A PARTITION OF THE NON-NEGATIVE INTEGERS, WITH APPLICATIONS

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Abstract

We describe a particular partition of the non-negative integers which consists of infinitely many translates of an infinite set. This partition is used to show that a certain van der Waerden-like theorem has no simple canonical version. The partition is also used to give a lower bound for one of the classical van der Waerden functions, namely $w(3; m)$,the smallest positive integer such that every m-coloring of $[1, w(3; m)]$ produces a monochromatic 3-term arithmetic progression. Several open questions are mentioned.

1 Introduction

Let S denote the set of all distinct sums of odd powers of 2, including 0 as the empty sum, and let T denote the set of all distinct sums of even powers of 2, including 0 as the empty sum. Then every non-negative integer can be written uniquely in the form $s+t$, where $s \in S$ and $t \in T$. Thus $\{s+T : s \in S\}$ is a partition of $\omega = \{0, 1, 2, ...\}$ into translates of T.

It is more convenient to describe this partition as a coloring f of ω. Thus for each $n \in \omega$, we write $n = s + t$, $s \in S$, $t \in T$, and define $f(n) = s$. In other words, if $n = \sum_{i \text{ odd}} 2^i$ $+ \sum_{i \text{ even}} 2^i$, then $f(n) = \sum_{i \text{ odd}} 2^i$. For this coloring f, the set of colors is S, and for each $s \in S$, f is constant on the "color class" $s + T$.

2 A van der Waerden-like theorem, and its canonical version

We need the following definition.

Definition 1. If $A = \{a_1 < a_2 < \cdots < a_n\} \subset \omega = \{0, 1, 2, ...\}$, $n \geq 2$, the *gap size* of A is $gs(A) = \max\{a_{j+1} - a_j : 1 \leq j \leq n-1\}$. If $|A| = 1$, $gs(A) = 1$.

Theorem 1. If ω is finitely colored, there exist a fixed $d \geq 1$ (d depends only on the coloring) and arbitrarily large (finite) monochromatic sets A with $gs(A) = d$.

This fact first appeared in [3]. Various applications appear in [4, 6, 11, 13]. Theorem 1 is somewhat similar in form to van der Waerden's theorem on arithmetic progressions [15]. (Van der Waerden's theorem says that for every k, every finite coloring of the positive integers produces a monochromatic k-term arithmetic progression.) However, Theorem 1 differs in a number of ways: Van der Waerden's theorem does not imply Theorem 1, since the d in the conclusion of Theorem 1 is independent of the size of the monochromatic sets A. Beck [1] showed the existence of a 2-coloring of ω such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < 2\log d$. Hence the presence of large monochromatic arithmetic progressions, which is guaranteed by van der Waerden's theorem, is not enough to imply Theorem 1. Somewhat earlier, Justin [10] found an explicit coloring such that if A is any monochromatic arithmetic progression with common difference d, then $|A| < h(d)$; in his example, the coloring is explicit but the function $h(d)$ is not. Theorem 1 (which has a simple proof) does not imply van der Waerden's theorem in a simple way. (In Chapter 14 of [8], Hindman and Strauss give a proof that Fact 1 does in fact imply van der Waerden's theorem - and at this point in their book, the proof *does* seem simple - however, a fair amount of machinery has been developed by this point.) Theorem 1 does not have a density version corresponding to Szemerédi's theorem [14]. That is, there exists a set $X \subset \omega$ with positive upper density for which there do *not* exist a fixed $d \geq 1$ and arbitrarily large sets $A = \{a_1 < a_2 < \cdots < a_n\} \subset X$ with $\max\{a_{j+1} - a_j : 1 \leq j \leq n-1\} = d$. For an example of such a set X, see [2]. Finally, no "canonical version" of this result is known. The Erdős-Graham canonical version of van der Waerden's theorem ([7]) states that if $g : \omega \to \omega$ is an arbitrary coloring of ω (using finitely many or infinitely many colors) then there exist arbitrarily large arithmetic progressions A such that either g is constant on A, i. e. $|g(A)| = 1$, or g is one-to-one on A, i. e. $|g(A)| = |A|$. We show that there is no such canonical version of Theorem 1. This is Corollary 1 below. A very brief sketch of an outline of a proof of the following result has appeared in [5]. It seems worthwhile to fill in some of the missing details.

Theorem 2. For every $A \subset \omega$ (with f as described in the introduction),

$$\frac{1}{4}\sqrt{|A|/gs(A)} < |f(A)| < 4\sqrt{|A|gs(A)}.$$

Corollary 1. For the coloring f above, there do not exist a fixed d and arbitrarily large sets A with $gs(A) = d$ on which f is either constant or 1-1.

Proof of Corollary 1. If $16gs(A) \leq |A|$, then by Theorem 2, $1 < |f(A)| < |A|$. To prove Theorem 2, we need the following definition.

Definition 2. For $k \geq 0$, an *aligned block* of size 4^k is a set of 4^k consecutive non-negative integers whose smallest element is $m4^k$, for some $m \geq 0$.

Proof of Theorem 2. Note that the first aligned block of size 4^k, namely $[0, 4^k - 1] = [0, 2^{2k} - 1]$, is in 1-1 correspondence with the set of all binary sequences of length $2k$. ¿From this we see (by the definition of f) that for $n \in [0, 2^{2k}-1]$, there are 2^k possible values of $f(n)$, and each value occurs exactly 2^k times. It is easy to see (using the definition of f) that the same is true for any aligned block $[m4^k, m4^k+4^k-1]$. We express this more simply by saying that "*each aligned block of size* 4^k *has* 2^k *colors, each appearing exactly* 2^k *times.*" Now we can establish the upper bound in Theorem 2. Let $A = \{a_0 < a_1 < a_2 < \cdots < a_n\} \subset \omega$. Then $a_n \le a_0 + n \cdot gs(A) = a_0 + (|A|-1)gs(A)$, or

$$a_n - a_0 < |A|gs(A).$$

Choose s minimal so that A is contained in the union of two adjacent aligned blocks of size 4^s. (Two blocks are necessary in case A contains both $m4^s-1$ and $m4^s$ for some m.) Then

$$4^{s-1} < a_n - a_0.$$

Since each aligned block of size 4^s has 2^s colors,

$$|f(A)| \le 2 \cdot 2^s.$$

Putting these three inequalities together gives

$$|f(A)| < 4\sqrt{|A|gs(A)}.$$

Next, we establish the lower bound for $|f(A)|$, which requires a bit more care. We will use the following Lemma.

Lemma 1. For each $k \ge 0$, any two aligned blocks of size 4^k (consecutive or not) are either colored identically, or have no color in common.

Proof of Lemma 1. Consider the aligned blocks $[p4^k, p4^k+4^k-1]$ and $[q4^k, q4^k+4^k-1]$. By the definition of f (and since 4^k is an even power of 2), $f(p4^k) = f(p)4^k$, so that $f(p4^k) = f(q4^k)$ if and only if $f(p) = f(q)$. Also, for $0 \le j \le 4^k-1$, $f(p4^k+j) = f(p4^k)+f(j)$. This last equality obviously holds if $p=0$, and for $p>0$ it holds since then each power of 2 which occurs in j is less than each power of 2 which occurs in $p4^k$. Thus the blocks $[p4^k, p4^k+4^k-1]$ and $[q4^k, q4^k+4^k-1]$ are colored identically if $f(p) = f(q)$, and have no color in common if $f(p) \ne f(q)$. Proceeding with the lower bound in Theorem 2, we note that for $k \ge 1$, the colors of any aligned block of size 4^k have the form $UUVV$, where U and V are blocks (of colors) of size 4^{k-1}. To see this, let the given aligned block be $[m4^{k+1}, m4^{k+1}+4^{k+1}-1]$, where now $k \ge 0$. Divide this block into four consecutive aligned blocks of size 4^k, namely those blocks of size 4^k whose first elements are respectively $m4^{k+1}$, $m4^{k+1}+4^k$, $m4^{k+1}+2\cdot4^k$, $m4^{k+1}+2\cdot4^k+4^k$. Since $f(m4^{k+1}) = f(m4^{k+1}+4^k) \ne f(m4^{k+1}+2\cdot4^k) = f(m4^{k+1}+2\cdot4^k+4^k)$, Lemma 1 and its proof imply that the colors of these four blocks have the forms U, U, V, V. Next, we note that any block of size 4^k, aligned or not, contains *at least* 2^kcolors. For let A be any block of size 4^k. Let the first element of A lie in the aligned block S of size 4^k, and let T be the aligned block of size 4^k which immediately succeeds S. If S and T are colored identically, then the elements of $f(A)$ are just a cyclic permutation of the elements of $f(S)$, and hence the block A contains exactly 2^k colors. By Lemma 1, the remaining case is when S, T have no color in common. In this case, by the preceding paragraph,

$f(S)f(T) = UUVVXXYY$, where no two of U, V, X, Y have a color in common, and U, V, X, Y are of size 4^{k-1}. Then $f(A)$, which has size 4^k, contains either UV or VX or XY, and so has at least $2^{k-1} + 2^{k-1} = 2^k$ colors. Finally, we note that for $s \geq 1$, $k \geq 1$, every set of 4^s consecutive aligned blocks of size 4^k contains at least 2^s blocks of size 4^k, no two of which have a common color. This follows from the fact that these 4^s blocks have the form $[p4^k, p4^k + 4^k - 1]$, $t \leq p \leq t + 4^s - 1$, for some t. The block $f([t, t + 4^s - 1])$ has at least 2^s colors, by the preceding paragraph. If $f(p) \neq f(q)$, where $t \leq p < q \leq t + 4^s - 1$, then $f(p4^k) \neq f(q4^k)$, so by Lemma 1 the two blocks $[p4^k, p4^k + 4^k - 1]$ and $[q4^k, q4^k + 4^k - 1]$ have no color in common. Now let $A \subset \omega$ be given. Choose k so that $4^{k-1} \leq gs(A) < 4^k$. Choose t minimal so that A is contained in the union of t consecutive aligned blocks of size 4^k. Then A meets each of these blocks (by the choice of k), and

$$|A| \leq t4^k.$$

Choose s so that $4^s \leq t < 4^{s+1}$. Then among the t consecutive aligned blocks of size 4^k are at least 2^s blocks of size 4^k, no two of which have a color in common. Since each of the t blocks meets A, we have

$$2^s \leq |f(A)|.$$

Thus $|A| \leq t4^k < 4 \cdot 4^s \cdot 4 \cdot 4^{k-1} \leq 4|f(A)|^2 \cdot 4 \cdot gs(A)$, so $\frac{1}{4}\sqrt{|A|/gs(A)} < |f(A)|$.

3 A bound for a van der Waerden function

Definition 3. For $m \geq 1$, let $w(3; m)$ denote the smallest positive integer such that every m-coloring of $[1, w(3; m)]$ produces a monochromatic 3-term arithmetic progression.

Theorem 3. For all $m \geq 1$, $w(3; m) > \frac{1}{2}m^2$.

Proof. For $k \geq 1$, the coloring f described in the introduction colors the interval $[0, 2^{2k+1} - 1]$ with 2^k colors. The colors are the sums (including 0 as the empty sum) of distinct elements of the set $\{2^1, 2^3, 2^5, ..., 2^{2k-1}\}$. The color classes are subsets of the translates (by the 2^k colors) of the set S_k of sums (including 0 as the empty sum) of distinct elements of the set $\{2^0, 2^2, 2^4, ..., 2^{2k}\} = \{4^0, 4^1, 4^2, ..., 4^k\}$. It is easy to see that S_k contains no 3-term arithmetic progression. Hence, with respect to the coloring f, there is no monochromatic 3-term arithmetic progression in $[0, 2^{2k+1} - 1]$. The coloring f shows that for $k \geq 1$, $w(3; 2^k) > 2^{2k+1}$. For a general m, choose k so that $2^k \leq m < 2^{k+1}$. Then $w(3; m) \geq w(3; 2^k) > 2^{2k+1} = \frac{1}{2}2^{2k+2} > \frac{1}{2}m^2$.

4 Remarks

1. The lower bound in Theorem 3 is not the best possible. Indeed, in the standard reference Ramsey Theory (by R. L. Graham, B. L. Rothschild, and J. H. Spencer, 2nd

edition, 1990, John Wiley & Sons, New York), the authors show that for some positive constant $c, w(3;m) > m^(clogm)$.

2. It would be nice to be able to say *something* about general colorings along the lines of Theorem 1. Perhaps the following is true: If $g : \omega \to \omega$ is an arbitrary coloring of ω, then there exist a fixed $d \geq 1$ and arbitrarily large (finite) sets A with $gs(A) = d$ such that either (a) at most $\sqrt{|A|}$ distinct colors appear in $g|_A$; or (b) each color appears in $g|_A$ at most $\sqrt{|A|}$ times. Note that for the particular coloring f, if we take $d = 1$, and let $A = [0, 4^k - 1]$, then exactly $\sqrt{|A|}$ distinct colors appear in $f|_A$, and each color appears in $f|_A$ exactly $\sqrt{|A|}$ times.

3. We have used a particular partition of ω. We would get another partition of ω (into infinitely many translates of an infinite set) by replacing the odds and evens by arbitrary A and B, where $\{A, B\}$ is any partition of $\{1, 2, 3, ...\}$ into two infinite sets. Perhaps it's possible to describe *all* of the partitions of ω into infinitely many translates of an infinite set.

5 References

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