ON ODD PERFECT NUMBERS AND EVEN 3-PERFECT NUMBERS

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Abstract

An idea used in the characterization of even perfect numbers is used, first, to derive new necessary conditions for the existence of an odd perfect number and, second, to show that there are no even 3-perfect numbers of the form 2^aM, where M is odd and squarefree and $a \leq 718$, besides the six known examples.

–In memory of John Selfridge

1. Introduction

Some elementary background, quickly. If σ denotes the sum-of-divisors function, then a natural number N is *perfect* if $\sigma(N) = 2N$ and 3-*perfect* if $\sigma(N) = 3N$.

All known perfect numbers are even. Euler showed that such numbers are necessarily of the form $2^a(2^{a+1} - 1)$, where $2^{a+1} - 1$ is prime, and 47 of these have been found to this time. For more, see *Wikipedia*: wikipedia.org/wiki/Perfect_ number. Odd perfect numbers, if there are any, are known to satisfy a great many restrictions in their size and factorization. Many such restrictions are listed in the website just given.

There are six known 3-perfect numbers, all found more than 360 years ago, the last in 1643. They are:

$$120 = 2^3 3 \cdot 5,$$
$$672 = 2^5 3 \cdot 7,$$
$$523776 = 2^9 3 \cdot 11 \cdot 31,$$
$$459818240 = 2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73,$$

$$1476304896 = 2^{13}3 \cdot 11 \cdot 43 \cdot 127,$$
$$51001180160 = 2^{14}5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.$$

At a Western Number Theory Conference some years ago, John Selfridge quickly wrote these up by considering even 3-perfect numbers in the form 2^aM, M odd, and factorizing $2^{a+1} - 1$ for $a \leq 14$; he went on to ask what was known otherwise of such numbers. It is generally assumed, for example by Achim Flammenkamp in his *Multiply Perfect Numbers Page* (www.uni-bielefeld.de/∼achim/mpn.html), that there are no others, but this presupposes that no odd perfect numbers exist. This is a consequence of the following result; the proof is easy.

Lemma 1 *An odd number N is perfect if and only if $2N$ is 3-perfect.*

In this note we will use an idea based on a simple proof of Euler's theorem on even perfect numbers as an aid in showing that there are no 3-perfect numbers 2^aM, with M odd and squarefree, besides the six above, for $a \leq 718$. (The reason for the bound 718 is given in Section 4.) The same idea is used in determining new necessary conditions for the existence of an odd perfect number.

Regarding notation, Roman letters denote positive integers and p, with or without a subscript, is prime. The notation $m \parallel n$ means m is a unitary divisor of n, that is, $(m, n/m) = 1$. Note in particular that $p^e \parallel n$ $(e \geq 0)$ means $p^e \mid n$ but $p^{e+1} \nmid n$.

We make extensive use of the following simple lemma, detailing two properties of the function σ.

Lemma 2 (i) *For positive integers m and n, if $m \mid n$, then*

$$\frac{\sigma(m)}{m} \leq \frac{\sigma(n)}{n},$$

with equality if and only if $m = n$.

(ii) *If $m \parallel n$ (in particular, if n is squarefree and $m \mid n$), then $\sigma(m) \mid \sigma(n)$.*

Proof. (i) Notice that

$$\frac{\sigma(m)}{m} = \sum_{d|m} \frac{1}{d} \leq \sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n};$$

the statement on equality is clear.

(ii) Notice first that, if n is squarefree and $m \mid n$, then, necessarily, $m \parallel n$. Since σ is multiplicative, we have

$$\sigma(n) = \sigma\left(\frac{n}{m} \cdot m\right) = \sigma\left(\frac{n}{m}\right)\sigma(m),$$

and the result follows. $\square$

A natural number m is called *abundant* if $\sigma(m) > 2m$ and *deficient* if $\sigma(m) < 2m$. It follows from Lemma 2(i) that all proper divisors of a perfect number are deficient.

A proof that an even perfect number N must have the form $N = 2^a(2^{a+1} - 1)$, with $2^{a+1} - 1$ prime, may now be given as follows. We suppose that $N = 2^a M$, where M is odd. Then, since $\sigma(N) = 2N$,

$$(2^{a+1} - 1)\sigma(M) = 2^{a+1} M.$$

For any prime p, we have $(p^a, \sigma(p^a)) = 1$, so $2^{a+1} - 1 \mid M$ and, using Lemma 2(i),

$$\frac{2^{a+1}}{2^{a+1} - 1} = \frac{\sigma(M)}{M} \geq \frac{\sigma(2^{a+1} - 1)}{2^{a+1} - 1} \geq \frac{(2^{a+1} - 1) + 1}{2^{a+1} - 1} = \frac{2^{a+1}}{2^{a+1} - 1}.$$

It follows that there must be equality in both places, so that, first, $M = 2^{a+1} - 1$ and, second, $2^{a+1} - 1$ is prime.

This proof, based on an idea by Wayne McDaniel, was given in Cohen [3].

2. On Odd Perfect Numbers

It is another result of Euler's that an odd perfect number N, if one exists, must have the prime decomposition $N = \prod_{i=0}^{t} p_i^{a_i}$, where all p_i are odd, $p_0 \equiv a_0 \equiv 1 \pmod 4$ and $a_i \equiv 0 \pmod 2$ for $i = 1, \ldots, t$. It is easy to see that $t > 0$. (In fact, Nielsen [7] has recently shown that $t \geq 8$.) We refer to the $p_i^{a_i}$ as *components* and to p_0 as the *special prime* in the factorization of N.

Since $2N = \sigma(N) = \prod_{i=0}^{t} \sigma(p_i^{a_i})$, it is clear that $\sigma(p_i^{a_i}) \mid N$ for $i = 1, \ldots, t$. From an earlier remark, then all such divisors of N are deficient. That is, $\sigma(\sigma(p_i^{a_i})) < 2\sigma(p_i^{a_i})$ for $i = 1, \ldots, t$. It is our intention in the first place to give a slight improvement of such results based on the approach in the proof of Euler's theorem on even perfect numbers, given above. The result is contained in the following theorem.

Theorem 3 *Let p^a be a component in an odd perfect number. If p is the special prime, then*

$$\sigma(\sigma(p^a)) \leq 3p^a - 1.$$

Otherwise,

$$\sigma(\sigma(p^a)) \leq 2p^a - 2.$$

Proof. Let $N = p^a M$ be an odd perfect number, where $p \nmid M$. Since $\sigma(N) = 2N$, we have $\sigma(p^a)\sigma(M) = 2p^a M$.

Suppose that p is the special prime. Then $\sigma(p^a) \mid 2M$ and, using Lemma 2(i),

$$\frac{p^a}{\sigma(p^a)} = \frac{\sigma(M)}{2M} = \frac{\sigma(2M)}{3 \cdot 2M} \geq \frac{\sigma(\sigma(p^a))}{3\sigma(p^a)}. \tag{1}$$

Hence $\sigma(\sigma(p^a)) \leq 3p^a$. We will show that equality is not possible here. Since p is the special prime, we can write $\sigma(p^a) = 2n$, where n is odd. Then $\sigma(\sigma(p^a)) = 3\sigma(n)$. If we suppose that $\sigma(\sigma(p^a)) = 3p^a$, then $\sigma(n) = p^a$. However, it is shown in Dandapat *et al.* [4] (Theorem 1) that the equations $\sigma(n) = p^a$ and $\sigma(p^a) = 2n$ can both hold only for $p = 2^{b+1} - 1$, $b \geq 1$. Then $p \equiv 3 \pmod 4$, a contradiction. This shows that $\sigma(\sigma(p^a)) \leq 3p^a - 1$.

Next, suppose that p is not the special prime. Then $\sigma(p^a) \mid M$ and, this time,

$$\frac{2p^a}{\sigma(p^a)} = \frac{\sigma(M)}{M} \geq \frac{\sigma(\sigma(p^a))}{\sigma(p^a)}.$$

Hence $\sigma(\sigma(p^a)) \leq 2p^a$. It is known that equality is not possible here, a result due to Suryanarayana [8]; see also [4]. Furthermore, if $\sigma(\sigma(p^a)) = 2p^a - 1$, an odd number, then $\sigma(p^a) = y^2$ for some odd number y, but it follows from a result of Ljunggren [5] that this cannot happen in our situation. (In fact, it follows from Ljunggren's result that $\sigma(\sigma(p^a))$ must be even here, except for the one possibility $p^a = 3^4$.)

That completes the proof. $\square$

A computer run, testing prime powers p^a with $p \equiv a \equiv 1 \pmod 4$, for $p < 10^5$ and $p^a < 10^{15}$, produced the following cases in which $\sigma(\sigma(p^a)) > 3p^a$ and $a > 1$:

$$5^5*, \quad 5^{17}*, \quad 41^9, \quad 89^5, \quad 109^5, \quad 149^5, \quad 269^5, \quad 373^5*,$$
$$389^5, \quad 509^5, \quad 569^5, \quad 769^5, \quad 809^5, \quad 829^5, \quad 929^5;$$

and the following in which $\sigma(\sigma(p)) > 3p$:

1889, 4409, 5669, 10709, 11549, 11969, 13229, 13649, 14489,
15749, 16829∗, 18269, 20789, 25409, 28349, 30029, 30869, 34649,
38609∗, 40949, 42209, 43889, 44549∗, 44729, 45989, 51869, 52289,
53129, 53549, 57329, 59669∗, 63629, 67829, 69929, 73709, 77489,
80849, 84629, 85049, 85469, 90089, 94049∗, 94709, 95549, 97649.

By Theorem 1, none of these can be components in an odd perfect number.

It has long been known that an odd number divisible by $3 \cdot 5 \cdot 7$ cannot be perfect. For each prime power p^a above (including the second list where $a = 1$) which is *not* marked with an asterisk, and only for those, it is the case that $3 \cdot 5 \cdot 7 \mid \sigma(p^a)$, and this implies another way to show that those cannot be components of an odd perfect number. There are other prime powers in the range of p and p^a considered here that can also be excluded in that way; but those marked with an asterisk indicate that Theorem 1 may be useful in conjunction with other approaches in any investigation of the factorization of an odd perfect number.

Somewhat more is available relating to the second list. Suppose that $N = p^a M$ is an odd perfect number, with p the special prime and $p \nmid M$. Since $p \equiv a \equiv 1 \pmod 4$, we have $2 \parallel p+1$ and $p+1 = \sigma(p) \mid \sigma(p^a)$. But $\sigma(p^a) \mid 2N$, so $\frac{1}{2}(p+1) \mid N$. Thus, $\frac{1}{2}(p+1)$ is deficient: $\sigma(\frac{1}{2}(p+1)) < 2 \cdot \frac{1}{2}(p+1) = p+1$. Suppose that $\sigma(\frac{1}{2}(p+1)) = p$. Since the right-hand side is prime, we must have $\frac{1}{2}(p+1) = q^b$, for some prime q. In that case,

$$\frac{q^{b+1}-1}{q-1} = 2q^b - 1,$$

and it is easy to see that the only solution of this is $q = 2$. But then $p = 2^{b+1} - 1 \equiv 3 \pmod 4$, a contradiction. Therefore, $\sigma(\frac{1}{2}(p+1)) \le p-1$. Since $2 \parallel p+1$, we have the following result.

Theorem 4 *If p is the special prime in an odd perfect number, then*

$$\sigma(p+1) \le 3(p-1). \tag{2}$$

In (2), equality holds when $p = 5$, so it is unlikely that this result can be improved in a similarly straightforward manner. It follows that none of the 45 primes in the second list above can be the special prime in an odd perfect number. (Up to 10^7, a further 2992 primes congruent to 1 (mod 4) may be similarly shown to be ineligible as the special prime in an odd perfect number.)

Theorem 1, as it relates to non-special primes, seems to have less practical value. We will indicate later how large p might be in order that $\sigma(\sigma(p^2)) > 2p^2 - 2$, but, for interest, will first consider $\sigma(\sigma(p^a))$ for certain odd values of a and any odd prime p. (Divisibility properties quoted to the end of this section are based on results in, for example, Nagell [6], Chapter 5.)

We will show first that $\sigma(p^a)$ is abundant when $a \equiv 3 \pmod 4$. For $a = 3$,

$$\begin{aligned}\sigma(\sigma(p^3)) = \sigma((p+1)(p^2+1)) &> (p+1)(p^2+1) + \frac{p+1}{2}(p^2+1) + (p+1)\frac{p^2+1}{2} \\ &= \left(1 + \frac{1}{2} + \frac{1}{2}\right)(p+1)(p^2+1) = 2\sigma(p^3);\end{aligned}$$

and the same approach works in general since $(p+1)(p^2+1) \mid \sigma(p^a)$ for all such a.

Now take $a = 5$. It may be checked directly that $\sigma(3^5)$ is abundant. For $p > 3$,

$3 \mid p^4+p^2+1$ so

$$\begin{aligned}\sigma(\sigma(p^5)) &= \sigma((p+1)(p^4+p^2+1))\\ &> (p+1)(p^4+p^2+1) + \frac{p+1}{2}(p^4+p^2+1) + (p+1)\frac{p^4+p^2+1}{3}\\ &\quad + \frac{p+1}{2}\frac{p^4+p^2+1}{3}\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)(p+1)(p^4+p^2+1) = 2\sigma(p^5).\end{aligned}$$

An alternative approach makes use of Lemma 2(i): since 6 is a proper divisor of $(p+1)(p^4+p^2+1) = \sigma(p^5)$ for $p>3$, $\sigma(\sigma(p^5))/\sigma(p^5) > \sigma(6)/6 = 2$.

The same is true for $p>3$ and any $a \equiv 5 \pmod{12}$, but the situation is less clear for $p=3$ here and for other values of $a \equiv 1 \pmod 4$. For example, it may be checked that $\sigma(\sigma(3^9)) > 2\cdot 3^9$, but $\sigma(3^9)$ is not abundant.

The situation is similarly different when a is even, particularly so if $a+1$ is prime since then $\sigma(p^a)$ has no algebraic factorization. We can, for example, indicate as follows how the smallest p such that $\sigma(\sigma(p^2))$ is abundant might be estimated. It is known that $\sigma(p^2)$ is divisible only by 3 perhaps, but never by 3^2, and by primes congruent to $1 \pmod 3$. Some experimenting shows that the smallest product B of such primes satisfying $\sigma(B)/B > 2$ is $B = 3\cdot 7^4 13\cdot 19\cdots 73$. Then, if p is a prime such that $1+p+p^2$ is divisible by B, we have, by Lemma 2(i),

$$\frac{\sigma(\sigma(p^2))}{\sigma(p^2)} \geq \frac{\sigma(B)}{B} > 2.$$

A useful estimate for the smallest p such that $\sigma(\sigma(p^2)) > 2p^2-2$ is therefore $\sqrt{B} > 10^8$, so there is little point for the present purposes in carrying out the estimation to greater precision.

3. On Even 3-Perfect Numbers

We begin with some simple properties, given in the following two lemmas. These and other tools will be used in an investigation of 3-perfect numbers N of the form $2^a M$, where M is odd and squarefree. Three-perfect numbers of this form are called *flat* by Broughan and Zhou [2], who have derived certain other necessary properties, distinct from those obtained here.

Lemma 5 (i) *If N is a 3-perfect number, then kN is not 3-perfect for any $k>1$.*

(ii) *If N is an even perfect number, then kN is not 3-perfect for any k with $(k,N)=1$.*

(iii) *If $N = 2^a M$ is 3-perfect, where M is odd and squarefree, then $a \geq 3$ and $2^{a+1}-1$ is not prime.*

Proof. (i) Since $N \mid kN$ and $k > 1$, by Lemma 2(i),

$$\frac{\sigma(kN)}{kN} > \frac{\sigma(N)}{N} = 3,$$

so kN is not 3-perfect.

(ii) If kN is 3-perfect where $(k, N) = 1$, then $3kN = \sigma(kN) = \sigma(k)\sigma(N) = 2\sigma(k)N$. This implies that $2 \mid k$, a contradiction since N is even and $(k, N) = 1$.

(iii) Suppose first that $a = 1$. Then, by Lemma 1, M is perfect, which is not possible for M odd and squarefree, as we have seen in Section 2. So $a \geq 2$. Suppose next that $2^{a+1} - 1$ is prime. We have $\sigma(N) = 3N$, so $(2^{a+1} - 1)\sigma(M) = 2^a 3M$. Thus $2^{a+1} - 1 \mid M$, as $a \geq 2$ and $2^{a+1} - 1$ is prime. Then we have a contradiction, using (ii), since M is squarefree and $2^a(2^{a+1} - 1)$ is perfect when $2^{a+1} - 1$ is prime. Finally, then we must have $a \geq 3$ since $2^3 - 1$ is prime. $\square$

Lemma 6 *Let $N = 2^a M$ be 3-perfect, where M is odd and squarefree.*

(i) *Suppose that $p^2 \mid \sigma(2^{a+1} - 1)$ for some odd prime p. Then $3 \parallel M$, a is even, and $p = 3$. Furthermore, then $3^2 \parallel \sigma(2^{a+1} - 1)$.*

(ii) *Also, $2^{a+1} - 1$ is squarefree except if $a \equiv 5 \pmod 6$, in which case $3^2 \parallel 2^{a+1} - 1$ and $\frac{1}{9}(2^{a+1} - 1)$ is squarefree.*

Proof. Since $\sigma(N) = 3N$, so $(2^{a+1} - 1)\sigma(M) = 2^a 3M$.

(i) We show first that $3 \parallel M$. If that is not the case, then $3M$ is squarefree and $2^{a+1} - 1 \parallel 3M$. By Lemma 2(ii), $\sigma(2^{a+1} - 1) \mid \sigma(3M) = 4\sigma(M)$, so $\sigma(2^{a+1} - 1) \mid 2^{a+2} 3M$. Then we cannot have $p^2 \mid \sigma(2^{a+1} - 1)$ for any odd p. Hence $3 \parallel M$. Now consider three cases.

(a) If a is even, then $3 \nmid 2^{a+1} - 1$, so $2^{a+1} - 1 \parallel M$. Then $\sigma(2^{a+1} - 1) \mid \sigma(M) \mid 2^a 3M$, so, if $p^2 \mid \sigma(2^{a+1} - 1)$ for p odd, then $p = 3$ and clearly $3^3 \nmid \sigma(2^{a+1} - 1)$.

(b) If $a \equiv 1$ or $3 \pmod 6$, then $3 \parallel 2^{a+1} - 1$. From

$$\frac{2^{a+1} - 1}{3}\sigma(M) = 2^a M,$$

we have $\frac{1}{3}(2^{a+1} - 1) \parallel M$ and

$$\sigma(2^{a+1} - 1) = \sigma\left(3 \cdot \frac{2^{a+1} - 1}{3}\right) = 4\sigma\left(\frac{2^{a+1} - 1}{3}\right) \mid 4\sigma(M) \mid 2^{a+2} M.$$

Then we cannot have $p^2 \mid \sigma(2^{a+1} - 1)$ for any odd p.

(c) If $a \equiv 5 \pmod 6$, then $3^2 \parallel 2^{a+1} - 1$. (Notice that $3^3 \nmid 2^{a+1} - 1$ since M is squarefree.) From

$$\frac{2^{a+1} - 1}{3^2}\sigma(M) = 2^a \frac{M}{3}, \tag{3}$$

we have $\frac{1}{9}(2^{a+1} - 1) \parallel \frac{1}{3}M$, and, using (3) again,

$$\sigma(2^{a+1} - 1) = 13\sigma\left(\frac{2^{a+1} - 1}{3^2}\right) \mid 13\sigma\left(\frac{M}{3}\right) = \frac{13}{4}\sigma(M) \mid 2^{a-2} 13 \frac{M}{3}.$$

Hence, if p is odd and $p^2 \mid \sigma(2^{a+1}-1)$, then $p = 13$. But, in that case, $13 \parallel M$ so $7 \mid \sigma(13) \mid \sigma(M)$ and it follows that $7 \parallel M$. Then, since $\sigma(3 \cdot 7) = 2^5$, we have $2^5 3 \cdot 7 \mid N$, and we note that $2^5 3 \cdot 7$ is 3-perfect. By Lemma 3(i), we cannot also have $13 \parallel N$, since that would give a contradiction. Hence $p^2 \mid \sigma(2^{a+1}-1)$ is not possible for odd p.

(ii) This is evident from arguments within the proof of (i). □

In the remainder of this paper, assume that M is odd and squarefree and $N = 2^a M$ is 3-perfect. We will confine our attention to values of a satisfying $3 \leq a \leq 718$. By Lemma 3(iii), we may assume $a \notin S = \{4, 6, 12, 16, 18, 30, 60, 88, 106, 126, 520, 606\}$, since, for these values of a, $2^{a+1}-1$ is prime. The eight cases corresponding to whether or not 3, 5 and 7 are factors of N will be considered separately, and in this regard (for N as stated) the following will be useful:

$$\begin{aligned}
3 \parallel 2^{a+1}-1 \quad &\text{if and only if} \quad a \equiv 1 \text{ or } 3 \pmod 6, \\
5 \parallel 2^{a+1}-1 \quad &\text{if and only if} \quad a \equiv 3 \pmod 4, \\
7 \parallel 2^{a+1}-1 \quad &\text{if and only if} \quad a \equiv 2 \pmod 3, \\
3^2 \parallel 2^{a+1}-1 \quad &\text{if and only if} \quad a \equiv 5 \pmod 6.
\end{aligned}$$

We also make use of the following sets of primes:

$$\begin{aligned}
T_3 &= \{p > 5 : p \equiv 2 \pmod 3\}, \\
T_5 &= \{p : p \equiv 4 \pmod 5\}, \\
T_7 &= \{p : p \equiv 6 \pmod 7\}.
\end{aligned}$$

`Cases 1 and 2.` $3 \cdot 5 \parallel N$, $7 \parallel N$; $3 \cdot 5 \parallel N$, $7 \nmid N$.

By Lemma 3(iii), $2^3 \mid N$. We note that $2^3 3 \cdot 5$ is 3-perfect, so, by Lemma 3(i), there is no other 3-perfect number N with $3 \cdot 5 \parallel N$. The possibilities $7 \parallel N$ and $7 \nmid N$ are both covered here.

`Case 3.` $3 \cdot 7 \parallel N$, $5 \nmid N$.

The argument in this case is similar to that just given, producing the 3-perfect number $2^5 3 \cdot 7$. (The argument was in fact part of the proof of Lemma 4.)

`Case 4.` $3 \parallel N$, $5 \nmid N$, $7 \nmid N$.

It is convenient to consider two subcases: (i) $11 \parallel N$, and (ii) $11 \nmid N$.

(i) Write $N = 2^a 3 \cdot 11 M$, where $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, M) = 1$. Since $\sigma(N) = 3N$, we have $(2^{a+1}-1) \cdot 2^2 \cdot 2^2 3 \cdot \sigma(M) = 2^a 3^2 11 M$, so

$$(2^{a+1}-1)\sigma(M) = 2^{a-4} 3 \cdot 11 M. \tag{4}$$

Therefore, $a \geq 4$ and $2^{a+1}-1 \parallel 3 \cdot 11 M$. Since $3 \nmid M$ then $3^2 \nmid 2^{a+1}-1$, so $a \not\equiv 5 \pmod 6$. Similarly, since $5 \nmid M$, $a \not\equiv 3 \pmod 4$; and since $7 \nmid M$, $a \not\equiv 2 \pmod 3$.

Also, by Lemma 2(ii), $\sigma(2^{a+1}-1) \mid \sigma(3\cdot 11M) = 2^4 3\sigma(M)$, and from (4) it follows that $\sigma(2^{a+1}-1) \mid 2^a 3^2 11M$.

Also from (4),

$$\frac{2^{a-4}}{2^{a+1}-1} = \frac{\sigma(M)}{3\cdot 11M} = \frac{\sigma(3\cdot 11M)}{2^4 3\cdot 3\cdot 11M} \geq \frac{\sigma(2^{a+1}-1)}{2^4 3(2^{a+1}-1)},$$

by Lemma 2(i). So $\sigma(2^{a+1}-1) \leq 2^a 3$.

A computer program was run to determine values of a, $4 \leq a \leq 718$, such that $a \notin S$; $a \equiv 0$, 1, 4, 6, 9 or 10 (mod 12); Lemma 4 is satisfied; $\sigma(2^{a+1}-1)$ is not divisible by 2^{a+1}, 5 or 7; and $\sigma(2^{a+1}-1) \leq 2^a 3$.

Furthermore, the program required that $\sigma(2^{a+1}-1)$ not be divisible by any $p \in T_5 \cup T_7$. For suppose that $p \parallel \sigma(2^{a+1}-1)$ for any such prime. (The possibility $p^2 \mid \sigma(2^{a+1}-1)$ is excluded by Lemma 4, since this lemma is applied first.) Then $p \parallel M$, since $\sigma(2^{a+1}-1) \mid 2^a 3^2 11M$, so $\sigma(p) = p+1 \mid \sigma(M)$. It follows from (4) and the definitions of T_5 and T_7 that $5 \mid M$ or $7 \mid M$; but this is not possible.

The values of a that were found were: 9, 13, 22, 25, 33, 37, 48, and 121.

For $a = 9$, write $N = 2^9 3\cdot 11M$ and note that $\sigma(2^9) = 3\cdot 11\cdot 31$. Since $\sigma(N) = 3N$, we have $3\cdot 11\cdot 31\cdot 2^4 3\sigma(M) = 2^9 3^2 11M$ so that $31\sigma(M) = 2^5 M$. Hence $31 \parallel M$. The number $2^9 3\cdot 11\cdot 31$ is 3-perfect, so, by Lemma 3(i), there is no other 3-perfect number N with $2^9 3\cdot 11 \parallel N$.

For $a = 13$, write $N = 2^{13} 3\cdot 11M$ and note that $\sigma(2^{13}) = 3\cdot 43\cdot 127$. We then proceed as in the preceding paragraph, noting that $2^{13} 3\cdot 11\cdot 43\cdot 127$ is 3-perfect.

For the remaining values of a, we use factor chains to arrive at a divisor of M that is not possible—a contradiction of the statement that N is 3-perfect. The approach in part extends the use of the set $T_5 \cup T_7$ to further iterations of the sum-of-divisors function applied to $\sigma(2^a)$ and is similar to that used in many problems to do with odd perfect numbers, such as in Brent *et al.* [1].

Suppose that $a = 22$, and that $N = 2^{22} 3\cdot 11M$ is 3-perfect, where, as usual here, $(2\cdot 3\cdot 5\cdot 7\cdot 11, M) = 1$. We obtain a number of necessary prime factors of M by noting that: $\sigma(2^{22}) = 47\cdot 178481$, so $178481 \parallel M$; $\sigma(178481) = 2\cdot 3\cdot 151\cdot 197$ so $151 \parallel M$; $\sigma(151) = 2^3 19$, so $19 \parallel M$; $\sigma(19) = 2^2 5$, so $5 \mid M$. But this is a contradiction.

This argument is summarized in the following, where arguments for the remaining values of a are given. Each ends in a contradiction. There is a slightly different form of argument for $a = 33$, but the approach for each a should be self-explanatory.

Here, and in later similar lists, primes of 15 digits or more are indicated by p, q, r, $\ldots$. They retain their values for any particular value of a, but vary between values of a.

`Case 4(i).` $N = 2^a 3\cdot 11M,\ M$ squarefree, $(2\cdot 3\cdot 5\cdot 7\cdot 11, M) = 1$

$a = 22,\ \sigma(2^a) = 47\cdot 178481$:

$178481 \parallel M \Rightarrow \sigma(178481) = 2 \cdot 3 \cdot 151 \cdot 197 \mid \sigma(M) \Rightarrow 151 \parallel M$
$\Rightarrow \sigma(151) = 2^3 19 \mid \sigma(M) \Rightarrow 19 \parallel M \Rightarrow \sigma(19) = 2^2 5 \mid \sigma(M) \Rightarrow 5 \mid M;$

$a = 25,\ \sigma(2^a) = 3 \cdot 2731 \cdot 8191$:
$2731 \parallel M \Rightarrow \sigma(2731) = 2^2 683 \mid \sigma(M) \Rightarrow 683 \parallel M \Rightarrow \sigma(683) = 2^2 3^2 19 \mid \sigma(M)$
$\Rightarrow 3 \mid M;$

$a = 33,\ \sigma(2^a) = 3 \cdot 43691 \cdot 131071$:
$43691 \parallel M \Rightarrow \sigma(43691) = 2^2 3 \cdot 11 \cdot 331 \mid \sigma(M)\ \&\ 3 \mid \sigma(2^a) \Rightarrow 3 \mid M;$

$a = 37,\ \sigma(2^a) = 3 \cdot 174763 \cdot 524287$:
$174763 \parallel M \Rightarrow \sigma(174763) = 2^2 43691 \mid \sigma(M) \Rightarrow 43691 \parallel M$ (see $a = 33$);

$a = 48,\ \sigma(2^a) = 127 \cdot 4432676798593$:
$4432676798593 \parallel M \Rightarrow \sigma(4432676798593) = 2 \cdot 11 \cdot 201485309027 \mid \sigma(M)$
$\Rightarrow 201485309027 \parallel M \Rightarrow \sigma(201485309027) = 2^2 3 \cdot 7589 \cdot 2212471 \mid \sigma(M)$
$\Rightarrow 7589 \parallel M$
$\Rightarrow \sigma(7589) = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 23 \mid \sigma(M) \Rightarrow 5 \mid M;$

$a = 121,\ \sigma(2^a) = 3 \cdot p \cdot q\ (p < q)$:
$p \parallel M \Rightarrow \sigma(p) = 2^2 2833 \cdot 37171 \cdot 1824726041 \mid \sigma(M) \Rightarrow 1824726041 \parallel M$
$\Rightarrow \sigma(1824726041) = 2 \cdot 3^4 17 \cdot 79 \cdot 8387 \mid \sigma(M) \Rightarrow 3 \mid M.$

`Case 4(ii)`. Write $N = 2^a 3M$, where $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, M) = 1$. Since $\sigma(N) = 3N$, we have $(2^{a+1} - 1)2^2\sigma(M) = 2^a 3^2 M$, so

$$(2^{a+1} - 1)\sigma(M) = 2^{a-2}3^2 M. \tag{5}$$

Then $2^{a+1} - 1 \mid 3^2 M$. We will consider separately three possibilities: (a) $3 \nmid 2^{a+1} - 1$, (b) $3 \parallel 2^{a+1} - 1$, (c) $3^2 \parallel 2^{a+1} - 1$.

`Subcase 4(ii)a`. Notice that a is even. Furthermore, $a \not\equiv 2$ or $8 \pmod{12}$, else $7 \mid 2^{a+1} - 1$. Since M is squarefree, we have $2^{a+1} - 1 \parallel M$ so $\sigma(2^{a+1} - 1) \mid \sigma(M) \mid 2^{a-2}3^2 M$. Also, using (5) and Lemma 2(i),

$$\frac{2^{a-2}3^2}{2^{a+1} - 1} = \frac{\sigma(M)}{M} \geq \frac{\sigma(2^{a+1} - 1)}{2^{a+1} - 1},$$

so $\sigma(2^{a+1} - 1) \leq 2^{a-2}3^2$.

A computer program was run to determine values of $a \equiv 0, 4, 6$ or $10 \pmod{12}$, $4 \leq a \leq 718$, such that $a \notin S$; Lemma 4 is satisfied; $\sigma(2^{a+1} - 1)$ is not divisible by 2^{a-1}, 5, 7, 11 or any prime in $T_5 \cup T_7$ (see Case 4(i) and the argument there); and $\sigma(2^{a+1} - 1) \leq 2^{a-2}3^2$. The only value of $a \leq 718$ that was found was 22, eliminated as for Case 4(i).

Subcase 4(ii)b. Notice that $a \equiv 1$ or $3 \pmod 6$. Further, $a \not\equiv 3 \pmod 4$, else $5 \mid 2^{a+1}-1$, and $a \not\equiv 9 \pmod{10}$, else $11 \mid 2^{a+1}-1$. From (5), we have $\frac{1}{3}(2^{a+1}-1) \parallel M$ so $\sigma(2^{a+1}-1) = 4\sigma(\frac{1}{3}(2^{a+1}-1)) \mid 4\sigma(M) \mid 2^a 3^2 M$. Using (5) and Lemma 2(i),

$$\frac{2^{a-2}3}{\frac{1}{3}(2^{a+1}-1)} = \frac{\sigma(M)}{M} \geq \frac{\sigma(\frac{1}{3}(2^{a+1}-1))}{\frac{1}{3}(2^{a+1}-1)},$$

so

$$\sigma(2^{a+1}-1) = 4\sigma\left(\frac{2^{a+1}-1}{3}\right) \leq 2^a 3.$$

A computer program was run to determine values of $a \leq 718$ such that $a \equiv 1$ or $9 \pmod{12}$ and $a \not\equiv 9 \pmod{10}$; $a \notin S$; Lemma 4 is satisfied; $\sigma(2^{a+1}-1)$ is not divisible by 2^{a+1}, 5, 7, 11 or any prime in $T_5 \cup T_7$ (see Case 4(i) and the argument there); and $\sigma(2^{a+1}-1) \leq 2^a 3$. The values found were 25, 37 and 121, all of which may be eliminated as for Case 4(i).

(c) Notice that $a \equiv 5 \pmod 6$, since $3^2 \parallel 2^{a+1}-1$. But $a \not\equiv 2 \pmod 3$, else $7 \mid 2^{a+1}-1$, so this subcase is empty.

Case 5. $5 \cdot 7 \parallel N$, $3 \nmid N$.

Write $N = 2^a 5 \cdot 7M$, where M is squarefree and $(2 \cdot 3 \cdot 5 \cdot 7, M) = 1$. Since $\sigma(N) = 3N$, we have $(2^{a+1}-1) \cdot 2 \cdot 3 \cdot 2^3 \cdot \sigma(M) = 2^a 3 \cdot 5 \cdot 7M$, so

$$(2^{a+1}-1)\sigma(M) = 2^{a-4} 5 \cdot 7M. \tag{6}$$

Therefore, $2^{a+1}-1 \parallel 5 \cdot 7M$. We must have $3 \nmid 2^{a+1}-1$, so a is even. By Lemma 2(ii), $\sigma(2^{a+1}-1) \mid \sigma(5 \cdot 7M) = 2^4 3\sigma(M)$, and, from (6), it follows that $\sigma(2^{a+1}-1) \mid 2^a 3 \cdot 5 \cdot 7M$. Notice that $3^2 \nmid \sigma(2^{a+1}-1)$.

Also from (6),

$$\frac{2^{a-4}}{2^{a+1}-1} = \frac{\sigma(M)}{5 \cdot 7M} = \frac{\sigma(5 \cdot 7M)}{2^4 3 \cdot 5 \cdot 7M} \geq \frac{\sigma(2^{a+1}-1)}{2^4 3(2^{a+1}-1)},$$

by Lemma 2(i). So $\sigma(2^{a+1}-1) \leq 2^a 3$.

A computer program was run to determine even values of a, $4 \leq a \leq 718$, such that $a \notin S$; Lemma 4 is satisfied, except that $3^2 \nmid \sigma(2^{a+1}-1)$; $\sigma(2^{a+1}-1)$ is not divisible by 2^{a+1} or any prime in T_3; and $\sigma(2^{a+1}-1) \leq 2^a 3$. (If $p \mid \sigma(2^{a+1}-1)$ for any $p \in T_3$, then we see, along the lines of the corresponding argument in Case 4(i), that $3 \mid M$, a contradiction.) The values found were: 8, 14, 50, 64 and 66.

For $a = 8$, write $N = 2^8 5 \cdot 7M$ and note that $\sigma(2^8) = 7 \cdot 73$, $\sigma(73) = 2 \cdot 37$, $\sigma(37) = 2 \cdot 19$, and $\sigma(19) = 2^2 5$. It follows that $19 \cdot 37 \cdot 73 \parallel M$ and that $\sigma(2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73) = 2^8 3 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$. That is, $2^8 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$ is 3-perfect and, by Lemma 3(i), there is no other 3-perfect number N with $2^8 5 \cdot 7 \parallel N$.

For $a = 14$, write $N = 2^{14} 5 \cdot 7M$ and note that $\sigma(2^{14}) = 7 \cdot 31 \cdot 151$, $\sigma(151) = 2^3 19$ and $\sigma(19) = 2^2 5$. As above, we observe that $\sigma(2^{14} 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151) = 2^{14} 3 \cdot 5 \cdot 7 \cdot$

$19 \cdot 31 \cdot 151$. That is, $2^{14}5 \cdot 7 \cdot 19 \cdot 31 \cdot 151$ is 3-perfect and, by Lemma 3(i), there is no other 3-perfect number N with $2^{14}5 \cdot 7 \parallel N$.

The remaining values may be treated as follows.

Case 5. $N = 2^a 5 \cdot 7M,\ M$ squarefree, $(2 \cdot 3 \cdot 5 \cdot 7, M) = 1$

$a = 50,\ \sigma(2^a) = 7 \cdot 103 \cdot 2143 \cdot 11119 \cdot 131071$:

$103 \parallel M \Rightarrow \sigma(103) = 2^3 13 \mid \sigma(M)$

$\Rightarrow 13 \parallel M \Rightarrow \sigma(13) = 2 \cdot 7 \mid \sigma(M)\ \&\ 7 \mid \sigma(2^a) \Rightarrow 7 \mid M$;

$a = 64,\ \sigma(2^a) = 31 \cdot 8191 \cdot p$:

$p \parallel M \Rightarrow \sigma(p) = 2^5 19 \cdot 238972275589 \mid \sigma(M) \Rightarrow 19 \cdot 238972275589 \parallel M$

$\Rightarrow \sigma(19 \cdot 238972275589) = 2^3 5^2 23897227559 \mid \sigma(M) \Rightarrow 5 \mid M$;

$a = 66,\ \sigma(2^a) = 193707721 \cdot 761838257287$:

$193707721 \parallel M \Rightarrow \sigma(193707721) = 2 \cdot 13 \cdot 7450297 \mid \sigma(M) \Rightarrow 7450297 \parallel M$

$\Rightarrow \sigma(7450297) = 2 \cdot 23 \cdot 149 \cdot 1087 \mid \sigma(M)$

$\Rightarrow 149 \parallel M \Rightarrow \sigma(149) = 2 \cdot 3 \cdot 5^2 \mid \sigma(M) \Rightarrow 5 \mid M$.

Case 6. $5 \parallel N,\ 3 \nmid N,\ 7 \nmid N$.

Write $N = 2^a 5M$, where M is squarefree and $(2 \cdot 3 \cdot 5 \cdot 7, M) = 1$. Since $\sigma(N) = 3N$, we have $(2^{a+1} - 1)2 \cdot 3\sigma(M) = 2^a 3 \cdot 5M$, so

$$(2^{a+1} - 1)\sigma(M) = 2^{a-1} 5M. \tag{7}$$

Therefore, $2^{a+1} - 1 \parallel 5M$. We must have $3 \nmid 2^{a+1} - 1$, so a is even; and $a \not\equiv 2 \pmod 3$, since $7 \nmid 2^{a+1} - 1$. By Lemma 2(ii), $\sigma(2^{a+1} - 1) \mid \sigma(5M) = 2 \cdot 3\sigma(M)$, and, from (7), it follows that $\sigma(2^{a+1} - 1) \mid 2^a 3 \cdot 5M$. Notice that $3^2 \nmid \sigma(2^{a+1} - 1)$.

Also from (7),

$$\frac{2^{a-1}}{2^{a+1} - 1} = \frac{\sigma(M)}{5M} = \frac{\sigma(5M)}{2 \cdot 3 \cdot 5M} \geq \frac{\sigma(2^{a+1} - 1)}{2 \cdot 3(2^{a+1} - 1)},$$

by Lemma 2(i). So $\sigma(2^{a+1} - 1) \leq 2^a 3$.

A computer program was run to determine values of $a \equiv 0, 4, 6 \text{ or } 10 \pmod{12}$, $4 \leq a \leq 718$, such that $a \notin S$; Lemma 4 is satisfied, except that $3^2 \nmid \sigma(2^{a+1} - 1)$; $\sigma(2^{a+1} - 1)$ is not divisible by 7, 2^{a+1} or any prime in $T_3 \cup T_7$; and $\sigma(2^{a+1} - 1) \leq 2^a 3$. The only value found was 64, which may be treated as in Case 5.

Case 7. $7 \parallel N,\ 3 \nmid N,\ 5 \nmid N$.

Write $N = 2^a 7M$, where M is squarefree and $(2 \cdot 3 \cdot 5 \cdot 7, M) = 1$. Since $\sigma(N) = 3N$, we have $(2^{a+1} - 1)2^3\sigma(M) = 2^a 3 \cdot 7M$, so

$$(2^{a+1} - 1)\sigma(M) = 2^{a-3} 3 \cdot 7M. \tag{8}$$

Therefore, $2^{a+1}-1 \parallel 3\cdot 7M$ and, since $3\nmid M$, $a \not\equiv 5 \pmod 6$. Also, $a\not\equiv 3 \pmod 4$, else $5 \mid 2^{a+1}-1$. By Lemma 2(ii), $\sigma(2^{a+1}-1) \mid \sigma(3\cdot 7M) = 2^5\sigma(M)$, and, from (8), it follows that $\sigma(2^{a+1}-1) \mid 2^{a+2}3\cdot 7M$. Notice that $3^2 \nmid \sigma(2^{a+1}-1)$.

Also from (8),

$$\frac{2^{a-3}}{2^{a+1}-1} = \frac{\sigma(M)}{3\cdot 7M} = \frac{\sigma(3\cdot 7M)}{2^5 3\cdot 7M} \geq \frac{\sigma(2^{a+1}-1)}{2^5(2^{a+1}-1)},$$

by Lemma 2(i). So $\sigma(2^{a+1}-1) \leq 2^{a+2}$.

A computer program was run to determine values of a, $4\leq a\leq 718$, such that $a\not\equiv 5 \pmod 6$ and $a\not\equiv 3 \pmod 4$; $a\notin S$; Lemma 4 is satisfied, but $3^2\nmid\sigma(2^{a+1}-1)$; $\sigma(2^{a+1}-1)$ is not divisible by 5, 2^{a+3} or $p\in T_5$; and $\sigma(2^{a+1}-1)\leq 2^{a+2}$. The 13 values found may be shown as follows not to lead to 3-perfect numbers.

`Case 8`. $N = 2^a 7M$, M squarefree, $(2\cdot 3\cdot 5\cdot 7, M) = 1$

$a = 8$, $\sigma(2^a) = 7\cdot 73$:

$73 \parallel M \Rightarrow \sigma(73) = 2\cdot 37 \mid \sigma(M) \Rightarrow 37 \parallel M \Rightarrow \sigma(37) = 2\cdot 19 \mid \sigma(M) \Rightarrow 19 \parallel M$
$\Rightarrow \sigma(19) = 2^2 5 \mid \sigma(M) \Rightarrow 5 \mid M$;

$a = 9$, $\sigma(2^a) = 3\cdot 11\cdot 31$:

$11 \parallel M \Rightarrow \sigma(11) = 2^2 3 \mid \sigma(M)$ & $3 \mid \sigma(2^a) \Rightarrow 3 \mid M$;

$a = 13$, $\sigma(2^a) = 3\cdot 43\cdot 127$:

$43 \parallel M \Rightarrow \sigma(43) = 2^2 11 \mid \sigma(M) \Rightarrow 11 \parallel M$ (see $a = 9$);

$a = 24$, $\sigma(2^a) = 31\cdot 601\cdot 1801$:

$1801 \parallel M \Rightarrow \sigma(1801) = 2\cdot 17\cdot 53 \mid \sigma(M) \Rightarrow 17 \parallel M \Rightarrow \sigma(17) = 2\cdot 3^2 \mid \sigma(M)$
$\Rightarrow 3 \mid M$;

$a = 25$ (see Case 4(i));

$a = 33$ (see Case 4(i));

$a = 37$ (see Case 4(i));

$a = 48$ (see Case 4(i));

$a = 66$ (see Case 5);

$a = 84$, $\sigma(2^a) = 31\cdot 131071\cdot p$:

$p \parallel M \Rightarrow \sigma(p) = 2^5 13\cdot 23\cdot 6163\cdot 161461131023 \mid \sigma(M) \Rightarrow 6163 \parallel M$
$\Rightarrow \sigma(6163) = 2^2 23\cdot 67 \mid \sigma(M)$ & $23 \mid \sigma(p) \Rightarrow 23^2 \mid M$;

$a = 121$ (see Case 4(i));

$a = 253$, $\sigma(2^a) = 3\cdot p\cdot q$ $(p<q)$:

$p \parallel M \Rightarrow \sigma(p) = 2^2 11\cdot 251\cdot 4051\cdot 229668251\cdot r \mid \sigma(M) \Rightarrow 251 \parallel M$
$\Rightarrow \sigma(251) = 2^2 3^2 7 \mid \sigma(M) \Rightarrow 3 \mid M$;

$a = 266,\ \sigma(2^a) = 7 \cdot 78903841 \cdot 28753302853087 \cdot p \cdot q\ (p < q):$

$$q \parallel M \Rightarrow \sigma(q) = 2 \cdot 877 \cdot 10643483 \cdot r \mid \sigma(M)$$
$$\Rightarrow 877 \parallel M \Rightarrow \sigma(877) = 2 \cdot 439 \mid \sigma(M)$$
$$\Rightarrow 439 \parallel M \Rightarrow \sigma(439) = 2^3 5 \cdot 11 \mid \sigma(M) \Rightarrow 5 \mid M.$$

Case 9. $3 \nmid N,\ 5 \nmid N,\ 7 \nmid N.$

Since $\sigma(N) = 3N$, we have $(2^{a+1} - 1)\sigma(M) = 2^a 3M$, where $N = 2^a M$, M is squarefree and $(2 \cdot 3 \cdot 5 \cdot 7, M) = 1$. We consider two subcases: (i) $3 \parallel 2^{a+1} - 1$, and (ii) $3 \nmid 2^{a+1} - 1$.

(i) Notice that $a \equiv 1$ or $3 \pmod 6$. Further, $a \not\equiv 3 \pmod 4$, else $5 \mid 2^{a+1} - 1$, and $a \not\equiv 2 \pmod 3$, else $7 \mid 2^{a+1} - 1$. We have $\frac{1}{3}(2^{a+1} - 1) \parallel M$ so $\sigma(2^{a+1} - 1) = 4\sigma(\frac{1}{3}(2^{a+1} - 1)) \mid 4\sigma(M) \mid 2^{a+2}3M$. Using Lemma 2(i),

$$\frac{2^a}{\frac{1}{3}(2^{a+1} - 1)} = \frac{\sigma(M)}{M} \geq \frac{\sigma(\frac{1}{3}(2^{a+1} - 1))}{\frac{1}{3}(2^{a+1} - 1)},$$

so $\sigma(2^{a+1} - 1) \leq 2^{a+2}$.

A computer program was run to determine values of $a \leq 718$ such that $a \equiv 1$ or $9 \pmod{12}$; $a \notin S$; Lemma 4 is satisfied; $\sigma(2^{a+1} - 1)$ is not divisible by 2^{a+3}, 5, 7 or $p \in T_5 \cup T_7$; and $\sigma(2^{a+1} - 1) \leq 2^{a+2}$. The values found were 9 and 13 (eliminated as in Case 7), and 25, 33, 37 and 121 (eliminated as in Case 4(i)).

(ii) Notice that a is even. Furthermore, $a \not\equiv 2$ or $8 \pmod{12}$, else $7 \mid 2^{a+1} - 1$. Since M is squarefree, we have $2^{a+1} - 1 \parallel M$ so $\sigma(2^{a+1} - 1) \mid \sigma(M) \mid 2^a 3M$. Also, using Lemma 2(i),

$$\frac{2^a 3}{2^{a+1} - 1} = \frac{\sigma(M)}{M} \geq \frac{\sigma(2^{a+1} - 1)}{2^{a+1} - 1},$$

so $\sigma(2^{a+1} - 1) \leq 2^a 3$. A computer program was run to determine values of $a \equiv 0$, 4, 6 or $10 \pmod{12}$, such that $a \notin S$; Lemma 4 is satisfied, but $3^2 \nmid \sigma(2^{a+1} - 1)$; $\sigma(2^{a+1} - 1)$ is not divisible by 2^{a+1}, 5, 7 or $p \in T_5 \cup T_7$; and $\sigma(2^{a+1} - 1) \leq 2^a 3$. The only value of $a \leq 718$ that was found was 48, which may be eliminated as in Case 4(i).

We have completed the proof of our main theorem:

Theorem 7 *The only 3-perfect numbers of the form $2^a M$, for M odd and squarefree and $a \leq 718$, are the six known in the literature.*

4. Further Comments

Almost certainly, if there were another such 3-perfect number in the range of a considered here then it would have been discovered in the wider searches for multiperfect numbers (or *multiply perfect* numbers, satisfying $\sigma(N) = kN$ for integer $k \geq 1$), as described on Flammenkamp's website, mentioned in Section 1. But none of these searches, even the latest hugely intensive and extensive ones, indicate how exhaustive they are in any aspect.

Our Theorem 3 has a small amount of personal intervention in its proof, as is evident from the displayed lists where alternative approaches are available, probably in all cases. It would not have been difficult to allow a single program, along the lines used here or with a more direct factor-chain approach, to complete the whole job. Our approach was adopted firstly to maintain the spirit of dependence on ideas discussed in Section 1, and secondly to see the brevity of the computations when "extraneous" information (here, whether 3, 5, 7 are factors or not of an even 3-perfect number) is introduced. In this sense, it is remarkable that no values of a, for $267 \leq a \leq 718$, needed special consideration. (It is likely that the notion of extraneous information could be used profitably in searches such as that of Brent *et al.* [1].)

Furthermore, the search in the present paper could no doubt be carried further, perhaps even to the extent of the Cunningham Project's complete factorizations of $2^{a+1} - 1$ for all a to around 920, and, with gaps, to 1928. (For the Cunningham tables, see homes.cerias.purdue.edu/∼ssw/cun/.) The Cunningham Project's factorizations for larger a were uploaded to our program, as were factorizations of $\sigma(2^{a+1} - 1)$, as accumulated by the second author. We considered only those values of a for which complete factorizations of both $2^{a+1} - 1$ and $\sigma(2^{a+1} - 1)$ were available. Our bound of 718 for a reflects the fact that $\sigma(2^{720} - 1)$ has a composite factor of 164 digits (with unknown factorization). Partial factorizations, of either $2^{a+1} - 1$ or $\sigma(2^{a+1} - 1)$, or both, would no doubt produce results in many cases, but we have chosen not to continue the work to that extent.

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