FURTHER ANALOGUES OF THE ROGERS-RAMANUJAN FUNCTIONS WITH APPLICATIONS TO PARTITIONS

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Abstract

In this paper, we establish several modular relations involving two functions analogous to the Rogers-Ramanujan functions. These relations are analogous to Ramanujan's famous forty identities for the Rogers-Ramanujan functions. Also, by the notion of colored partitions, we extract partition theoretic interpretations from some of our relations.

–Dedicated to Professor Ron Graham

1. Introduction

Throughout the paper, we assume $|q| < 1$ and for positive integers n, we use the standard notation

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

The famous Rogers-Ramanujan identities ([20], [16], [17, pp. 214–215]) are

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{1}$$

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and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2}$$

$G(q)$ and $H(q)$ are known as the Rogers-Ramanujan functions. Ramanujan [19] found forty modular relations for $G(q)$ and $H(q)$, which are called "Ramanujan's forty identities." In 1921, Darling [10] proved one of the identities in the Proceedings of London Mathematical Society. In the same issue of the journal, Rogers [21] established ten of the forty identities, including the one proved by Darling. In 1933, Watson [24] proved eight of the identities, two of which had been previously established by Rogers. In 1977, Bressoud [7], in his doctoral thesis, proved fifteen more. In 1989, Biagioli [5] proved eight of the remaining nine identities by invoking the theory of modular forms. Recently, Berndt et. al. [4] have found proofs of thirty-five of the identities in the spirit of Ramanujan's mathematics. For the remaining five identities, they also offered heuristic arguments showing that both sides of the identity have the same asymptotic expansions as $q \to 1^-$.

Two identities analogous to (1) and (2) are the so-called Göllnitz-Gordon identities [11], [12], given by

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \tag{3}$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}. \tag{4}$$

$S(q)$ and $T(q)$ are known as the Göllnitz-Gordon functions. Huang [15] and Chen and Huang [9] found twenty-one modular relations involving only the Göllnitz-Gordon functions, nine relations involving both the Rogers-Ramanujan and Göllnitz-Gordon functions, and one new relation for the Rogers-Ramanujan functions. They used the methods of Rogers [21] , Watson [24], and Bressoud [7] to derive the relations. They also extracted partition theoretic results from some of the relations. Baruah, Bora, and Saikia [2] also found new proofs for the relations which involve only the Göllnitz-Gordon functions by using Schröter's formulas and some theta-function identities found in Ramanujan's notebooks [18]. In the process, they also found some new relations. In [13] - [14], H. Hahn defined the septic analogues, $A(q)$, $B(q)$, and $C(q)$ below, of the Rogers-Ramanujan functions, and L.J. Slater [22] established the following identities:

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n (-q;q)_{2n}} = \frac{(q^7;q^7)_\infty (q^3;q^7)_\infty (q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{5}$$

$$B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n (-q;q)_{2n}} = \frac{(q^7;q^7)_\infty (q^2;q^7)_\infty (q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{6}$$

and

$$C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n (-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty (q;q^7)_\infty (q^6;q^7)_\infty}{(q^2;q^2)_\infty}. \tag{7}$$

Hahn found several modular relations involving only A(q), B(q), and C(q), as well as relations which are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions.

More recently, the authors [1] established many modular relations involving the following nonic analogues of the Rogers-Ramanujan functions and other analogous functions defined above:

$$D(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^4;q^9)_\infty (q^5;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{8}$$

$$E(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} (1-q^{3n+2})}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty (q^7;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{9}$$

$$F(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1} q^{3n(3n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty (q^8;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{10}$$

where the three equalities are also due to Slater [22].

In this paper, we consider the following two analogous functions of the Rogers-Ramanujan functions:

$$X(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n (1-q^{n+1}) q^{n(n+2)}}{(q;q)_{2n+2}} = \frac{(q;q^{12})_\infty (q^{11};q^{12})_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{11}$$

$$Y(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_{n-1} (1+q^n) q^{n^2}}{(q;q)_{2n}} = \frac{(q^5;q^{12})_\infty (q^7;q^{12})_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{12}$$

also established in [22]. By applying various results on Ramanujan's theta-functions and methods of Blecksmith et al. [6] and Bressoud [7], we find several modular relations for $X(q)$ and $Y(q)$. Some of these relations are connected with the Rogers-Ramanujan functions and their analogues defined in (1)-(10).

In Section 2, we present some definitions and preliminary results. In Section 3, we present the modular relations involving $X(q)$ and $Y(q)$ and other analogous Rogers-Ramanujan-type functions. In Sections 4-7, we present proofs of the modular relations. In our last section, we apply some of the modular relations to the theory of partitions.

2. Definitions and Preliminary Results

Ramanujan's general theta-function is

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \; |ab| < 1. \tag{13}$$

In the following lemma, we state a basic identity satisfied by $f(a,b)$.

Lemma 2.1 [3, p. 34, Entry 18(iv)] *If n is an integer, then*

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{14}$$

We state Jacobi's famous triple product identity in our next lemma.

Lemma 2.2 [3, p. 35, Entry 19] *We have*

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{15}$$

In the next lemma, we state three important special cases of $f(a,b)$.

Lemma 2.3 [3, p. 36, Entry 22] *If $|q| < 1$, then*

$$\phi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{16}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{17}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n-1)/2} + \sum_{n=1}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty, \tag{18}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{19}$$

The product representations in (16)-(18) arise from (15). Also, note that if $q = e^{\pi i \tau}$, then $\phi(q) = \vartheta_3(0,\tau)$, where $\vartheta_3(z,\tau)$ denotes the classical theta-function in standard notation [25, p. 464]. Again, if $q = e^{2\pi i \tau}$, then $f(-q) = e^{-\pi i \tau/12} \eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function. The last equality in (18) is a statement of Euler's famous pentagonal number theorem.

Invoking (15) and (18) in (11) and (12), we readily arrive at the following result.

Lemma 2.4 *We have*

$$X(q) = \frac{f(-q,-q^{11})}{f(-q)}, \qquad \text{and} \qquad Y(q) = \frac{f(-q^5,-q^7)}{f(-q)}. \tag{20}$$

Throughout the remainder of the paper, we shall use f_n to denote $f(-q^n)$. The following lemma is a consequence of (15) and Entry 24 of [3, p. 39].

Lemma 2.5 *We have*

$$\phi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \tag{21}$$

$$\phi(-q) = \frac{f_1^2}{f_2}, \quad \psi(-q) = \frac{f_1 f_4}{f_2}, \quad f(q) = \frac{f_2^3}{f_1 f_4}, \quad \text{and} \quad \chi(q) = \frac{f_2^2}{f_1 f_4}. \tag{22}$$

The following three lemmas, from [3], will be useful.

Lemma 2.6 *We have*

$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3). \tag{23}$$

and

$$f(a,b) - f(-a,-b) = 2af(b/a, a^5b^3). \tag{24}$$

Lemma 2.7 *We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{25}$$

Lemma 2.8 *We have*

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}. \tag{26}$$

We state one more lemma, which is from [23].

Lemma 2.9 *For* $|ab| < 1$*,*

$$f^3(ab^2, a^2b) - bf^3(a, a^2b^3) = \frac{f(-b^2, -a^3b)}{f(b, a^3b^2)} f^3(-ab). \tag{27}$$

3. Modular Relations for $X(q)$ and $Y(q)$

In this section, we present a list of modular relations involving some combinations of $X(q)$, $Y(q)$, and other analogous Rogers-Ramanujan-type functions. For simplicity, for a positive integer n, we set $X_n := X(q^n)$, $Y_n := Y(q^n)$, $G_n := G(q^n)$, $H_n := H(q^n)$, $S_n := S(q^n)$, $T_n := T(q^n)$, $A_n := A(q^n)$, $B_n := B(q^n)$, $C_n := C(q^n)$, $D_n := D(q^n)$, $E_n := E(q^n)$, and $F_n := F(q^n)$. We also note that more relations can easily be obtained by replacing q by $-q$

in each of the following relations:

$$Y_1 + qX_1 = \frac{f_2^3}{f_1^2 f_4}, \tag{28}$$

$$Y_1 - qX_1 = \frac{f_4 f_6^5}{f_2^2 f_3^2 f_{12}^2}, \tag{29}$$

$$X_1Y_2 + qX_2Y_1 = \frac{f_3 f_{24}}{f_1 f_2}, \tag{30}$$

$$X_1Y_3 + q^2X_3Y_1 = \frac{f_4 f_6^5 f_9 f_{36}}{f_2^2 f_3^3 f_{12}^2 f_{18}}, \tag{31}$$

$$Y_1^3 + q^3X_1^3 = \frac{f_4^3 f_6^3}{f_1^3 f_3 f_{12}^2}, \tag{32}$$

$$Y_1Y_2 + q^3X_1X_2 = \frac{f_2 f_4 f_4}{f_1 f_1 f_8} - q\frac{f_3 f_{24}}{f_1 f_2}, \tag{33}$$

$$X_1Y_3 - q^2X_3Y_1 = \frac{f_2 f_{12}^3}{f_1 f_3 f_4 f_6} - q\frac{f_4 f_6 f_{18} f_{72}}{f_1 f_2 f_3 f_{36}}, \tag{34}$$

$$Y_3^2 + q^6X_3^2 = q\frac{f_9^2 f_4^2}{f_{18} f_2 f_3^2} - q^2\frac{f_{18}^4}{f_1^2 f_6^2}, \tag{35}$$

$$Y_1Y_5 + q^6X_1X_5 = \frac{f_4 f_4 f_5}{f_1 f_2 f_{10}} - q^2\frac{f_3 f_{12} f_{15} f_{60}}{f_1 f_5 f_6 f_{30}}, \tag{36}$$

$$X_1Y_8 + q^8X_8Y_1 = \frac{f_{32}}{f_8} + q^2\frac{f_3 f_{12} f_{24} f_{72}}{f_1 f_6 f_8 f_{48}}, \tag{37}$$

$$X_2Y_7 + q^5X_7Y_2 = \frac{f_1 f_{56}}{f_2 f_7} + q\frac{f_6 f_{21} f_{24} f_{84}}{f_2 f_7 f_{12} f_{42}}, \tag{38}$$

$$X_1Y_{11} + q^{10}X_{11}Y_1 = \frac{f_4 f_{44}}{f_2 f_{22}} + q^3\frac{f_3 f_{12} f_{33} f_{132}}{f_1 f_6 f_{22} f_{66}}, \tag{39}$$

$$Y_1Y_{14} + q^{15}X_1X_{14} = \frac{f_7 f_8}{f_1 f_{14}} - q^5\frac{f_3 f_{12} f_{42} f_{168}}{f_1 f_6 f_{14} f_{84}}, \tag{40}$$

$$Y_1Y_{20} + q^{21}X_1X_{20} = \frac{f_5 f_{16}}{f_1 f_{20}} - q^7\frac{f_3 f_{12} f_{60} f_{240}}{f_1 f_6 f_{20} f_{120}}, \tag{41}$$

$$q^3X_7Y_5 + qX_5Y_7 = \frac{f_{15} f_{21} f_{60} f_{84}}{f_5 f_7 f_{42} f_{30}} - \frac{f_1 f_4 f_{35} f_{140}}{f_2 f_5 f_7 f_{70}}, \tag{42}$$

$$Y_1Y_{35} + q^{36}X_1X_{35} = q^{12}\frac{f_3 f_{12} f_{105} f_{420}}{f_1 f_6 f_{35} f_{210}} - \frac{f_5 f_7 f_{20} f_{28}}{f_1 f_{10} f_{14} f_{35}}. \tag{43}$$

The identities (44)-(47) involve quotients of the functions $X(q)$ and $Y(q)$:

$$\frac{Y_{15} - q^5(f_{27}f_{108})/(f_{15}f_{54}) - q^{15}X_{15}}{Y_3 - q(f_9f_{36})/(f_3f_{18}) - q^3X_3} = \frac{f_1f_4f_{10}f_{15}}{f_2f_3f_5f_{20}}, \tag{44}$$

$$\frac{X_5Y_{31} - q^7(f_{15}f_{60}f_{93}f_{372})/(f_5f_{30}f_{31}f_{186}) + q^{26}X_{31}Y_5}{Y_1Y_{155} + q^{52}(f_3f_{12}f_{465}f_{1860})/(f_1f_6f_{155}f_{930}) + q^{156}X_1X_{155}} = \frac{f_1f_{155}}{f_5f_{31}}, \tag{45}$$

$$\frac{X_7Y_{29} - q^5(f_{21}f_{84}f_{87}f_{348})/(f_7f_{29}f_{42}f_{174}) + q^{22}X_{29}Y_7}{Y_1Y_{203} + q^{68}(f_3f_{12}f_{609}f_{2436})/(f_1f_6f_{203}f_{1218}) + q^{204}X_1X_{203}} = \frac{f_1f_{203}}{f_7f_{29}}, \tag{46}$$

$$\frac{X_1Y_{275} - q^{91}(f_1f_{12}f_{825}f_{3300})/(f_1f_6f_{275}f_{1650}) + q^{274}X_{275}Y_1}{X_{11}Y_{25} - q(f_{33}f_{75}f_{132}f_{300})/(f_{11}f_{25}f_{66}f_{150}) + q^{14}X_{25}Y_{11}} = \frac{f_{11}f_{25}}{f_1f_{275}}. \tag{47}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the Rogers-Ramanujan functions $G(q)$ and $H(q)$:

$$\frac{G_7G_8 + q^3H_7H_8}{Y_2Y_7 + q^3(f_6f_{21}f_{24}f_{84})/(f_2f_7f_{12}f_{42}) + q^9X_2X_7} = \frac{f_2}{f_8}, \tag{48}$$

$$\frac{G_9G_{16} + q^5H_9H_{16}}{Y_3Y_{12} + q^5(f_9f_{36}^2f_{144})/(f_3f_{12}f_{18}f_{72}) + q^{15}X_3X_{12}} = \frac{f_9f_{16}}{f_3f_{12}}, \tag{49}$$

$$\frac{G_8G_{27} + q^7H_8H_{27}}{Y_3Y_{18} + q^7(f_9f_{36}f_{54}f_{216})/(f_3f_{18}^2f_{72}) + q^{21}X_3X_{18}} = \frac{f_3f_{18}}{f_8f_{27}}. \tag{50}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the Göllnitz-Gordon functions $S(q)$ and $T(q)$:

$$\frac{Y_8Y_1 + q^3(f_3f_{24})/(f_1f_2) + q^9X_1X_8}{S_4S_2 + q^3T_4T_2} = \frac{f_1f_8}{f_2f_{16}}, \tag{51}$$

$$\frac{Y_{15} - q^5(f_{45}f_{180})/(f_{15}f_{90}) - q^{15}X_{15}}{S_{15}S_1 + q^8T_{15}T_1} = \frac{f_1f_4f_6f_{60}}{f_2f_3f_{12}f_{30}}, \tag{52}$$

$$\frac{X_1Y_{23} - q^7(f_3f_{12}f_{69}f_{276})/(f_1f_6f_{23}f_{138}) + q^{22}Y_1X_{23}}{T_1S_{23} - qT_{23}S_1} = \frac{f_4f_{12}f_{92}f_{176}}{f_2f_8f_{46}f_{184}}, \tag{53}$$

$$\frac{Y_{95}Y_1 + q^{32}(f_3f_{12}f_{195}f_{780})/(f_1f_6f_{95}f_{390}) + q^{96}X_1X_{95}}{S_{19}T_5 - q^7S_5T_{19}} = \frac{f_5f_{19}f_{20}f_{76}}{f_1f_{10}f_{38}f_{95}}, \tag{54}$$

$$\frac{Y_1Y_{119} + q^{40}(f_3f_{12}f_{357}f_{1428})/(f_1f_6f_{119}f_{714}) + q^{120}X_1X_{119}}{S_{17}T_7 - q^5T_{17}S_7} = \frac{f_7f_{17}f_{28}f_{68}}{f_1f_{14}f_{34}f_{119}}, \tag{55}$$

$$\frac{X_5Y_{19} - q^3(f_{15}f_{60}f_{57}f_{228})/(f_5f_{19}f_{30}f_{114}) + q^{19}Y_5X_{19}}{S_{95}S_1 + q^{48}T_{95}T_1} = \frac{f_1f_4f_{95}f_{380}}{f_2f_5f_{19}f_{190}}, \tag{56}$$

$$\frac{X_3Y_{21} - q^5(f_9f_{36}f_{63}f_{252})/(f_3f_{18}f_{21}f_{126}) + q^{18}Y_3X_{21}}{S_{63}S_1 + q^{32}T_{63}T_1} = \frac{f_1f_4f_{63}f_{252}}{f_2f_3f_{21}f_{126}}, \tag{57}$$

$$\frac{X_7Y_{17} - q(f_{21}f_{51}f_{84}f_{204})/(f_7f_{17}f_{42}f_{102}) + q^{10}Y_7X_{17}}{S_{119}T_1 - q^{59}S_1T_{119}} = \frac{f_1f_4f_{119}f_{476}}{f_2f_7f_{17}f_{238}}. \tag{58}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the septic analogues $A(q)$, $B(q)$, and $C(q)$:

$$\frac{Y_5Y_4 + q^3(f_{12}f_{15}f_{48}f_{60})/(f_4f_5f_{24}f_{30}) + q^9X_5X_4}{A_5A_{16} + q^3B_5B_{16} + q^9C_5C_{16}} = \frac{f_{32}f_{10}}{f_4f_5}, \tag{59}$$

$$\frac{Y_1Y_{98} + q^{33}(f_3f_{12}f_{294}f_{1176})/(f_1f_6f_{98}f_{588}) + q^{99}X_1X_{98}}{A_{56}C_7 - q^5B_{56}A_7 + q^{22}C_{56}B_7} = \frac{f_{14}f_{112}}{f_1f_{98}}. \tag{60}$$

The following identities are relations involving some combinations of $X(q)$ and $Y(q)$ with the nonic analogues $D(q)$, $E(q)$, and $F(q)$:

$$\frac{Y_1Y_2 + q(f_3f_{24})/(f_1f_2) + q^3X_1X_2}{D_1D_8 + q + q^3E_1E_8 + q^6F_1F_8} = \frac{f_3f_{24}}{f_1f_2}, \tag{61}$$

$$\frac{X_1Y_{50} - q^{16}(f_3f_{12}f_{150}f_{600})/(f_1f_6f_{50}f_{300}) + q^{49}Y_1X_{50}}{D_{25}E_8 - q - q^{11}E_{25}F_8 + q^{14}F_{25}D_8} = \frac{f_{24}f_{75}}{f_1f_{50}}. \tag{62}$$

4. Proofs of (28), (29), and (32)

Proof of (28)*:* Putting $a = q$ and $b = q^2$ in (23) and (24), we find that

$$f(q, q^2) + f(-q, -q^2) = 2f(q^5, q^7) \tag{63}$$

and

$$f(q, q^2) - f(-q, -q^2) = 2qf(q, q^{11}), \tag{64}$$

respectively. Subtracting (64) from (63), and then replacing q by $-q$, we find that

$$f(-q^5, -q^7) + qf(-q, -q^{11}) = f(q, -q^2) = f(q), \tag{65}$$

where the last equality follows from (18).

Dividing both sides of (65) by $f(-q)$, we arrive at

$$\frac{f(-q^5, -q^7)}{f(-q)} + q\frac{f(-q, -q^{11})}{f(-q)} = \frac{f(q)}{f(-q)}. \tag{66}$$

Employing (20) and (22) in (66), we easily arrive at (28).

Proof of (29)*:* Adding (63) and (64), and then replacing q by $-q$, we obtain

$$f(-q^5, -q^7) - qf(-q, -q^{11}) = f(-q, q^2) = \frac{\phi(q^3)}{\chi(q)}, \tag{67}$$

where the last equality follows from (26).

Employing (20), (21), and (22) in (67), we easily deduce (29).

Alternative proof. From [3, Entry 31, Corollary(ii)], we have

$$f(q^{15}, q^{21}) + q^3 f(q^3, q^{33}) = \psi(q) - q\psi(q^9). \tag{68}$$

Replacing q^3, by $-q$, and employing (25), (26), (20) in (68), we easily arrive at (29).

Proof of (32)*:* Setting $a = q$ and $b = q^3$ in (27), we obtain

$$f^3(q^5, q^7) - q^3 f^3(q, q^{11}) = \frac{\phi(-q^6)}{\psi(q^3)} f^3(-q^4). \tag{69}$$

Replacing q, by $-q$, in (69), we find that

$$f^3(-q^5, -q^7) + q^3 f^3(-q, -q^{11}) = \frac{\phi(-q^6)}{\psi(-q^3)} f^3(-q^4). \tag{70}$$

Using (22) and (20) in (70), we easily arrive at (32).

5. Proofs of (33) - (35)

To present proofs of (33) - (35), we use a formula of R. Bleckmith, J. Brillhart, and I. Gerst [6, Theorem 2], providing a representation for a product of two theta functions as a sum of m products of pair of theta functions, under certain conditions. This formula generalizes formulas of H. Schröter [3, p. 65-72]. Define, for $\epsilon \in \{0, 1\}$ and $|ab| < 1$,

$$f_\epsilon(a, b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n} (ab)^{n^2/2} (a/b)^{n/2}. \tag{71}$$

Theorem 5.1 *Let a, b, c, and d denote positive numbers with $|ab|$, $|cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}. \tag{72}$$

Let ϵ_1, $\epsilon_2 \in \{0, 1\}$, and define δ_1, $\delta_2 \in \{0, 1\}$ by

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 (mod\ 2) \quad and \quad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 (mod\ 2), \tag{73}$$

respectively, where $p = m - \alpha\beta$. Then if R denotes any complete residue system modulo m,

$$f_{\epsilon_1}(a, b) f_{\epsilon_2}(c, d) = \sum_{r \in R} (-1)^{\epsilon_2 r} c^{r(r+1)/2} d^{r(r-1)/2} f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha} \right)$$
$$\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a\beta)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p} \right). \tag{74}$$

Proof of (33)*:* We apply Theorem 5.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^4$, $c = 1$, $d = q$, $\alpha = 2$, $\beta = 1$, $m = 6$. Consequently, we find that

$$2\phi(-q^4)\psi(q) = 2\{f(-q^7,-q^5)f(-q^{14},-q^{10})$$
$$q\psi(-q^3)\psi(-q^6) + q^3 f(-q,-q^{11})f(-q^2,-q^{22})\}. \tag{75}$$

Now, using (20) and (22), we deduce (33).

In a similar way, we prove the identities (34) and (35). To prove (34), we apply Theorem 5.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^9$, $c = q$, $d = q^2$, $\alpha = 1$, $\beta = 1$, $m = 4$ and to prove (35), we again apply Theorem 5.1 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$ $a = b = q^{18}$, $c = q^4$, $d = 1$, $\alpha = 3$, $\beta = 1$, $m = 6$.

6. Proofs of (30), (31), and (36) - (43)

We will apply the method given by Bressoud in his thesis [7]. Here, we use f_n instead of P_n, and the variable q instead of x, which stands for q^2 in [7]. The letters α, β, m, n, p always denote positive integers, and m must be odd. Following Bressoud [7], we define

$$\underline{g}_\alpha^{(p,n)} = \left\{q^{(12n^2-12n+3-p(p-1)/2)/(12p)\alpha}\right\}\frac{(q^{(p+1-2n)\alpha};q^{2p\alpha})_\infty(q^{(p-1+2n)\alpha};q^{2p\alpha})_\infty}{(q^\alpha;q^{2\alpha})_\infty}. \tag{76}$$

Proposition 6.1 [7, Proposition 5.8] *We have*

$$\underline{g}_\alpha^{(p,n)} = \underline{g}_\alpha^{(p,-n+1)}, \qquad \textit{and} \qquad \underline{g}_\alpha^{(p,n)} = -\underline{g}_\alpha^{(p,n-p)}, \qquad \textit{and} \qquad \underline{g}_\alpha^{(p,n)} = -\underline{g}_\alpha^{(p,p-n+1)}. \tag{77}$$

Proposition 6.2 [7, Proposition 5.9] *We have*

$$\underline{g}_\alpha^{(2,1)} = 1, \tag{78}$$

$$\underline{g}_\alpha^{(4,1)} = q^{-11\alpha/48}\frac{f_{4\alpha}}{f_{8\alpha}}S(q^\alpha), \qquad \textit{and} \qquad \underline{g}_\alpha^{(4,2)} = q^{13\alpha/48}\frac{f_{4\alpha}}{f_{8\alpha}}T(q^\alpha), \tag{79}$$

where $S(q)$ and $T(q)$ are as defined in (3) and (4), respectively.

Proposition 6.3 *We have*

$$\underline{g}_\alpha^{(6,1)} = q^{-5\alpha/12}Y_\alpha\frac{f_{2\alpha}}{f_{12\alpha}}, \tag{80}$$

$$\underline{g}_\alpha^{(6,2)} = q^{-\alpha/12}\frac{f_{2\alpha}f_{3\alpha}}{f_\alpha f_{6\alpha}}, \tag{81}$$

$$\underline{g}_\alpha^{(6,3)} = q^{7\alpha/12}X_\alpha\frac{f_{2\alpha}}{f_{12\alpha}}. \tag{82}$$

Proof. Take $p = 6$, and $n = 1$ in (76). Then

$$\underline{g}_\alpha^{(6,1)} = q^{-5\alpha/12} \frac{(q^{5\alpha}; q^{12\alpha})_\infty (q^{7\alpha}; q^{12\alpha})_\infty (q^{12\alpha}; q^{12\alpha})_\infty}{(q^\alpha; q^{2\alpha})(q^{12\alpha}; q^{12\alpha})_\infty}. \tag{83}$$

Employing (15) and (20) in (83), we readily deduce (80). Similarly we can prove (81) and (82).

Theorem 6.4 [7, Proposition 5.4] *For odd $p > 1$,*

$$\phi_{\alpha,\beta,m,p} = 2q^{\alpha+\beta/24} f(-q^\alpha) f(-q^\beta) \left(\sum_{n=1}^{(p-1)/2} g_\beta^{(p,n)} g_\alpha^{(p,(2mn-m+1)/2)} \right). \tag{84}$$

Lemma 6.5 [7, Corollary 5.5 and 5.6] *If $\phi_{\alpha,\beta,m,p}$ is defined by* (84)*, then*

$$\phi_{\alpha,\beta,m,1} = 0, \tag{85}$$

and

$$\phi_{\alpha,\beta,1,3} = 2q^{(\alpha+\beta)/24} f(-q^\alpha) f(-q^\beta). \tag{86}$$

Lemma 6.6 [7, Lemma 6.5] *We have*

$$\phi_{\alpha,\beta,1,5} = 2q^{(\alpha+\beta)/40} f(-q^\alpha) f(-q^\beta) \{G(q^\beta)G(q^\alpha) + q^{(\alpha+\beta)/5} H(q^\beta)H(q^\alpha)\}. \tag{87}$$

Lemma 6.7 [13, Lemma 6.6] *We have*

$$\phi_{\alpha,\beta,1,7} = 2q^{(\alpha+\beta)/56} f(-q^{2\alpha}) f(-q^{2\beta}) \{A_\beta A_\alpha + q^{(\alpha+\beta)/7} B_\beta B_\alpha + q^{(3\alpha+3\beta)/7} C_\beta C_\alpha\}, \tag{88}$$

$$\phi_{\alpha,\beta,5,7} = 2q^{(25\alpha+\beta)/56} f(-q^{2\alpha}) f(-q^{2\beta}) \{A_\beta C_\alpha - q^{(-3\alpha+\beta)/7} B_\beta A_\alpha + q^{(-2\alpha+3\beta)/7} C_\beta B_\alpha\}. \tag{89}$$

Lemma 6.8 [1] *We have*

$$\begin{aligned}\phi_{\alpha,\beta,1,9} = &2q^{(\alpha+\beta)/72} f(-q^{3\alpha}) f(-q^{3\beta}) \{D_\alpha D_\beta + q^{(\alpha+\beta)/9} + q^{(\alpha+\beta)/3} E_\alpha E_\beta \\ &+ q^{2(\alpha+\beta)/3} F_\alpha F_\beta\}.\end{aligned} \tag{90}$$

$$\begin{aligned}\phi_{\alpha,\beta,5,9} = &2q^{(25\alpha+\beta)/72} f(-q^{3\alpha}) f(-q^{3\beta}) \{D_\beta E_\alpha - q^{(\beta-2\alpha)/9} - q^{(\alpha+\beta)/3} E_\beta F_\alpha \\ &+ q^{(2\beta-\alpha)/3} F_\beta D_\alpha\}.\end{aligned} \tag{91}$$

Theorem 6.9 [7, Proposition 5.10] *For even p,*

$$\phi_{\alpha,\beta,m,p} = 2q^{(2p-1)(\alpha+\beta)/24} \left\{ \sum_{n=1}^{p/2} \underline{g}_\beta^{(p,n)} \underline{g}_\alpha^{(p,mn-((m-1)/2))} \right\} \frac{f_{2p\alpha} f_{2p\beta} f_\alpha f_\beta}{f_{2\alpha} f_{2\beta}}. \tag{92}$$

Lemma 6.10 [7, Corollary 5.11] *If α and β are even positive integers, then*

$$\phi_{\alpha,\beta,1,2} = 2q^{(\alpha+\beta)/16}\frac{f(-q^{2\alpha})f(-q^{2\beta})f(-q^{\alpha/2})f(-q^{\beta/2})}{f(-q^{\alpha})f(-q^{\beta})}. \tag{93}$$

Lemma 6.11 [15, Lemma 5.1] *We have*

$$\phi_{\alpha,\beta,1,4} = 2q^{(\alpha+\beta)/32}\{S(q^{\beta/2})S(q^{\alpha/2}) + q^{(\alpha+\beta)/4}T(q^{\beta/2})T(q^{\alpha/2})\}\frac{f_{2\alpha}f_{2\beta}f_{\alpha}f_{\beta}}{f_{\alpha}f_{\beta}}. \tag{94}$$

$$\phi_{\alpha,\beta,3,4} = 2q^{(9\alpha+\beta)/32}\{S(q^{\beta/2})T(q^{\alpha/2}) - q^{(\beta-\alpha)/2}S(q^{\alpha/2})T(q^{\beta/2})\}\frac{f_{2\alpha}f_{2\beta}f_{\alpha}f_{\beta}}{f_{\alpha}f_{\beta}}. \tag{95}$$

Theorem 6.12 [7, Proposition 5.10]

$$\phi_{\alpha,\beta,5,2} = -2q^{(\alpha+\beta)/8}\frac{f_{4\alpha}f_{4\beta}f_{\alpha}f_{\beta}}{f_{2\alpha}f_{2\beta}}. \tag{96}$$

Proof. Applying equation (92) with $m = 5$ and $p = 2$, we have

$$\phi_{\alpha,\beta,5,2} = 2q^{(\alpha+\beta)/8}\underline{g}_{\beta}^{(2,5)}\underline{g}_{\alpha}^{(2,3)}\frac{f_{4\alpha}f_{4\beta}f_{\alpha}f_{\beta}}{f_{2\alpha}f_{2\beta}}. \tag{97}$$

Now, using (77) and (78) in (97), we obtain the result.

Proposition 6.13

$$\phi_{\alpha,\beta,1,6} = 2q^{(\alpha+\beta)/24}f_{\alpha}f_{\beta}\left\{Y_{\alpha}Y_{\beta} + q^{(\alpha+\beta)/3}\frac{f_{3\alpha}f_{3\beta}f_{12\alpha}f_{12\beta}}{f_{\alpha}f_{\beta}f_{6\alpha}f_{6\beta}} + q^{(\alpha+\beta)}X_{\alpha}X_{\beta}\right\}. \tag{98}$$

$$\phi_{\alpha,\beta,3,6} = 2q^{(9\alpha+\beta)/24}\frac{f_{\beta}f_{3\alpha}f_{12\alpha}}{f_{6\alpha}}\left\{Y_{\beta} - q^{\beta/3}\frac{f_{3\beta}f_{12\beta}}{f_{\beta}f_{6\beta}} - q^{\beta}X_{\beta}\right\}. \tag{99}$$

$$\phi_{\alpha,\beta,5,6} = 2q^{(\alpha+\beta)/24}f_{\alpha}f_{\beta}\left\{q^{\alpha}X_{\alpha}Y_{\beta} - q^{(\alpha+\beta)/3}\frac{f_{3\alpha}f_{3\beta}f_{12\alpha}f_{12\beta}}{f_{\alpha}f_{\beta}f_{6\alpha}f_{6\beta}} + q^{\beta}X_{\beta}Y_{\alpha}\right\}. \tag{100}$$

Proof. Applying equation (92) with $m = 1$ and $p = 6$, we have

$$\phi_{\alpha,\beta,1,6} = 2q^{(\alpha+\beta)/24}\left\{\underline{g}_{\beta}^{(6,1)}\underline{g}_{\alpha}^{(6,1)} + \underline{g}_{\beta}^{(6,2)}\underline{g}_{\alpha}^{(6,2)} + \underline{g}_{\beta}^{(6,3)}\underline{g}_{\alpha}^{(6,3)}\right\}\frac{f_{12\alpha}f_{12\beta}f_{\alpha}f_{\beta}}{f_{2\alpha}f_{2\beta}}. \tag{101}$$

Now, using (80), (81), (82), in (101), we obtain the result after simplification. The equation (99) and (100) can be proved in a similar way by applying equation (77) with $m = 3$ and 5, respectively, and $p = 6$.

Theorem 6.14 [7, Corollary 7.3] *Let α_i, β_i, m_i, p_i, be positive integers, for $i = 1, 2$, with m_1, m_2 odd. Let $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$. If the conditions*

$$\lambda_1 = \lambda_2,\ \alpha_1\beta_1 = \alpha_2\beta_2,\ \text{and}\ \alpha_1 m_1 = \pm\alpha_2 m_2 (\text{mod } \lambda_1)$$

hold, then $\phi_{\alpha_1,\beta_1,m_1,p_1} = \phi_{\alpha_2,\beta_2,m_2,p_2}$.

We next give proofs of several of the modular relations from Section 3.

Proof of (30)*:* From [13, Proposition 6.23], we have

$$\phi_{p+1,4p^2,5,p+5} = \phi_{p,4p(p+1),1,p}, \tag{102}$$

where p is a positive integer. Setting $p = 1$ in (102), we easily arrive at (30) with the help of (100) and (85).

Proof of (31)*:* If p is even, then from [15, Proposition 6.3], we note that

$$\phi_{6,4p+10,p+3,p+4} = \phi_{2,12p+30,1,2}. \tag{103}$$

Setting $p = 2$ in (103), we obtain (31) by employing (100) and (93).

Proof of (36)*:* If p in an integer greater than 1, then from [15, Proposition 5.4], we have

$$\phi_{1,p-1,1,p} = q^{1/4} f(1, q^2) f(-q^{p-1}, -q^{p-1}). \tag{104}$$

setting $p = 6$ in (104), we obtain (36) with the help of (98) and (22).

Proofs of (37) *and* (38)*:* From Propositions 6.2 and 6.3 of [15], we have

$$\phi_{2,3p+10,p+3,p+4} = \phi_{1,6p+20,1,3} \tag{105}$$

and

$$\phi_{4,3p+8,p+3,p+4} = \phi_{1,12p+32,1,3}, \tag{106}$$

respectively, where p is even. Setting $p = 2$ in (105) and (106), we readily deduce (37) and (38), respectively, with the aid of (100) and (86).

Proof of (39)*:* From Proposition 6.15 of [13], we have

$$\phi_{1,p^2+10p,5,p+5} = \phi_{p+10,p,1,2}, \tag{107}$$

where p is a positive integer. We set $p = 1$ in (107) to obtain (39) with the aid of (100) and (93).

Alternative proof of (3.12). From [3, p. 69, (36.10)], we note that

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \sum_{m=0}^{\mu/2-1} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2\nu m}), \tag{108}$$

where μ is even.

We set $\mu = 6$, $\nu = 5$ in (108), and then employ(14) to arrive at

$$2q\psi(q)\psi(q^{11}) = 2q\left\{f(q,q^{11})f(q^{55},q^{77}) + q^3\psi(q^3)\psi(q^{33}) + q^{10}f(q^5,q^7)f(q^{11},q^{121})\right\}. \quad (109)$$

Replacing q, by $-q$, and dividing both sides by $f(-q)f(-q^{11})$, and using (20) and (22), we deduce (39).

Proofs of (40) *and* (41)*:* From Propositions 6.13 and 6.19 of [13], we have

$$\phi_{2,p(p+3),1,p+2} = \phi_{p+3,2p,1,3} \quad (110)$$

and

$$\phi_{2,p^2+3p,1,p+1} = \phi_{2p+6,p,1,3}, \quad (111)$$

respectively, where p is a positive integer. We set $p = 4$ and 5 in (110) and (111), respectively, to deduce (40) and (41) with the aid of (98) and (86).

Proof of (42)*:* From Theorem 6.14, we obtain

$$\phi_{7p,5p,5,6} = \phi_{p,35p,5,2}, \quad (112)$$

where p is a positive integer. We set $p = 1$ in (112) to deduce (42) with the help of (100) and (96).

Proof of (43)*:* From Theorem 6.14, we find that

$$\phi_{p+1,p+3,1,2} = \phi_{1,p^2+4p+3,1,p+2}, \quad (113)$$

where p is a positive integer. We set $p = 4$ in (113) to deduce (43) with the aid of (98) and (93).

7. Proofs of (44)–(62):

Proofs of (44)–(47)*:* The following identities hold by Theorem 6.14:

$$\phi_{5p,9p,3,6} = \phi_{p,45p,3,6}, \quad (114)$$

$$\phi_{p,155p,1,6} = \phi_{5p,31p,5,6}, \quad (115)$$

$$\phi_{p,203p,1,6p} = \phi_{7p,29p,5,6p}, \quad (116)$$

$$\phi_{p,275p,5,6} = \phi_{11p,25p,5,6}, \quad (117)$$

where p is a positive integer. Now, we set $p = 1$ in (114), (115), (116), and (117), and then use (98), (99), and (100) to complete the proofs.

Proofs of (48)–(50)*:* The following identities hold by Theorem 6.14:

$$\phi_{4,p(p+5),1,p+4} = \phi_{p+5,4p,1,5}. \tag{118}$$

$$\phi_{6,p^2+5p,1,p+3} = \phi_{2p+10,3p,1,5}. \tag{119}$$

$$\phi_{6,p^2+5p,1,p+2} = \phi_{3p+15,2p,1,5}, \tag{120}$$

where p is a positive integer. Note that (119) and (120) were deduced by Hahn [14, Propositions 3.4.11 and 3.4.21]. Setting $p = 2$, 3, and 4 in (118), (119), and (120), respectively, and then employing (98) and (87) we complete the proofs.

Proofs of (51) *and* (52)*:* For a positive integer p, the following identities hold by Theorem 6.14:

$$\phi_{16p,8p,1,4} = \phi_{32p,4p,1,6}, \tag{121}$$

$$\phi_{1,4p+3,p,p+3} = \phi_{1,4p+3,1,4}. \tag{122}$$

set $p = 1$ and 3 in (121) and (122), respectively, and then employ (98), (99), and (94) to finish the proofs.

Proof of (53)*:* Hahn [13, Proposition 6.20] deduced the following identity from Theorem 6.14. If p is a positive integer, then

$$\phi_{1,8p+7,2p+3,p+4} = \phi_{1,8p+7,2p+1,p+2}. \tag{123}$$

We set $p = 2$ in (123) to obtain

$$\phi_{1,23,7,6} = \phi_{1,23,5,4}. \tag{124}$$

Employing (92) and (77) in (124), we find that

$$2q^{11}\{-\underline{g}_{23}^{(6,1)}\underline{g}_{1}^{(6,3)} + \underline{g}_{23}^{(6,2)}\underline{g}_{1}^{(6,2)} - \underline{g}_{23}^{(6,3)}\underline{g}_{1}^{(6,1)}\} = q^7\{-\underline{g}_{23}^{(4,1)}\underline{g}_{1}^{(4,2)} + \underline{g}_{23}^{(4,2)}\underline{g}_{1}^{(4,1)}\}. \tag{125}$$

Applying (80), (81), (82), and (79) in (125), we readily arrive at (53).

Proofs of (54) *and* (55)*:* In her thesis, Hahn [13, Propositions 3.4.3 and 3.4.23] deduced the following identities from Theorem 6.14. If p is a positive integer, then

$$\phi_{15p+80,p,1,p+5} = \phi_{5p,3p+16,3,3p+1}, \tag{126}$$

$$\phi_{p,2p^2+27p+90,1,p+5} = \phi_{p+6,2p^2+15p,3,p+3}. \tag{127}$$

We set $p = 1$ in (126) and (127), and then use (98) and (95) to arrive at the desired identities.

Proofs of (56) – (58)*:* For p even, Huang [15, Propositions 6.8, 6.7, and 6.9] deduced that

$$\phi_{5,4p+11,p+3,p+4} = \phi_{1,20p+55,1,4}, \tag{128}$$

$$\phi_{3,4p+13,p+3,p+4} = \phi_{1,12p+39,1,4}, \tag{129}$$

$$\phi_{7,4p+19,p+3,p+4} = \phi_{1,28p+63,3,4}. \tag{130}$$

Setting $p = 2$ in (128), (129), and (130), and then using (100), (94), and (95), we readily arrive at the desired identities.

Proof of (59)*:* From Theorem 6.14, we find that

$$\phi_{10p,8p,1,6p} = \phi_{16p,5p,1,7p}, \tag{131}$$

where p is a positive integer. We set $p = 1$ in (131)to arrive at (59) with the help of (98) and (88).

Proofs of (60) *and* (61)*:* In her thsis, Hahn [14, Propositions 3.4.7 and 3.4..25] deduce that

$$\phi_{2,5p^2+23p+24,1,p+2} = \phi_{p+3,10p+16,5,7} \tag{132}$$

and

$$\phi_{p,2p+18,1,p+6} = \phi_{p+9,2p,1,p+3}, \tag{133}$$

where p is a positive integer. Setting $p = 3$ in (132) and (133), and then employing (89), (90) and (98) in the resulting identities, we readily arrive at the desired results.

Proof of (62)*:* For a positive integer p, we can easily verify by Theorem 6.14 that

$$\phi_{2p,100p,5,6} = \phi_{8p,25p,5,9}. \tag{134}$$

Setting $p = 1$ in (134), and then employing (100) and (91), we obtain (62).

8. Applications to the Theory of Partitions

The identities stated in Section 3 have applications to the theory of partitions. We demonstrate this by giving combinatorial interpretations of some of these identities. For simplicity, we adopt the standard notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

We also need the notion of colored partitions. A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called *colored partitions*. For example, if 1 is allowed to have 2 colors, say r (*red*), and g (*green*), then all colored partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, and $1_r + 1_g$. We note that

$$\frac{1}{(q^u; q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to u (mod v) and have k colors.

Theorem 8.1 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±2, ±4, 6 (mod 12) with ±2, 6 (mod 12) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±2, ±4, ±5, 6 (mod 12) with ±2, 6 (mod 12) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±5 (mod 12) with ±1, ±5 (mod 12) having two colors. Then, for any positive integer $n \geq 1$, $p_1(n) + p_2(n-1) = p_3(n)$.*

Proof. Identity (28) is equivalent to

$$\frac{(q^{5\pm}; q^{12})(q^{12}; q^{12})}{(q; q)} + q\frac{(q^{1\pm}; q^{12})(q^{12}; q^{12})}{(q; q)} = \frac{(q^2; q^2)^3}{(q; q)^2(q^4; q^4)}. \tag{135}$$

Rewriting the products of the above identity subject to the common base q^{12}, we deduce that

$$\frac{1}{(q^{1\pm,2\pm,2\pm,4\pm,6,6}; q^{12})} + \frac{q}{(q^{2\pm,2\pm,4\pm,5\pm,6,6}; q^{12})} = \frac{1}{(q^{1\pm,1\pm,3\pm,5\pm,5\pm}; q^{12})}. \tag{136}$$

The three quotients of (136) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (138) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n on both sides yields the desired result.

Example. The following table illustrates the case $n = 9$ in Theorem 8.1.

$p_1(5) = 7$	$p_2(4) = 4$	$p_3(5) = 11$
$4+1$	$2_r + 2_r$	5_r
$2_r + 2_r + 1$	$2_r + 2_g$	5_g
$2_r + 2_g + 1$	$2_g + 2_g$	$3 + 1_r + 1_r$
$2_g + 2_g + 1$	4	$3 + 1_r + 1_g$
$2_g + 1 + 1 + 1$		$3 + 1_g + 1_g$
$2_r + 1 + 1 + 1$		$1_r + 1_r + 1_r + 1_r + 1_r$
$1+1+1+1+1$		$1_r + 1_r + 1_r + 1_r + 1_g$
		$1_r + 1_r + 1_r + 1_g + 1_g$
		$1_r + 1_r + 1_g + 1_g + 1_g$
		$1_r + 1_g + 1_g + 1_g + 1_g$
		$1_g + 1_g + 1_g + 1_g + 1_g$

Theorem 8.2 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 5, ± 6, ± 7, ± 9 (mod 24). Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 6, ± 9, ± 10, ± 11 (mod 24). Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 5, ± 7, ± 10, ± 11, (mod 24). Then, for any positive integer $n \geq 2$, $p_1(n) + p_2(n-2) = p_3(n)$.*

Proof. Identity (30) is equivalent to

$$\frac{(q^{1\pm};q^{12})(q^{12};q^{12})(q^{10\pm};q^{24})(q^{24};q^{24})}{(q;q)(q^2;q^2)} + q^2\frac{(q^{5\pm};q^{12})(q^{12};q^{12})(q^{2\pm};q^{24})(q^{24};q^{24})}{(q;q)(q^2;q^2)} = \frac{(q^3;q^3)(q^{24};q^{24})}{(q;q)(q^2;q^2)}. \tag{137}$$

This identity can be written as

$$\frac{(q^{1\pm},q^{12};q^{12})(q^{10\pm};q^{24})}{(q^3;q^3)} + q^2\frac{(q^{5\pm},q^{12};q^{12})(q^{2\pm};q^{24})}{(q^3;q^3)} = 1. \tag{138}$$

Rewriting all the products by the common base q^{24}, for examples, writing $(q^{1\pm};q^{12})_\infty$ as $(q^{1\pm};q^{24})_\infty(q^{11\pm};q^{24})_\infty$ and $(q^3;q^3)_\infty$ as $(q^{3\pm,6\pm,9\pm,12,24};q^{24})$ and cancelling the common terms, we obtain

$$\frac{1}{(q^{2\pm,3\pm,5\pm,6\pm,7\pm,9\pm};q^{24})} + q^2\frac{1}{(q^{1\pm,3\pm,6\pm,9\pm,10\pm,11\pm};q^{24})} = \frac{1}{(q^{1\pm,2\pm,5\pm,7\pm,10\pm,11\pm};q^{24})}. \tag{139}$$

The three quotients of (139) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (139) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^2\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{140}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (140), we arrive at the desired result.

Example. The following table illustrates the case $n = 9$ in Theorem 8.2.

$p_1(9) = 5$	$p_2(7) = 4$	$p_3(9) = 9$
9	1+1+1+1+1+1+1	2+2+2+2+1
5+2+2	3+1+1+1+1	5+2+2
7+2	6+1	7+2
3+3+3	3+3+1	2+2+2+1+1+1
3+2+2+2		7+1+1
		5+1+1+1+1
		5+2+1+1
		2+2+1+1+1+1+1
		2+1+1+1+1+1+1+1

Theorem 8.3 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 3, ± 5, ± 6, ± 7, ± 17, 18 (mod 36) with parts congruent to ± 6, 18 (mod 36) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 6, ± 11, ± 13, ± 15, 18 (mod 36) and parts congruent to ± 6, 18 (mod 36) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 10, ± 14, ± 15 (mod 36) with parts congruent to ± 3, ± 15 (mod 36) having two colors. Then, for any positive integer $n \geq 2$, $p_1(n) + p_2(n-2) = p_3(n)$.*

Proof. Identity (31) is equivalent to

$$\frac{(q^{1\pm};q^{12})(q^{12};q^{12})(q^{15\pm};q^{36})(q^{36};q^{36})}{(q;q)(q^3;q^3)} + q^2\frac{(q^{5\pm};q^{12})(q^{12};q^{12})(q^{3\pm};q^{36})(q^{36};q^{36})}{(q;q)(q^3;q^3)} = \frac{(q^4;q^4)(q^6;q^6)^5(q^9;q^9)(q^{36};q^{36})}{(q^2;q^2)^2(q^3;q^3)^3(q^{12};q^{12})^2(q^{18};q^{18})}. \tag{141}$$

Rewriting all the products in (141) by the common base q^{36}, for examples, writing $(q^{1\pm},q^{12})_\infty$ as $(q^{1\pm},q^{11\pm},q^{13\pm};q^{36})_\infty$ and $(q^3;q^3)_\infty$ as $(q^3,q^6)_\infty(q^6;q^6)_\infty$ and cancelling the common terms, we obtain

$$\frac{1}{(q^{3\pm,5\pm,6\pm,6\pm,7\pm,17\pm,18,18};q^{36})} + \frac{q^2}{(q^{1\pm,6\pm,6\pm,11\pm,13\pm,15\pm,18,18};q^{36})} = \frac{1}{(q^{2\pm,3\pm,3\pm,10\pm,14\pm,15\pm,15\pm};q^{36})}. \tag{142}$$

The three quotients of (142) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (142) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^2\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{143}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (143), we obtain the required result.

Example. The following table illustrates the case $n = 16$ in Theorem 8.3.

$p_1(16) = 6$	$p_2(14) = 8$	$p_3(16) = 14$
$3+3+3+7$	6_r+6_r+1+1	$14+2$
$5+5+3+3$	6_r+6_g+1+1	$10+3_r+3_r$
6_r+5+5	6_g+6_g+1+1	$10+3_g+3_g$
6_g+5+5	$11+1+1+1$	$10+3_g+3_r$
6_r+7+3	$13+1$	$10+2+2+2$
6_g+7+3	$6_r+1+1+1+1+1+1+1+1$	$3_r+3_r+3_r+3_r+2+2$
	$6_g+1+1+1+1+1+1+1+1$	$3_r+3_r+3_r+3_g+2+2$
	$1+1+1+1+1+1+1+1+1+1+1+1+1+1$	$3_r+3_r+3_g+3_g+2+2$
		$3_r+3_g+3_g+3_g+2+2$
		$3_g+3_g+3_g+3_g+2+2$
		$3_r+3_r+2+2+2+2+2$
		$3_r+3_g+2+2+2+2+2$
		$3_g+3_g+2+2+2+2+2$
		$2+2+2+2+2+2+2+2$

Theorem 8.4 *Let $p_1(n)$ denote the number of partitions of n into parts congruent to ±1, ±4 (mod 12) having three colors, and parts congruent to 6 (mod 12) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ±4, ±5 (mod 12) having three colors, and and parts congruent to 6 (mod 12) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±5 (mod 12) with ±1, ±5 (mod 12) having three colors. Then, for any positive integer $n > 3$, $p_1(n) + p_2(n-3) = p_3(n)$.*

Proof. Identity (32) is equivalent to

$$\frac{(q^{5\pm};q^{12})^3(q^{12};q^{12})^3}{(q;q)^3} + q^3\frac{(q^{1\pm};q^{12})^3(q^{12};q^{12})^3}{(q;q)^3} = \frac{(q^6;q^6)^3(q^4;q^4)^3}{(q;q)^3(q^3;q^3)(q^{12};q^{12})^2}. \tag{144}$$

Noting that $(q^6;q^6)_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty$, and rewriting all the products by the common base q^{12}, and cancelling the common terms, we can rewrite (144) as

$$\frac{1}{(q^{1\pm,1\pm,1\pm,4\pm,4\pm,4\pm,6,6};q^{12})} + \frac{q^3}{(q^{4\pm,4\pm,4\pm,5\pm,5\pm,5\pm,6,6};q^{12})} = \frac{1}{(q^{1\pm,1\pm,1\pm,3\pm,5\pm,5\pm,5\pm};q^{12})}. \tag{145}$$

The three quotients of (145) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (145) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n + q^3\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n, \tag{146}$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating the coefficients of q^n on both sides of (146), we obtain the desired result.

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