p = t_2 > 0$, the factor w from Lemma 18 item (B) is empty. Thus $\varphi_\beta^m(0p) = \varphi_\beta^3(02)$ and $\varphi_\beta^m(04) = \varphi_\beta^3(04)$ are the only sources of CRWs of type (B). We get two CRWs:

$$s = 1 \quad \to \quad 03001001002001003,$$
$$s = 2 \quad \to \quad 0300100100200100100200l003.$$

Having all CRWs of type (B), we can see there are no CRWs of type (C) for both $\varphi_\beta^2(00100100200100)$ and $\varphi_\beta^2(00100100200100100200100)$ contain 03 as a factor.

Example 23. Let $d_\beta(1) = 2000(1)^\omega$. The substitution φ_β then reads: $0 \mapsto 001, 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 04$ and we have $m = 4, p = 1, t = 0$, and $0^t m = 4$.

There is only one CRW of type (A): 40010012001001230010012001001234.

As for type (B), there are four multiples of $p = 1$ less than $m + p = 5$: $y = sp, s = 1, 2, 3, 4$. But for $s \geq 2$, the only factor of the form $0wy$ is $012 \cdots y$, i.e., not admissible as it contains $p = 1$ between 0 and y. Hence, we have only one CRW of type (B): the suffix 404 of $\varphi_\beta^4(01)$.

For this β, there exist CRWs of type (C). Take 404; then $w' = 0$ (see Lemma 18 item (C)) and $\varphi_\beta^p(0) = 001$ does not contain 4. Hence, we get a CRW of type (C): 400104. Now take this factor and again apply $\varphi_\beta^p = \varphi_\beta$. This yields another CRW of type (C): 4001001200104. Doing the same again, we get the third CRW of type (C): 400100120010012300100120010 4. And this is the last one since the word $\varphi_\beta(0010012001001230010012001 0)$ does contain 4.

6.2. Bispecial Factors of Type (*I*)

Now let $u^{(n)}$ be the f^n-image of an $(a-c, b-d)$-bispecial factor 0^s, where p does not divide $a-b$, $0 < s < t_1$, and $t_{c\oplus 1}t_{c\oplus 2}\cdots \preceq t_{d\oplus 1}t_{d\oplus 2}\cdots$. Then we have by Lemma 16

$$u^{(n)} = \varphi_\beta^n(0^s)\varphi_\beta^n(c)(c\oplus n)^{-1}.$$

Lemma 24. *The root factor of* $u^{(n)}$ *is* $\varphi_\beta^n(0)$.

Proof. We proceed by induction on n. The case of $n = 0$ is trivial. For a greater n, the statement is a direct consequence of the simple fact that any factor starting in $0^{t_1}1$ and ending in a nonzero digit has a unique φ_β-preimage: Assume that $u^{(n)} = \bar{w}^\ell \bar{w}'$, where $\bar{w}$ is shorter than $\varphi_\beta^n(0)$ and $\bar{w}'$ is a proper prefix of $\bar{w}$. Then $\bar{w}$ must begin in $0^{t_1}1$ and end in a nonzero letter. Hence, there exists a unique φ_β-preimage of $\bar{w}$ which is shorter than $\varphi_\beta^{n-1}(0)$ and such that $u^{(n-1)}$ is a power of it. A contradiction. $\square$

We will prove in the sequel that $\varphi_\beta^n(0)u^{(n)}$ is a factor of $\mathbf{u}_\beta$. Hence, we will have shown that $(u^{(n)}, \varphi_\beta^n(0)) \in \mathcal{B}(\mathbf{u}_\beta)$.

Now, let us look at the case when $a = 0$ and b is a multiple of p. This assumption means that the f^m-image begins in $0^t m$. Then the f^n-image of $(a-c, b-d)$-bispecial factor 0^s reads

$$v^{(n)} = z^{(n)}\varphi_\beta^n(0^s)\varphi_\beta^n(c)(c\oplus n)^{-1}.$$

For $n < m$, $z^{(n)}$ is empty and hence $v^{(n)}$ equals $u^{(n)}$ defined above. For $n \geq m$, by Lemma 6, any w such that $(v^{(n)}, w) \in \mathcal{B}(\mathbf{u}_\beta)$ must be a LRW of $z^{(n)}$ and since $z^{(n)}$ is followed by $\varphi_\beta^n(0)$, the shortest such w must be a LRW of type (A), namely $z^{(n)}\varphi_\beta^n(0)(z^{(n)})^{-1}$, a conjugate of $\varphi_\beta^n(0)$.

Let us summarize what we know so far: if a BS factor, which is an f^n-image of some initial block of zeros, begins in $\varphi_\beta^n(0)$, then $\varphi_\beta^n(0)$ must be its root factor; if it begins in $z^{(n)}\varphi_\beta^n(0^s)$, the root factor is $z^{(n)}\varphi_\beta^n(0)(z^{(n)})^{-1}$. Since both of these root factors are of the same length, the greatest index is attained by the longest one of such BS factors. Altogether, we have found the root factors of a significant subset of all BS factors; these BS factors will be defined as being of type (I).

Definition 25. If v is the f^n-image of an $(a-c, b-d)$-bispecial factor 0^s, where $0 < s < t_1$ and either p does not divide $a-b$ or $a = 0$ and b is a multiple of p, then v is said to be of type (I). If v is not of type (I), it is of type (II).

The complicated definition of type (I) can be reformulated: either p does not divide $a-b$ or the f^m-image of 0^s begins in $0^t m$. This is not satisfied, e.g., for $(1-c, (p+1)-d)$-bispecial factors for their f^{m-1}-image already begins in $0^t m$. We will study the BS factors of type (II) in the next section. In fact, the following holds: the root factors of $(0-c, kp-d)$ BS factors of type (I) are the LRWs of $z^{(n)}$

of type (A), for a BS factor v of type (II) even the LRWs of $z^{(n)}$ of type (B) or (C) can form together with v a pair from $\mathcal{B}(\mathbf{u}_\beta)$.

Definition 26. We define

$$\mathrm{E}_I(\mathbf{u}_\beta) = \sup\{\mathrm{ind}(w) \mid (v,w) \in \mathcal{B}(\mathbf{u}_\beta),\ v\text{of type } (I)\}.$$

In order to identify the longest BS factors of type (I), we will use some technical results. We know that the longest common prefix of $\varphi_\beta^n(c)$ and $\varphi_\beta^n(d)$ equals $\varphi_\beta^n(c)(c \oplus n)^{-1}$, where $t_{c\oplus 1}t_{c\oplus 2}\cdots \preceq t_{d\oplus 1}t_{d\oplus 2}\cdots$. Due to the Parry condition

$$t_{c\oplus 1}t_{c\oplus 2}\cdots \preceq t_1t_2\cdots \quad \text{for all } c \in \mathcal{A}\setminus\{0\},$$

we get that the longest common prefix of $\varphi_\beta^n(c)$ and $\varphi_\beta^n(0)$ equals $\varphi_\beta^n(c)(c\oplus n)^{-1}$ for all nonzero c. Therefore, to get the longest $u^{(n)}$ and $v^{(n)}$, the initial factor must be an $(a-c, b-d)$-bispecial factor 0^{t_1-1}. In order for the condition $(B2)$ from Definition 3 to be satisfied for this initial factor and its f^n-images, 0 must be one of the left extension, say $a = 0$. This implies that $0^{t_1}c$ must be a factor of $\mathbf{u}_\beta$; this is always true for $c = 1$. To get the longest f^n-images, the choice $d = 0$ is the best possible and c has to be chosen so that $t_{c\oplus 1}t_{c\oplus 2}\cdots$ is the greatest possible with respect to the lexicographical order. But since either $t_c = t_1$ or $t_{m+p} = t_1$ and $c = m$, the lexicographical maximum is obtained by $c = 1$ due to the Parry condition. So, the a, c, and d are fixed, b is to be chosen so that the f^m-image of $(0-1, b-0)$ begins in 0^tm. This always happens for $b = p$, but any multiple of p will do the same job.

The above explanation implies that if we prove validity of the condition $(B2)$ from Definition 3, we will have the right to replace in the definition of $\mathrm{E}_I(\mathbf{u}_\beta)$ "v of type (I)" by "v is the f^n-image of the $(0-1, p-0)$-bispecial factor 0^{t_1-1}".

Lemma 27. *$z^{(n)}\varphi_\beta^n(0^{t_1}1)$ is always a factor of $\mathbf{u}_\beta$.*

Proof. For $n < m$ the statement is trivial, so let $n \geq m$. We have already proved that the factor $\varphi_\beta^n(0wp)$ contains $z^{(n)}$ as a suffix, where w is the same w as in Lemma 18. The proof then follows from the fact that $0wp0^{t_1}1$ is a factor of $\mathbf{u}_\beta$. $\square$

Definition 28. For all $n \in \mathbb{N}$ denote the factor $z^{(n)}\varphi_\beta^n(0)(z^{(n)})^{-1}$ by $w_I^{(n)}$ and the f^n-image of the $(0-1, p-0)$-bispecial factor 0^{t_1-1} by $v_I^{(n)}$.

Proposition 29. *Let $t_1 \geq 2$. Then*

$$\mathrm{E}_I(\mathbf{u}_\beta) = \sup\{\mathrm{ind}(w_I^{(n)}) \mid n \in \mathbb{N}\},$$

and hence

$$\mathrm{E}_I(\mathbf{u}_\beta) = \sup_{n\in\mathbb{N}}\left\{t_1 + \frac{|z^{(n)}| + |\varphi_\beta^n(1)| - 1}{|\varphi_\beta^n(0)|}\right\}.$$

Proof. The proof follows from the simple fact that $w_I^{(n)}$ is a conjugate of $\varphi_\beta^n(0)$ and so $|w_I^{(n)}| = |\varphi_\beta^n(0)|$. □

In [9] we proved that the only β for which $\mathbf{u}_\beta$ has an affine factor complexity is the one with $d_\beta(1) = t_1(0\cdots 0(t_1-1))^\omega$. For these special βs, it is easy to simplify the formula for $\mathrm{E}_I(\mathbf{u}_\beta)$. By induction, one can prove that $\varphi_\beta^n(0)(z^{(n)})^{-1} = 0$, i.e., $|z^{(n)}| = |\varphi_\beta^n(0)| - 1$.

Corollary 30. *If $d_\beta(1) = t_1(0\cdots 0(t_1-1))^\omega$, then*

$$\mathrm{E}_I(\mathbf{u}_\beta) = \sup_{n\in\mathbb{N}} \left\{ t_1 + 1 + \frac{|\varphi_\beta^n(1)| - 2}{|\varphi_\beta^n(0)|} \right\}.$$

6.3. Bispecial Factors of Type (*II*)

We have split the set of all BS factors of $\mathbf{u}_\beta$ into two subsets: BS factors of type (I) and of type (II). From the point of view of computing the critical exponent, we are done with those of type (I). Regarding the type (II), we define an analogue of $\mathrm{E}_I(\mathbf{u}_\beta)$:

Definition 31. We let

$$\mathrm{E}_{II}(\mathbf{u}_\beta) = \sup\{\mathrm{ind}(w) \mid (v,w) \in \mathcal{B}(\mathbf{u}_\beta),\ v \text{ of type } (II)\}.$$

In what follows, we will find a condition under which it holds that $\mathrm{E}_{II}(\mathbf{u}_\beta) < \mathrm{E}_I(\mathbf{u}_\beta)$ and so $\mathrm{E}(\mathbf{u}_\beta) = \mathrm{E}_I(\mathbf{u}_\beta)$.

As a first step we will show that any BS factor of type (II) is the f^n-image of the empty word. According to the definition, a BS factor of type (II) is the f^n-image either of the empty word or of an $(a-c, b-d)$-bispecial factor $0^s, s > 0$, with $p|(a-b)$ and $a \neq 0$. For simplicity, assume that $c \neq m$ and $c \geq a$ (the other cases are similar, but more technical); then $a0^sc = a0^{t_c}c$ is a factor of $\mathbf{u}_\beta$. Consequently, $(a-1)(c-1)$ is a factor as well. If $a-1$ is not zero (and again for simplicity $c-1 \neq m$), then $(a-2)(c-2)$ is a factor. Continuing in the same manner, we get $0(c-a)$ is a factor and the $(a-c, b-d)$-bispecial factor 0^s is the f^a-image of the $(0-(c-a), b'-d')$-bispecial factor ϵ (for certain $b', d' \in \mathcal{A}$). This is the idea of the proof of the following lemma.

Lemma 32. *For any $(a-c, b-d)$-bispecial factor v of type (II) with $p|(a-b)$, there exist n and $b', c', d' \in \mathcal{A}$ such that v is the f^n-image of $(0-c', b'-d')$-bispecial factor ϵ, where, moreover, b' is divisible by p.*

Let v be an f^n-image of $(0-c, b-d)$-bispecial factor ϵ, with $p \mid b$. For $n < m$ we have either $v = \epsilon$ or v is the f^ℓ-image of 0^s for some $0 \leq \ell < n$ (if there exists $\ell > 0$ such that $\varphi_\beta^{n-\ell}(c) = 0^s(c \oplus (n-\ell))$). These cases are not interesting since such BS factors are, in fact, of type (I). The interesting cases are those when $n \geq m$. For such n, the BS factor v begins in $z^{(n)}$ and hence it can only be a power of a LRW of $z^{(n)}$. But, we have already described all LRWs of $z^{(n)}$; in particular, for those of type (A) we know also their maximal power. Thus, given a BS factor of type (II), we look for the shortest LRW of $z^{(n)}$ of type (B) or (C) such that the BS factor v is its power.

Lemma 33. *Let w be a LRW of $z^{(n)}$ of type (B) or (C). Then:*

(i) *the index of w in $\mathbf{u}_\beta$ is less than* 4*,*

(ii) *if, moreover, p does not divide z, then the index is less than* 3*.*

Proof. First assume that $n = m$ (and so $z^{(m)} = 0^t m$) and that w is of type (B). Then we must have that $w0^t m$ is a suffix of $X0^{t_m} mv0^{t_{m+p}} m$ for some $v \in \mathcal{A}^*$ not containing m (see Lemma 18). This factor is always followed by a RRW of $0^t m$ of type (C) which is known to be different from w of type (B). Therefore, the longest power of w which can appear in $\mathbf{u}_\beta$ is at most as long as $0^t mv0^{t_{m+p}} mv0^{t_{m+p}}$. Hence the index of w is less than 3.

If w is a LRW of $0^t m$ of type (C), then its index must be less than 3 as well since we know that LRWs of type (C) are φ_β^p-images of those of type (B) (in the sense of Lemma 18).

Now assume $n > m$, w of type (B), and the case when p does not divide z. In such a case $z^{(n)} = \varphi_\beta^{n-m}(0^t m)$ and so the maximal power of w is at most the factor $\varphi_\beta^{n-m}(0^t mv0^{t_{m+p}} mv0^{t_{m+p}} m)$ without the last letter, where v is as above. Clearly, the index is still less than three. If w is of type (C), we can use the same argument as previously and so prove (ii).

The remaining case is when p divides z. Then $z^{(n)}$ is not just a φ_β-image of $0^t m$ but it is longer. However, if w is of type (B), then it is still true that the maximal power of it is at most as long as $z^{(n)} \varphi_\beta^{n-m}(v0^{t_{m+p}} mv0^{t_{m+p}} m)$ without the last letter. We know that $z^{(n)}$ is a suffix of $z^{(n)} \varphi_\beta^{n-m}(v0^{t_{m+p}} m)$ but since it can be possibly longer than $\varphi_\beta^{n-m}(v0^{t_{m+p}} m)$, the index of w might be greater than 3 (an example of such a situation is given below). However, the index cannot get over 4. The case of w of type (C) can be again brushed off by the argument used before. $\square$

This result gives us the following upper bound on $\mathrm{E}_{II}(\mathbf{u}_\beta)$.

Corollary 34. *The following holds:*

(i) $\mathrm{E}_{II}(\mathbf{u}_\beta) \leq 4$*,*

(ii) *if, moreover, p does not divide z, then* $\mathrm{E}_{II}(\mathbf{u}_\beta) \leq 3$*.*

As we promised, here is an example of the situation, where a LRW of $z^{(n)}$ of type (B) or (C) can have the index strictly greater than 3, in other words that $\mathrm{E}_{II}(\mathbf{u}_\beta) > 3$ can happen.

Example 35. Let $d_\beta(1) = 22(01)^\omega$. The substitution φ_β then reads: $0 \mapsto 001, 1 \mapsto 002, 2 \mapsto 3, 3 \mapsto 02$ and we have $m = 2, p = 2, t = 1, z = 2$, and $0^t m = 02$. There is only one CRW of 02 of type (B), namely the suffix of $\varphi_\beta^2(02) = \cdots 00202 = \cdots 0^{t_m} m 0^{t_{m+p}} m$. Hence, the factor v from the previous proof is the empty word and, since 0020200 is a factor, the maximal power of 02 reads 02020. Now consider $z^{(4)} = 02\varphi_\beta^2(02) = 0200100100202$. If we compute $\varphi_\beta^2(0020200)$, we see that

$$02\varphi_\beta^2(02)\varphi_\beta^2(02)0010010020 = (02001001002)^3 0$$

is a factor and so the index of the LRW of $z^{(4)} = 02001001002$ is $3 + \frac{1}{11}$.

Having divided the set of all BS factors into two disjoint subsets - those of type (I) and of type (II) - it obviously holds

$$\mathrm{E}(\mathbf{u}_\beta) = \max\{\mathrm{E}_I(\mathbf{u}_\beta), \mathrm{E}_{II}(\mathbf{u}_\beta)\}. \tag{12}$$

Since we know that $\mathrm{E}_I(\mathbf{u}_\beta) > t_1$, Theorem 17 is a simple consequence of Corollary 34 and Proposition 29.

The assumptions of Theorem 17 are quite strong and, in fact, we were able to weaken them slightly. More precisely, we managed to identify the cases when $\mathrm{E}_{II}(\mathbf{u}_\beta) < 2$ and so to prove that in such cases $\mathrm{E}(\mathbf{u}_\beta) = \mathrm{E}_I(\mathbf{u}_\beta)$ even if $t_1 = 2$. Unfortunately, the proof of this result is so technical that it becomes almost unreadable. Moreover, there still remain some βs such that we are not able to decide whether $\mathrm{E}(\mathbf{u}_\beta)$ is equal to $\mathrm{E}_I(\mathbf{u}_\beta)$ or to $\mathrm{E}_{II}(\mathbf{u}_\beta)$. In order to show that the latter can happen, we give the following proposition.

Proposition 36. *Let the following conditions be satisfied:* $\varphi_\beta^m(p) = m$, $0p \in \mathcal{L}(\mathbf{u}_\beta)$, $t_1 = 2$, *and* $|\varphi_\beta^n(m)| \geq |\varphi_\beta^n(1)|$, *for* $n = 1, 2, \ldots, m-1$. *Then*

$$\mathrm{E}(\mathbf{u}_\beta) = \mathrm{E}_{II}(\mathbf{u}_\beta).$$

Proof. Obviously, $0p0$ is a factor of $\mathbf{u}_\beta$. Hence, $0^t m 0^{t_{m+p}} m 0^t = mm$ is a factor as well. Furthermore, $z^{(n)}\varphi_\beta^n(p)\varphi_\beta^n(p)(p \oplus n)^{-1}$ is also a factor (it follows from the observation that $\varphi_\beta^n(a)(a \oplus n)^{-1}$ is a prefix of $\varphi_\beta^n(0)$ for every letter a) and it is a power of $z^{(n)}\varphi_\beta^n(p)(z^{(n)})^{-1}$. This is a conjugate of $\varphi_\beta^n(p)$ and so of the same length. Clearly, $z^{(n)}\varphi_\beta^n(p)(z^{(n)})^{-1}$ is a LRW of $z^{(n)}$ of type (B) and so

$$\mathrm{E}_{II}(\mathbf{u}_\beta) \geq \sup_{n \geq m}\{\mathrm{ind}(z^{(n)}\varphi_\beta^n(p)(z^{(n)})^{-1})\}.$$

We now prove that $\mathrm{ind}(w_I^{(n)}) \leq \mathrm{ind}(z^{(n+m)}\varphi_\beta^{n+m}(p)(z^{(n+m)})^{-1})$ for all $n \in \mathbb{N}$, i.e., that

$$2 + \frac{|z^{(n+m)}| - 1}{|\varphi_\beta^{n+m}(p)|} \geq t_1 + \frac{|z^{(n)}| + |\varphi_\beta^n(1)| - 1}{|\varphi_\beta^n(0)|}.$$

The assumptions and the fact that $z^{(n+m)} = z^{(n)}\varphi_\beta^n(m)$ yield the inequality

$$\frac{|z^{(n)}| + |\varphi_\beta^n(m)| - 1}{|\varphi_\beta^n(m)|} \geq \frac{|z^{(n)}| + |\varphi_\beta^n(1)| - 1}{|\varphi_\beta^n(0)|}$$

equivalent to

$$(|z^{(n)}| - 1)(|\varphi_\beta^n(0)| - |\varphi_\beta^n(m)|) + |\varphi_\beta^n(m)|(|\varphi_\beta^n(0)| - |\varphi_\beta^n(1)|) \geq 0.$$

And this is always true since for $n \geq m$ all members are nonnegative and for $n < m$ $|z^{(n)}| = 0$ and

$$|\varphi_\beta^n(0)| - |\varphi_\beta^n(m)| \leq |\varphi_\beta^n(0)| - |\varphi_\beta^n(1)|.$$

□

Example 37. Let $d_\beta(1) = 21(1200)^\omega$. The substitution φ_β then reads: $0 \mapsto 001, 1 \mapsto 02, 2 \mapsto 03, 3 \mapsto 004, 4 \mapsto 5, 5 \mapsto 2$ and we have $m = 2, p = 4, t = 0, z = 2$, and $0^t m = 2$. It holds that $\varphi_\beta^m(p) = \varphi_\beta^2(4) = m = 2$ and clearly $0p = 04$ is a factor of $\mathbf{u}_\beta$. We will show that $\mathrm{E}_{II}(\mathbf{u}_\beta) = 3$. Indeed, $0^t m\varphi_\beta^m(p)(p \oplus m)^{-1} = 2$ is a BS factor with left extensions 0 and 2 and right extensions 0 and 2 as well. The BS factors $\varphi_\beta^{n-2}(22)(2 \oplus (n-2))^{-1}$ are then powers of $\varphi_\beta^{n-2}(2)$. We have proved that $\mathrm{E}_{II}(\mathbf{u}_\beta) \geq 3$; the equality holds by Corollary 34.

What about $\mathrm{E}_I(\mathbf{u}_\beta)$ for this particular β? After some simple computation we get for $n \geq 2$ $w_I^{(n)} = \varphi_\beta^{n-2}(20010010)$ and $v_I^{(n)} = \varphi_\beta^{n-2}(20010010200103)(3 \oplus (n-2))^{-1}$. Hence

$$\mathrm{E}_I = \sup_{n \geq 2} \left\{ 3 - \frac{|\varphi_\beta^{n-2}(010)| - |\varphi_\beta^{n-2}(3)| + 1}{|\varphi_\beta^{n-2}(20010010)|} \right\} < 3.$$

7. The Ultimate Critical Exponent

In this section, we will find the ultimate critical exponent of $\mathbf{u}_\beta$ under the assumptions of Theorem 17. Using the formula for $\mathrm{E}(\mathbf{u}_\beta)$ – proven in the previous section – our task is to calculate the following limit

$$\mathrm{E}^*(\mathbf{u}_\beta) = \lim_{n \to \infty} \left(t_1 + \tfrac{|z^{(n)}| + |\varphi_\beta^n(1)| - 1}{|\varphi_\beta^n(0)|} \right).$$

7.1. Auxiliary Limits

In order to be able to compute the desired limit, we will need some technical results. For calculation of the lengths of $z^{(n)}$, $\varphi_\beta^n(0)$, and $\varphi_\beta^n(1)$, we will use the notions of the Parikh vector $\Psi(w)$ and the incidence matrix M of a primitive substitution φ. Recall that $\vec{e}$ stands for a column vector whose entries are all equal to one. As a simple consequence of (3), we get the following lemma.

Lemma 38. *For all $n \in \mathbb{N}$ and $w \in \mathcal{A}^*$ we have*

$$|\varphi^n(w)| = \Psi(w)M^n\vec{e}.$$

Since the matrix M is primitive, there exists a simple dominant eigenvalue $\beta \in \mathbb{R}$ such that any other eigenvalue is in modulus less than β. Denote $\vec{x}$ and $\vec{y}$ a left and right eigenvector for β (they can be chosen nonnegative), i.e.,

$$\vec{x}M = \beta\vec{x} \quad \text{and} \quad M\vec{y} = \beta\vec{y}.$$

Let $\mathcal{J}$ be the Jordan canonical form of M such that

$$M = P\mathcal{J}P^{-1} = P\begin{pmatrix} \beta & \vec{0} \\ \vec{0}^{\,T} & \mathcal{J}_{22} \end{pmatrix} P^{-1}, \tag{13}$$

where $\vec{0} = (0, \dots, 0)$ is a zero vector of the corresponding size and $\mathcal{J}_{22}$ contains the Jordan blocks corresponding to the eigenvalues different from β. With this notation, we see that P can be chosen so that the first column of P is $\vec{y}$ and the first row of P^{-1} is $\vec{x}$, but with the condition that for the eigenvectors in question we have $\vec{x}\vec{y} = 1$. Moreover, for any Jordan block in $\mathbb{R}^{d\times d}$ and an exponent $n \in \mathbb{N}$, one can prove by induction that

$$\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & \lambda^{n-1}\binom{n}{1} & \lambda^{n-2}\binom{n}{2} & \cdots & \lambda^{n-d+1}\binom{n}{d-1} \\ 0 & \lambda^n & \lambda^{n-1}\binom{n}{1} & \cdots & \lambda^{n-d+2}\binom{n}{d-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix}. \tag{14}$$

All these facts allow us to prove easily the following lemma.

Lemma 39. *Let M be a primitive nonnegative matrix M with the dominant eigenvalue β and P the matrix defined by* (13). *Then*

(i)
$$\lim_{n\to\infty} \frac{1}{\beta^n} M^n = P\begin{pmatrix} 1 & \vec{0} \\ \vec{0}^{\,T} & \Theta \end{pmatrix} P^{-1},$$

(ii)
$$\lim_{n\to\infty} \frac{1}{\beta^{s+nr}} \sum_{i=0}^{n-1} M^{s+ri} = \frac{1}{\beta^r - 1} P\begin{pmatrix} 1 & \vec{0} \\ \vec{0}^{\,T} & \Theta \end{pmatrix} P^{-1},$$

where Θ is a zero matrix of the corresponding size and $s, r \in \mathbb{N}$, r positive.

Proof. To compute the second limit, we consider the Jordan form (13) and (14). We get

$$\lim_{n\to\infty} \frac{1}{\beta^{s+nr}} \sum_{i=0}^{n-1} \mathcal{J}_{22}^{s+ri} = \Theta.$$

The proof then follows by the simple fact that

$$\frac{\sum_{i=0}^{n-1} \beta^{s+ri}}{\beta^{s+rn}} \longrightarrow \frac{1}{\beta^r - 1} \quad \text{as } n \to \infty.$$

The value of the first limit is obvious. □

In both cases, we got a very similar expression on the right-hand side. It can be even more simplified.

Lemma 40. *Let M be a primitive matrix and let $\vec{y}$ be a right eigenvector corresponding to the dominant eigenvalue β. Then*

$$P \begin{pmatrix} 1 & \vec{0}^{\,T} \\ \vec{0} & \Theta \end{pmatrix} P^{-1} \vec{e} = C_M \vec{y},$$

where P is given by (13) and C_M is a positive constant depending on the choice of $\vec{y}$.

Proof. As we said before, the first row of P^{-1} is the left eigenvector of the dominant eigenvalue such that $\vec{x}\vec{y} = 1$. Hence, we get

$$\begin{pmatrix} 1 & \vec{0}^{\,T} \\ \vec{0} & \Theta \end{pmatrix} P^{-1} \vec{e} = \begin{pmatrix} \vec{x} \\ \Theta' \end{pmatrix} \vec{e} = \vec{x}\vec{e} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}.$$

This, along with that the first column of P is $\vec{y}$, conclude the proof. Moreover, we get $C_M = \vec{x}\vec{e}$. □

7.2. The Case of $\mathbf{u}_\beta$

The incidence matrix M of the non-simple Parry substitution φ_β, having $\mathbf{u}_\beta$ as its fixed point, is defined in (8). As we have already mentioned, the components of its right eigenvector $\vec{y}_\beta$ corresponding to the eigenvalue β represent distances between the consecutive β-integers (see (6)), i.e.,

$$\vec{y}_\beta = (\triangle_0, \triangle_1, \triangle_2, \ldots, \triangle_{m+p-1})^T.$$

We have now at our disposal all we need to compute the limit equal to $E^*(\mathbf{u}_\beta)$. In order to simplify the notation, let us omit the index β in φ_β and $\vec{y}_\beta$. Let us start with calculation of the relevant limits.

Lemma 41. *We have*

(i)
$$\lim_{n\to\infty}\frac{|\varphi^n(1)|}{|\varphi^n(0)|}=\triangle_1=\beta-t_1,$$

(ii)
$$\lim_{n\to\infty}\frac{|z^{(n)}|}{|\varphi^n(0)|}=\begin{cases}\frac{1}{\beta^m-1}(t+\triangle_m) & \textit{if } p \textit{ divides } z,\\ \frac{1}{\beta^m}(t+\triangle_m) & \textit{otherwise.}\end{cases}$$

Proof. By Lemma 38 we have
$$\frac{|\varphi^n(1)|}{|\varphi^n(0)|}=\frac{\frac{1}{\beta^n}\Psi(1)M^n\vec{e}}{\frac{1}{\beta^n}\Psi(0)M^n\vec{e}}.$$
Due to Lemmas 39 and 40, this tends to
$$\frac{C_M\Psi(1)\vec{y}}{C_M\Psi(0)\vec{y}}=\triangle_1=\beta-t_1$$
as n goes to infinity.

We divide the proof of (ii) into two parts. First assume that p does not divide z. Then, using the same techniques as for (i), we get
$$\begin{aligned}\lim_{n\to\infty}\frac{|z^{(n)}|}{|\varphi^n(0)|} &= \lim_{n\to\infty}\frac{t|\varphi^{n-m}(0)|+|\varphi^{n-m}(m)|}{|\varphi^n(0)|}\\ &= \frac{1}{\beta^m}\frac{(t\Psi(0)+\Psi(m))\,\vec{y}}{\Psi(0)\vec{y}}=\frac{1}{\beta^m}(t+\triangle_m).\end{aligned}$$

Second, let p divide z and let $0\le k<m$. Then by Definition 15 we have
$$|z^{(nm+k)}|=t\sum_{j=0}^{n-1}|\varphi^{k+jm}(0)|+\sum_{j=0}^{n-1}|\varphi^{k+jm}(m)|.$$

Now, by Lemma 39 we get
$$\begin{aligned}\lim_{n\to\infty}\frac{|z^{(nm+k)}|}{|\varphi^{nm+k}(0)|} &= \lim_{n\to\infty}\frac{(t\Psi(0)+\Psi(m))\,\frac{1}{\beta^{nm+k}}\sum_{i=0}^{n-1}M^{k+mi}\vec{e}}{\Psi(0)\frac{1}{\beta^{nm+k}}M^{mn+k}\vec{e}}\\ &= \frac{1}{\beta^m-1}\frac{(t\Psi(0)+\Psi(m))\,\vec{y}}{\Psi(0)\vec{y}}=\frac{1}{\beta^m-1}(t+\triangle_m).\end{aligned}$$

Since the resulting expression does not depend on k, the proof is finished. □

Using the obtained limit values, we get the statement of Theorem 17 concerning the ultimate critical exponent:
$$\mathrm{E}^*(\mathbf{u}_\beta)=\begin{cases}\beta+\frac{1}{\beta^m-1}(t+\triangle_m) & \text{if } p \text{ divides } z,\\ \beta+\frac{1}{\beta^m}(t+\triangle_m) & \text{otherwise.}\end{cases}$$

8. Comments

- Our method for calculation of the critical exponent and the ultimate critical exponent can be used for any primitive substitution φ. It can be shown for any such substitution that all BS factors arise when applying the map f from Definition 11 repeatedly on a finite number of initial BS factors.

- The sequences of BS factors we have studied are, in terms of Lemma 2, maximal powers of some factor w minus the prefix w. Krieger in [10] considered directly sequences of maximal powers called π-sequences. There is a strong relation between these two sequences, it holds (omitting some technicalities) that $(v_i)_{i\geq 0}$ is a sequence of BS factors which are powers of w_i and $(v_i, w_i) \in \mathcal{B}(\mathbf{u})$ if and only if $(w_i v_i)_{i\geq 0}$ is a π-sequence. However, she studied the general case where she needed "only" to know that there are only a finite number of these π-sequences. The method used in this paper is a feasible way of how to identify all the π-sequences for a particular substitution.

- In [2] the values of the critical exponent for quadratic non-simple Parry numbers are studied. In this special case, the Rényi expansion of unity $d_\beta(1) = t_1 t_2^\omega$, hence the period length $p = 1$ and p divides then z automatically. In this case, we are able to decide when $E^*(\mathbf{u}_\beta) = E(\mathbf{u}_\beta)$ [2, Theorem 5.3].

- The exact value of the constant C_M from Lemma 40 was not necessary for calculation of our limits. However, its value is computed in [1] in case of canonical substitutions associated with simple and non-simple Parry numbers:

$$\begin{aligned} C_M &= \tfrac{\beta-1}{\beta^m(\beta^p-1)} P'(\beta) && \text{for non-simple Parry numbers,} \\ C_M &= \tfrac{\beta-1}{(\beta^m-1)} P'(\beta) && \text{for simple Parry numbers,} \end{aligned}$$

  where $P(x)$ is the Parry polynomial of β defined ibidem.

- An essential part of this paper is devoted to BS factors. This notion plays an important role in the study of many characteristics of infinite words, e.g., factor complexity, palindromic complexity, return words, abelian complexity, etc.

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