SOME PROBLEMS IN VIEW OF RECENT DEVELOPMENTS OF THE ERDŐS GINZBURG ZIV THEOREM

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Abstract

Two conjectures concerning the Erdős Ginzburg Ziv theorem were recently confirmed. Reiher and di Fiore proved independently the two dimension analogue of the EGZ theorem, as conjectured by Kemnitz, and Grynkiewicz proved the weighed generalization of the EGZ theorem as conjectured by Caro. These developments trigger some further problems. First, we will present computer experiments that at least for small numbers reveal very simple phenomena of zero sum theorems that seem to be difficult to prove. Next, we will examine the notion of generalization of Ramsey type theorems in the sense of a given zero sum theorem in view of the new developments.

1. Introduction

We begin by recalling the extended version of the Erdős Ginzburg Ziv theorem known today as the EGZ theorem.

Theorem 1

(a) *Suppose $G = (G, +, 0)$ is an abelian group of order n. Let I be a set of indices and let $\{a_i\}_{i \in I}$ be an indexed set of elements from G. If $|I| = 2n - 1$, then there exists a subset $J \subset I$, with $|J| = n$, such that $\sum_{j \in J} a_j = 0$.*

(b) *Furthermore, if $|I| = 2n - 2$ and there does not exist a subset $J \subset I$, with $|J| = n$, such that $\sum_{j \in J} a_j = 0$, then (b)(i) and (b)(ii) hold:*

    (i) *The group G is cyclic.*

(ii) Half of the a_i's are equal to $x \in G$, the other half of the a_i's are equal to $y \in G$, where $x - y$ is a generator of G.

Part (a) was originally proved in [20] for cyclic groups only, but it is not difficult to deduce the proof for any abelian group. Several other proofs appear in [2] and another one in [35]. Part (b) is known as a solution to an inverse problem. The proofs of parts (b)(i) and (b)(ii) can be deduced from [7] and [16], respectively. See also [40].

The author is interested mainly in generalizations of extremal theorems from combinatorics in the sense of a given zero sum theorem. But what is a zero sum theorem? To define rigorously what is a zero sum theorem, is quite a challenging task, and such an attempt will require the introduction of some intricate notation. But more disturbing will be that after our efforts, some new zero sum theorems in the philosophical sense will be discovered that are not covered by our definition. Thus, we prefer to accept the intuitive notion of a zero sum theorem. Similarly to Ramsey Theory being philosophically defined as "total disorder is impossible" we say that "avoiding zero configurations is impossible." First let's look at some zero sum theorems. The following one is known as the cave's man theorem, [21].

Theorem 2 *Suppose $G = (G, +, 0)$ is an abelian group. If S is a sequence of elements from G of length $|G|$, then S contains a subsequence all of whose elements add up to zero.*

Though, Theorem 2 falls into the category of a zero sum theorem, however in the foregoing we will be interested in theorems that resemble more the EGZ theorem. Here are three important characteristics of the EGZ theorem.

Observation 3

(a) The number of elements involved in the "zero sum configuration" is predetermined, say n. In the EGZ theorem the number n is the order of the group.

(b) If the same element from G appears in the sequence n times, then the conclusion of the theorem follows trivially.

(c) If the number of distinct elements in the sequence is bounded by some number, then the conclusion of the theorem follows easily. In the EGZ theorem this number is two, and in this case the EGZ theorem follows from the pigeon-hole principle.

We will demonstrate another zero sum theorem, Theorem 5, which has the above characteristics. But first we need to introduce some definitions.

Definition 4 *Let n and r be two positive integers. Define $Union(\mathbb{Z}_n^r)$ to be the union of r disjoint copies of $\mathbb{Z}_n$ where addition can be performed within each copy only.*

$$Union(\mathbb{Z}_n^r) \;=\; \underbrace{\mathbb{Z}_n \,\dot{\cup}\, \mathbb{Z}_n \,\dot{\cup}\, \ldots \,\dot{\cup}\, \mathbb{Z}_n}_{r \; times} \; and \; Union^{+1}(\mathbb{Z}_n^r) \;=\; \underbrace{\mathbb{Z}_n \,\dot{\cup}\, \mathbb{Z}_n \,\dot{\cup}\, \ldots \,\dot{\cup}\, \mathbb{Z}_n}_{r \; times} \cup \{1\},$$

where $1 \in \mathbb{Z}_n$.

Theorem 5

   (a) *If S is a sequence with elements from $Union(\mathbb{Z}_n^r)$, of length $2r(n-1)+1$, then S contains a subsequence of length n, all whose elements add up to zero.*

   (b) *If S is a sequence with elements from $Union^{+1}(\mathbb{Z}_n^r)$, of length $(2r+1)(n-1)+1$, then S contains a subsequence of length n, all whose elements add up to zero.*

The proof of Theorem 5 follows easily from the pigeon-hole principle and the EGZ theorem. Furthermore, if in part (a), $r = 1$, then we get the EGZ theorem. It is worthwhile to mention that Theorem 5 generalizes the following form of the pigeon principle: "If $2r(n-1)+1$ $((2r+1)(n-1)+1)$ pigeons are distributed into $2r$ in part (a) ($2r+1$ in part (b)) holes, then there are at least n pigeons in one of the holes."

In Section 2 we focus on three generalizations of the EGZ theorem. Two of them have been recently proved and the third one is still an open problem, but some recent progress has been made. In Section 3 we suggest a new line of zero sum theorems and make several conjectures. In Section 4 we introduce the notion of generalization in the sense of the EGZ theorem, but with some modification the reader can define the notion of generalization in the sense of other zero sum theorems.

2. On Three Generalizations of the EGZ Theorem

The general background reference to these kinds of problems in additive number theory is [39]. Many generalizations of the EGZ theorem have been proved and many others are still open problems. For example, see [11], [15], [24], [28], [44] and [46]. There have been many other recent developments of the EGZ theorem, for example see [1], [9], [25], [32], [33], [47] and [48]. In this section we will focus only on three generalizations, not mentioned above. We begin with the multiplicity conjecture conjectured in [5].

Conjecture 6 *Suppose s and n are two integers, where $n \geq 2$. If $a_1, a_2, \ldots, a_s$ is a sequence of elements from $\mathbb{Z}_n$, then there are at least $\binom{\lfloor \frac{s}{2} \rfloor}{n} + \binom{\lceil \frac{s}{2} \rceil}{n}$ subsets $I \subset \{1, \ldots, s\}$, such that $|I| = n$ and $\sum_{j \in I} a_j = 0$.*

If in Conjecture 6, we substitute $s = 2n - 1$, then the EGZ theorem follows. In [37] the conjecture has been proved if $n \in \{p^\alpha, p^\alpha q\}$, where p and q are primes. Furthermore, in [23] it has been proved asymptotically for a fixed n and $s \to \infty$. And recently it was proved in [30] for $s \leq 6\frac{1}{3}n$.

The second generalization of the EGZ theorem Theorem 7, was proved by Grykiewicz in [31]. The theorem was conjectured by Caro [17].

Theorem 7 (CG) *Let $w_1, w_2, \ldots, w_n \in \mathbb{Z}_n$ and suppose $\sum_{i=1}^{n} w_i = 0$. If $a_1, a_2, \ldots, a_{2n-1}$ is a sequence of elements from $\mathbb{Z}_n$, then there exists a subset $J \subset \{1, 2, \ldots, 2n-1\}$, with $|J| = n$, such that with some rearrangement of the corresponding subsequence we get: $\sum_{k \in I, i=1}^{n} a_k w_i = 0$.*

If in Theorem 7 we have $w_i = 1$ for every i, then the EGZ theorem follows.

The third generalization of the EGZ theorem was proved independently by Reiher [41] and di Fiore as reported in [45]. The theorem was conjectured by Kemnitz [36].

Theorem 8 (KFR) *Every sequence of elements from $\mathbb{Z}_n \oplus \mathbb{Z}_n$ of length $4n - 3$ contains a subsequence of length n, such that all of its elements add up to 0.*

It is worthwhile to comment that Theorem 8 can be viewed as a two dimensional generalization of the EGZ theorem. However, the innocent extrapolation for higher dimensions does not work, [19]. For a unified approach to the EGZ theorem and the KFR theorem see [18].

We conclude this section with the multiplicity weighted version of Theorem 8.

Conjecture 9 *Suppose s and n are two integers, where $n \geq 2$, and let $s = 4k+r$, where $r \in \{0, 1, 2, 3\}$. Furthermore, suppose $w_1, w_2, \ldots, w_n \in \mathbb{Z}_n$ satisfy $\sum_{i=1}^{n} w_i = 0$. If $a_1, a_2, \ldots, a_s$ is a sequence of elements from $\mathbb{Z}_n \oplus \mathbb{Z}_n$, then there are at least $(4-r)\binom{k}{n} + r\binom{k+1}{n}$ subsequences $I \subset \{1, 2, \ldots, s\}$, such that $|I| = n$, and $\sum_{k \in I, i=1}^{n} a_k w_i = 0$.*

3. Conjectures About Other Zero Sum Theorems

The novelty of Theorem 7 is that we look at $\mathbb{Z}_n$ as a ring or a module and not just a group and it suggests many other theorems along the same line. We make the following conjectures.

Conjecture 10 *Let $\mathbb{Z}_n$ be the ring of all residues modulo n. Define $f(n)$ to be the smallest positive integer such that every sequence of elements from $\mathbb{Z}_n$ of length $f(n)$ contains a subsequence $a_1, \ldots, a_n$ and a permutation σ such that $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i a_{\sigma(i)} = 0$. Then*

$$f(n) = \begin{cases} 2n+1 & \text{if } n = 3 \pmod 4 \\ 2n-1 & \text{otherwise} \end{cases}.$$

If $n = 3 \pmod 4$, then the extremal sequence is $0^{n-1}1^2 2^{n-1}$. If $n \neq 3 \pmod 4$, then the extremal sequence is $0^{n-1}1^{n-1}$.

The conjecture was confirmed for $n \leq 13$.

Conjecture 11 *Let $\mathbb{Z}_n$ be the ring of all residues modulo n, and let u_i and w_i, for $i = 1, 2, \ldots, n$ be elements of $\mathbb{Z}_n$, satisfying $\sum_{i=1}^{n} u_i = 0$ and $\sum_{i=1}^{n} w_i = 0$. Define $f(n)$ to be the smallest positive integer such that every sequence of elements from $\mathbb{Z}_n$ of length $f(n)$ contains a subsequence $a_1, a_2, \ldots, a_n$ and two permutations σ and θ such that $\sum_{i=1}^{n} u_i a_{\sigma(i)} = 0$ and $\sum_{i=1}^{n} w_i a_{\theta(i)} = 0$. Then*

$$f(n) = \begin{cases} 2n+1 & \text{if } n \text{ is odd} \\ 2n-1 & \text{if } n \text{ is even} \end{cases}.$$

If n is even, then the extremal sequence is $0^{n-1}1^{n-1}$. If n is odd, then the extremal sequence is $0^{n-1}1^{n-1}2^1(n-1)^1$.

The conjecture was confirmed for $n \leq 7$.

Conjecture 12 *Let $\mathbb{Z}_n$ be the ring of all residues modulo n. Define $f(n)$ to be the smallest positive integer such that every sequence of elements from $\mathbb{Z}_n$ of length $f(n)$ contains two disjoint subsequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$ and $\sum_{i=1}^{n} a_i b_{\sigma(i)} = 0$, for some permutation σ. Then $f(n) = 3n - 1$. The extremal sequence is given by $0^{2n-1}1^{n-1}$ and it is unique.*

The conjecture was confirmed for $n \leq 12$.

Conjecture 13 *Let $\mathbb{Z}_n$ be the ring of all residues modulo n. Define $f(n)$ to be the smallest positive integer such that every sequence of elements from $\mathbb{Z}_n$ of length $f(n)$ contains three pairwise disjoint subsequences $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, $c_1, c_2, \ldots, c_n$ and permutations $\alpha, \beta, \gamma, \sigma, \theta$ such that:*

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i b_{\alpha(i)} = \sum_{i=1}^{n} a_i c_{\beta(i)} = \sum_{i=1}^{n} b_i c_{\gamma(i)} = \sum_{i=1}^{n} a_i b_{\sigma(i)} c_{\theta(i)} = 0.$$

Then $f(n) = 4n - 1$. The extremal sequence is given by $0^{3n-1}1^{n-1}$, or by $0^{2n-1}1^{2n-1}$ and it is unique.

The conjecture was confirmed for $n \leq 9$.

Conjecture 14 *Let $\mathbb{Z}_n$ be the ring of all residues modulo n. Suppose $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ for $i = 1, 2, \ldots, n$ are sequences of elements from $\mathbb{Z}_n$, satisfying $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$. If n is even, then there exists a permutation α such that $\sum_{i=1}^{n} a_i b_{\alpha(i)} = 0$.*

The conjecture was confirmed for $n \leq 8$.

It is worthwhile to comment that if Conjecture 14 holds true, then for n even it puts Theorem 7 in a new light. We conclude with Conjecture 15 that may seem obscure at first glance, but its significance will be seen in the next section.

Conjecture 15 *Let $\mathbb{Z}_n$ denote the additive cyclic group of the set of residues $\{0, 1, ..., n-1\}$, with the metric $d(x, y) = \min\{|x - y|, |n - (x - y)|\}$. Denote by $f(n)$ the smallest integer such that such that every sequence of elements from $\mathbb{Z}_n$ of length $f(n)$ contains an n-element subsequence that can be rearranged as $a_1, a_2, \ldots, a_n$ $(a_{n+1} = a_1)$ to satisfy $\sum_{i=1}^{n} d(a_{i+1}, a_i) = 0$. Then*

$$f(n) = \begin{cases} 3n - 2 & \text{if } n \text{ is odd} \\ \frac{3}{2}n - 1 & \text{if } n \text{ is even.} \end{cases}$$

The lower bound is given by $0^{n-1}1^{n-1}2^{n-1}$ and $0^{n-1}1^{n/2-1}$ (not unique) for n odd and even respectively. The conjecture was confirmed for $n \leq 9$ and $n \leq 12$ for n odd and even respectively.

4. Generalization in the Sense of a Zero Sum Theorem

In this section, we first introduce the definitions of Ramsey numbers for graphs and generalized Rado numbers, as well as their zero sum counterparts. These definitions enable us to define generalization in the sence of Theorem 1.5. Similar definitions can be applied to other extremal numbers, like Turan numbers for example, their zero sum counterparts and generalizations in the sence of agiven zero sum theorem. The general reference for Ramsey theory is [26]. However we have a significant interest along the lines of [38]. Ramsey theory on the integers started by Schur and was developed by Rado in the thirties. It mainly deals with monochromatic solutions to general systems of linear equations. The interest of integrating linear inequalities into Ramsey theory on the integers is pretty new, [34] and [42]. After introducing the definitions we provide some references to generalizations in the sence of the Theorem 1.5, (mainly its particular case, the EGZ theorem). We will conclude the section with the notion of multiplicity Ramsey and Rado numbers, their zero sum counter parts, and generalizations in the sense of Conjecture 2.1.

Definition 16 (Ramsey numbers for graphs) *Let $\mathcal{G}$ be a family of graphs each having n edges and let t be a positive integer. Denote by $R(\mathcal{G},t)$ ($R_{EGZ}(\mathcal{G}, Union(\mathbb{Z}_n^{t/2}))$, if t is even and $R_{EGZ}(\mathcal{G}, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$) if t is odd) the minimum number s such that if the edges of the complete graph on s vertices, K_s, are colored by t colors, (are colored by $Union(\mathbb{Z}_n^{t/2})$ if t is even and by $Union^{+1}(\mathbb{Z}_n^{(t-1)/2})$ if t is odd), then there exists a copy in K_s that is isomorphic to an element of $\mathcal{G}$, such that it is monochromatic (the colors of its edges add up to zero).*

Definition 17 (Rado numbers) *Let $x_1, x_2, \ldots, x_{nk}$ be nk variables partitioned into k sets S_i, where $i = 1, 2, \ldots, k$, and suppose that $|S_i| = n$. Furthermore, let $\mathcal{L}$ be a system that consists of equations or inequalities in the variables $x_1, x_2, \ldots, x_{nk}$. For a positive integer t denote by $R(\mathcal{L},t)$ ($R_{EGZ}(\mathcal{L}, Union(\mathbb{Z}_n^{t/2}))$, if t is even and $R_{EGZ}(\mathcal{L}, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$) if t is odd) the minimum number s such that if the integers $1, 2, \ldots, s$ are colored by t colors, (are colored by $Union(\mathbb{Z}_n^{t/2})$ if t is even and by $Union^{+1}(\mathbb{Z}_n^{(t-1)/2})$ if t is odd), then there exist integers $x_1, x_2, \ldots, x_{nk}$, which satisfy $\mathcal{L}$ and each S_i is monochromatic, but not necessarily the same color (the colors of the variables in each S_i add up to zero).*

In order to convey the ideas clearly, we have limited the scope of our definitions. For instance our definition does neither include the off diagonal case nor a disjunctive system of equations and inequalities.

Definition 18 (Generalization in the sense of Theorem 5) *We say that a theorem concerning Ramsey numbers, (Rado numbers) admits a generalization in the sense of Theorem 5, if $R(\mathcal{G},t) = R_{EGZ}(\mathcal{G}, Union(\mathbb{Z}_n^{t/2}))$, if t is even*
and $R(\mathcal{G},t) = R_{EGZ}(\mathcal{G}, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ if t is odd,
($R(\mathcal{L},t) = R_{EGZ}(\mathcal{L}, Union(\mathbb{Z}_n^{t/2}))$, if t is even
and $R(\mathcal{L},t) = R_{EGZ}(\mathcal{L}, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ if t is odd.)

Generalizations in the sense of Theorem 1,(Theorem 1 is included in Theorem 5), that relate to graph theory appear in [4],[7],[8][22]and [43]. The two most handled systems of linear equeation are those which follow from the Schur equation $x + y = z$ and the arithmetic progression, known as the Van der Waerden problem. There have been various variations and modifications of the Schur equation and the arithmetic progression to inequalities, see [12], [3], [10], [14] and [29]. Generalizations of some of these systems in the sense of Theorem 5 appear in [6], [13], [10], and [27]. This list is far from being a complete list of references. However, we do not have general theorems which give sufficient conditions when a system of equations or inequalities admits a generalization in the sense of a given zero sum theorem. We Conclude with the notion of multiplicity.

Definition 19 (Multiplicity Ramsey numbers for graphs) *Let $\mathcal{G}$ be a family of graphs each having n edges and let r and t be positive integers. Denote by $M(\mathcal{G}, r, t)$*

$(M_{EGZ}(\mathcal{G}, r, Union(\mathbb{Z}_n^{t/2}))$, *if* t *is even and* $M_{EGZ}(\mathcal{G}, r, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ *if* t *is odd) the minimum number of graphs that are isomorphic to an element of* $\mathcal{G}$, *and are monochromatic (the colors of its edges add up to zero), where the minimum is taken over all possible colorings of the edges of* K_s *by* t *colors. (of the edges of* K_s *by* $Union(\mathbb{Z}_n^{t/2})$ *if* t *is even and by* $Union^{+1}(\mathbb{Z}_n^{(t-1)/2})$ *if* t *is odd).*

Definition 20 (Multiplicity Rado numbers) *Let* $x_1, x_2, \ldots, x_{nk}$ *be* nk *variables partitioned into* k *sets* S_i, *where* $i = 1, 2, \ldots, k$, *and suppose that* $|S_i| = n$. *Furthermore, let* $\mathcal{L}$ *be a system that consists of equations or inequalities in the variables* $x_1, x_2, \ldots, x_{nk}$. *For positive integers* t *and* r *denote by* $M(\mathcal{L}, r, t)$ $(M_{EGZ}(\mathcal{L}, r, Union(\mathbb{Z}_n^{t/2}))$, *if* t *is even and* $M_{EGZ}(\mathcal{L}, r, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ *if* t *is odd) the minimum number of* nk-*tuples* $x_1, \ldots, x_{nk}$ *which satisfy* $\mathcal{L}$ *and each* S_i *is monochromatic, but not necessarily the same color (the colors of the variables in each* S_i *add up to zero), where the minimum is taken over all possible colorings of* $1, 2, \ldots, r$ *by* t *colors. (of* $1, 2, \ldots, r$ *by* $Union(\mathbb{Z}_n^{t/2})$ *if* t *is even and by* $Union^{+1}(\mathbb{Z}_n^{(t-1)/2})$ *if* t *is odd.)*

Definition 21 *We say that a theorem concerning multiplicity Ramsey numbers, (multiplicity Rado numbers) admits a generalization in the sense of Conjecture 6, if*
$M(\mathcal{G}, r, t) = M_{EGZ}(\mathcal{G}, r, Union(\mathbb{Z}_n^{t/2}))$, *if* t *is even*
and $M(\mathcal{G}, r, t) = M_{EGZ}(\mathcal{G}, r, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ *if* t *is odd,*
$(M(\mathcal{L}, r, t) = M_{EGZ}(\mathcal{L}, r, Union(\mathbb{Z}_n^{t/2}))$, *if* t *is even*
and $M(\mathcal{L}, r, t) = M_{EGZ}(\mathcal{L}, r, Union^{+1}(\mathbb{Z}_n^{(t-1)/2}))$ *if* t *is odd.)*

Acknowledgement

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