



THE R^{TH} SMALLEST PART SIZE OF A RANDOM INTEGER
PARTITION

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Abstract

We study the size of the r th smallest part and the r th smallest distinct part in a random integer partition. This extends the research on partitions with no small parts of Nicolas and Sárközy.

– To the memory of Nicolaas Govert (Dick) de Bruijn

1. Introduction

Combinatorial objects can be decomposed into simpler objects called “prime,” “irreducible,” or “connected” components. This is a combinatorial analogue of the fact that integers decompose into products of primes. For example, permutations decompose into cycles, integer partitions into parts, polynomials into irreducible factors, and graphs into connected components.

The distribution of the largest and smallest components in combinatorial structures has been studied for several objects. The distribution of the *largest* component objects for several combinatorial problems has been studied, for example, by Stepanov [19] (see also Gourdon [12]). Gourdon expressed his results in terms of a generalization of the Dickman function [6]. This function underlies the study of numbers smaller than or equal to n with no primes larger than m ; see also [3, 20].

The distributions of the *smallest* components for several combinatorial objects including permutations and polynomials over finite fields have also been related to number theory [16]. In this case, the results are expressed in terms of a generalization of the Buchstab function [5]; this function underlies the study of numbers smaller than or equal to n with no primes smaller than m ; see also [2, 20].

In this paper we focus on integer partitions and study the size of their smallest parts. Some of the main contributions to the study of random integer partitions are due to Fristedt [11] and to Freiman and Pitman [10]. For the concrete problem of smallest parts that we focus on here, previous results are in [7, 15]. Our main results are asymptotic probability estimates for the size of the r th smallest part (Theorem 1) and the r th smallest distinct part (Theorem 2).

2. Integer Partitions with Restricted Parts

In this paper we consider integer partitions. In particular, we are interested in integer partitions where some parts appear a restricted number of times. For example, partitions with no parts smaller than m have a restricted pattern where parts $1, 2, \dots, m - 1$ cannot appear.

In general, let $A = \{a_1, a_2, \dots\}$ be a set of positive integers. We let $p_A(n)$ be the number of solutions of

$$n = l_1 a_1 + l_2 a_2 + \dots + l_m a_m, \quad l_i \geq 0,$$

where there are $l_1 + \dots + l_m$ components in this partition.

It is well-known that

$$\sum_{n \geq 0} p_A(n) x^n = \prod_{i \geq 0} \frac{1}{1 - x^{a_i}} = G(x).$$

Moreover if $q_A(n)$ denotes the number of partitions of n with distinct parts then

$$\sum_{n \geq 0} q_A(n) x^n = \prod_{i \geq 0} (1 + x^{a_i}).$$

The saddle point technique [4, 9] to estimate $p_A(n)$ starts with an application of Cauchy’s theorem stating

$$p_A(n) = \frac{1}{2\pi i} \int_C \frac{G(x)}{x^{n+1}} dx$$

where C is any oriented counterclockwise contour encircling the origin. One chooses the contour to be the circle $|x| = e^{-\sigma}$, where $\sigma = \sigma(n)$ is defined as the solution of

$$\frac{d(x^{-n-1}G(x))}{dx} = 0,$$

or

$$n = \sum_{j \geq 1} \frac{a_j}{e^{\sigma a_j} - 1}.$$

For $q_A(n)$ the analogous procedure gives

$$n = \sum_{j \geq 1} \frac{a_j}{e^{\sigma a_j} + 1}.$$

The asymptotic behaviour of $p_A(n)$ has been studied by Richmond [18] under the following assumptions: (1) if any finite subset of A is deleted, then the remaining sequence has gcd equal to one, and (2) the following limit exists

$$\lim_{j \rightarrow \infty} \frac{\ln a_j}{\ln j}.$$

Let

$$A_2(n) = A_2 = \sum_{j \geq 1} \frac{a_j^2 e^{\sigma a_j}}{(e^{\sigma a_j} - 1)^2}.$$

The following result is contained in Theorem 1.1 of [18]

$$p_A(n) = (2\pi A_2)^{-1/2} \exp\left(\sigma n - \sum_{j=1}^{\infty} (\ln(1 - e^{-\sigma a_j}))\right) (1 + O(\sigma)).$$

In the following we use

$$S = \left[\prod_{j \in J} j^{S(j)} \right]$$

to denote a pattern and $p_S(n)$ to denote the number of partitions of n which have exactly $S(j)$ parts of size j for each $j \in J$ (each $j \notin J$ may appear any number of times). For a pattern S satisfying

$$\sum_{j \in J} j = o(n^{1/2}), \quad |J| = o(n^{1/4}) \quad \text{and} \quad \sum_{j \in J} j S(j) = o(n^{1/2}), \quad (1)$$

it was shown in [8] that

$$p_S(n) \sim p(n) \prod_{j \in J} \frac{\pi^j}{\sqrt{6n}}. \quad (2)$$

We are interested in the r th smallest part size of a random partition. In this case the relevant sequence is $A = \{m, m + 1, \dots\}$; that is, partitions with no parts of size smaller than m . Dixmier and Nicolas [7] and Nicolas and Sárközy [15] defined $r(n, m)$ to be the number of partitions of n such that each part is bigger than or equal to m . They obtain an asymptotic formula for $r(n, m)$, where $1 \leq m \leq c_1 n / \log^{7k} n$, for any $k \geq 3$. When $m = o(n^{1/3})$, their formula simplifies to

$$r(n, m) \sim p(n) \left(\frac{C}{2\sqrt{n}}\right)^{m-1} (m-1)! \exp\left(-\frac{1}{4} \left(2C + \frac{1}{2C}\right) \frac{m^2}{\sqrt{n}}\right), \quad (3)$$

where $C = \pi\sqrt{2/3}$, and we have the well-known [1] estimate for the total number of partitions

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp(C\sqrt{n}).$$

(We observe that the formula for $r(n, m)$ on page 232 of [15] has a missing factor 2^{-m+1} that is correctly included in [7].)

We note that the formula in Equation (3) coincides with Nicolas and Sárközy’s formula when $m = o(n^{1/4})$. Indeed in this case, their formula simplifies to

$$r(n, m) \sim p(n) \left(\frac{C}{2\sqrt{n}}\right)^{m-1} (m-1)!. \quad (4)$$

It seems possible to relax the range for the pattern S ; however, as in [15], the asymptotic expression is more complicated and will contain a parameter which is defined by the saddle point equation.

3. The Size of the r^{th} Smallest Part

The result of Dixmier and Nicolas [7] and of Nicolas and Sárközy [15] shows that the probability that the smallest part of a random partition of n has size at least m , when $m = o(n^{1/3})$, is given by

$$\frac{r(n, m)}{p(n)} \sim \left(\frac{C}{2\sqrt{n}}\right)^{m-1} (m-1)! \exp\left(-\frac{1}{4} \left(2C + \frac{1}{2C}\right) \frac{m^2}{\sqrt{n}}\right).$$

We can use Equation (3) to derive the probability that the r th smallest part of a random partition of n has size at least m .

In the following, $p^{[r]}(n, m)$ is the number of partitions of n such that the size of its r th smallest part is at least m . Let $X_n^{[r]}$ denote the size of the r th smallest part in a random partition of size n .

Theorem 1. *The probability that the r th smallest part of a random partition of n has size at least $m = o(n^{1/3})$ satisfies*

$$\begin{aligned}
 P(X_n^{[r]} > m) &= \frac{p^{[r]}(n, m)}{p(n)} & (5) \\
 &\sim \binom{m+r-2}{r-1} (m-1)! \left(\frac{C}{2\sqrt{n}}\right)^{m-1} \exp\left(-\frac{1}{4}\left(2C + \frac{1}{2C}\right)\frac{m^2}{\sqrt{n}}\right).
 \end{aligned}$$

Proof. First we observe that $p(n-j) \sim p(n)$ for $0 \leq j < m$, where $m = o(n^{1/3})$. Considering the relation between the second smallest part and the first smallest part leads to

$$\begin{aligned}
 p^{[2]}(n, m) &= p^{[1]}(n, m) + \sum_{j=1}^{m-1} p^{[1]}(n-j, m) \\
 &\sim \sum_{j=0}^{m-1} p(n-j) \left(\frac{C}{2\sqrt{n-j}}\right)^{m-1} (m-1)! \exp\left(-\frac{1}{4}\left(2C + \frac{1}{2C}\right)\frac{m^2}{\sqrt{n-j}}\right) \\
 &\sim p(n) \left(\frac{C}{2\sqrt{n}}\right)^{m-1} (m-1)! \exp\left(-\frac{(2C + \frac{1}{2C})m^2}{4\sqrt{n}}\right) \sum_{j=0}^{m-1} \left(\frac{1}{\sqrt{1-j/n}}\right)^{m-1} \\
 &\sim mp^{[1]}(n, m),
 \end{aligned}$$

where we used the fact that in the range $0 \leq j \leq m = o(\sqrt{n})$,

$$\left(\frac{1}{\sqrt{1-j/n}}\right)^{m-1} \sim 1.$$

The above argument can be extended to general r . Let $p_r(n, m)$ be the number of partitions of n containing exactly $r-1$ parts (allowing repetition) in $\{1, 2, \dots, m-1\}$. Then we have $p^{[r]}(n, m) = p^{[r-1]}(n, m) + p_r(n, m)$.

Let $S(1), \dots, S(m-1)$ be a sequence of non-negative integers satisfying $S(1) + S(2) + \dots + S(m-1) = r-1$. There are $\binom{m+r-3}{r-1}$ such sequences. We have, provided that $\sum_j jS(j) = o(\sqrt{n})$,

$$p_r(n, m) = \sum_S p^{[1]}(n - \sum_j jS(j), m) \sim \binom{m+r-3}{r-1} p^{[1]}(n, m).$$

The condition $\sum_j jS(j) = o(\sqrt{n})$ clearly holds when r is a constant and m is in the range of the theorem. Indeed, r can be a function of n such that $r = O(\log n)$.

Now using induction on r , we obtain $p^{[r]}(n, m) \sim f_r(m)p^{[1]}(n, m)$, where $f_r(m)$ satisfies the following recursion

$$f_r(m) = f_{r-1}(m) + \binom{m+r-3}{r-1}, \quad f_1(m) = 1.$$

We have the solution

$$f_r(m) = \binom{m+r-2}{r-1},$$

and hence

$$\begin{aligned} p^{[r]}(n, m) &\sim \binom{m+r-2}{r-1} p^{[1]}(n, m) \\ &\sim \binom{m+r-2}{r-1} (m-1)! \left(\frac{C}{2\sqrt{n}}\right)^{m-1} \exp\left(-\frac{1}{4}\left(2C + \frac{1}{2C}\right)\frac{m^2}{\sqrt{n}}\right) p(n), \end{aligned}$$

and Equation (5) follows.

We observe that $f_r(m) \sim m^{r-1}/(r-1)!$ as $m \rightarrow \infty$ for any fixed r . □

Fristedt [11] lets $X_k(\lambda)$ be the number of parts of the partition λ that equal k , $k = 1, 2, \dots$. Thus, $kX_k(\lambda)$ is the contribution of the part k to the sum of the parts of λ , that is, $kX_k(\lambda) = l_k k$ in our notation. Theorem 2.1 in [11] states that if $k = o(n^{1/2})$ then

$$\lim_{n \rightarrow \infty} P_n \left(\frac{\pi}{\sqrt{6n}} k_n X_{k_n} \leq v \right) = 1 - e^{-v}.$$

Here P_n denotes the distribution of the k th part. Since $\frac{d(1-e^{-v})}{dv} = e^{-v}$ we have that e^{-v} is the probability that the k th part is v . Therefore

$$\int_0^\infty e^{-v} dv = 1$$

is the expected value of the k th part of a partition of n for $k = o(n^{1/2})$.

4. The Size of the r^{th} Smallest Part in Partitions with Distinct Parts

In this section we consider partitions with distinct parts. Let $q^{[r]}(n, m)$ be the number of partitions of n with distinct parts such that the size of the r th smallest part is at least m . Also, let $Y_n^{[r]}$ denote the size of the r th smallest part in a random partition with distinct parts.

Freiman and Pitman [10] obtained an asymptotic formula for $q^{[1]}(n, m)$ when $m = o(n \log^{-9} n)$. When $m = o(n^{1/3})$, their formula simplifies to

$$q^{[1]}(n, m) = 2^{-m-1} 3^{-1/4} n^{-3/4} \exp\left(\pi \left((n/3)^{1/2} + \frac{m(m-1)}{8} (3n)^{-1/2} \right)\right). \quad (6)$$

Hardy and Ramanujan [13] and Hua [14] give the following estimate for $q(n)$, the number of partitions of n into distinct parts

$$q(n) \sim \frac{1}{2^2 3^{1/4} n^{3/4}} \exp\left(\pi \left((n/3)^{1/2} \right)\right),$$

and so in the range $m = o(n^{1/3})$ we get

$$q^{[1]}(n, m) \sim \frac{1}{2^{m-1}} \exp\left(\pi \frac{m(m-1)}{8} (3n)^{-1/2}\right) q(n). \tag{7}$$

We give next the main result of this section.

Theorem 2. *The probability that the r th smallest part in a random partition with n into distinct parts has size at least $m = o(n^{1/3})$ satisfies*

$$P(Y_n^{[r]} > m) = \frac{q^{[r]}(n, m)}{q(n)} \sim \frac{1}{2^{m-1}} \frac{m^{r-1}}{(r-1)!} \exp\left(\pi \frac{m(m-1)}{8} (3n)^{-1/2}\right). \tag{8}$$

Proof. We start by considering

$$\begin{aligned} q^{[2]}(n, m) &= \sum_{j=0}^{m-1} q^{[1]}(n-j, m) \\ &\sim \frac{1}{2^{m+1} 3^{1/4}} \sum_{j=0}^{m-1} (n-j)^{-3/4} \exp\left(\pi \left(((n-j)/3)^{1/2} + \frac{m(m-1)}{8(3(n-j))^{1/2}} \right)\right) \\ &\sim m q^{[1]}(n, m). \end{aligned}$$

Using a similar argument to that in the proof of Theorem 1, and observing that all parts are distinct here, we obtain

$$q^{[r]}(n, m) \sim g_r(m) q^{[1]}(n, m),$$

where $g_r(m)$ satisfies the following recursion

$$g_r(m) = g_{r-1}(m) + \binom{m-1}{r-1}, \quad g_1(m) = 1.$$

We observe that $g_r(m)$ does not have a sum-free closed form expression (easily verified using Gosper’s algorithm [17]). We also note

$$g_r(m) = 1 + \binom{m-1}{1} + \dots + \binom{m-1}{r-1},$$

and $g_r(m) \sim m^{r-1}/(r-1)!$ as $m \rightarrow \infty$ for any fixed r . Therefore, putting all pieces together, we obtain Equation (8). □

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