A MODULAR IDENTITY FOR THE RAMANUJAN IDENTITY MODULO 35

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Abstract

In Rademacher's proof [Rad42] of the Ramanujan identities modulo 5 and 7, the function

$$L_k(z) = \eta(kz) \sum_{\mu=0}^{k-1} \left\{ \frac{1}{\eta\left(\frac{z+24\mu}{k}\right)} \right\}$$

appears. This is a modular function on $\Gamma_0(k)$, provided that $(k, 6) = 1$. The basic identities

$$p(5n+4) \equiv 0 \bmod 5,$$
$$p(7n+5) \equiv 0 \bmod 7,$$

(as well as some generalizations to higher powers of 5 and 7) are proved by examining the coefficients of the expansions of L_5 and L_7 at infinity. The two identities above immediately imply the following identity:

$$p(35n+19) \equiv 0 \bmod 35.$$

Rademacher asked if there was a corresponding modular identity. Such an identity is derived in this paper. Also, another identity is obtained from the expansion of L_{35} at 0.

1. Generalities

1.1 Basics

Throughout, $p(n)$ refers to the unrestricted partition function. Euler proved the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

We will see that this ties the partition function intimately to certain types of modular functions (and forms). Let $\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z})$, which is the special linear group modulo its center ($\pm I$). $\Gamma(1)$ is generated by the two transformations

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We'll have need also for the product TST, which we will call W. Then in $\Gamma(1)$, we have:

$$W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$\Gamma(1)$ acts faithfully on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ via linear fractional transformation. Throughout, we will identify a linear fractional transformation and its matrix.

We let $q = e^{2\pi i z}$ throughout. In what follows, Dedekind's function $\eta(z)$ will be used, so we remind the reader of some known facts. Recall $\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$, and note that $\eta(z)$ is periodic of period 24. It is pole-free and zero-free throughout $\mathcal{H}$. It is a modular form of weight one half, and has a multiplier which is a 24^{th} root of unity in its transformation equation. The root of unity $\varepsilon(a,b,c,d)$ is determined by the following formulas (provided $c \neq 0$):

$$\varepsilon(a,b,c,d) = \exp\left\{\pi i \left(\frac{a+d}{12c} + s(-d,c)\right)\right\},$$

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right), \quad (1)$$

where [ ] is the greatest integer function. For properties of the Dedekind $s(h,k)$ sum see [RW41, RG72]. We will often write just ε in place of the more cumbersome $\varepsilon(a,b,c,d)$. Thus, for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, $\eta(Az) = \varepsilon(cz+d)^{1/2}\eta(z)$. There is a relationship between η and p. Namely,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{q^{\frac{1}{24}}}{\eta(z)}.$$

Also, L_k has q-expansion given by:

$$L_k(z) = kq^{1+\left[\frac{k}{24}\right]} \prod_{m=1}^{\infty}(1 - q^{km}) \sum_{n=0}^{\infty} p(kn+\Delta)q^n,$$

$$\Delta \text{ least positive solution of } 24n \equiv 1 \bmod k. \quad (2)$$

It is clear that L_k has all its poles located at the cusps of $\Gamma_0(k)$. Note that when $k = 35$ (resp. $5,7$), $\Delta = 19$ (resp. $4,5$). This explains the form for the identities listed earlier. That $L_k(z)$ is a modular function on $\Gamma_0(k)$ may be seen as follows. One can show that $\frac{\eta(kz)}{\eta\left(\frac{z}{k}\right)}$ is a modular function on $\Gamma_0^0(k)$ using known facts about Dedekind sums (see particularly Theorems 17 and 18 of [RW41]). The matrices $\begin{pmatrix} 1 & 24\mu \\ 0 & 1 \end{pmatrix}$, $(0 \le \mu \le k-1)$ form a set of coset representatives for $\Gamma_0^0(k)$ in $\Gamma_0(k)$. Therefore, when we sum over this set of coset representatives to obtain $L_k(z)$, we obtain a modular function on $\Gamma_0(k)$.

Let $\omega \in \mathbb{Q} \bigcup \{i\infty\}$ be a cusp for $\Gamma_0(k)$. We will let H_ω be the set of modular functions on $\Gamma_0(k)$ having poles only at ω. It is with this space of functions that we will work. In particular we will consider functions in H_ω, where $\omega = 0$ or $i\infty$.

We mention also the Fricke involution $S_k(z) = \frac{-1}{kz}$. Let $\mathcal{F}$ denote the set of modular functions on $\Gamma_0(k)$, and $f \in \mathcal{F}$. $S_k(z)$ is an operator from $\mathcal{F}$ to itself. Moreover, it interchanges the polar orders of f at $i\infty$ and 0 in the appropriate uniformizing variables (i.e, the expansion of $f(S_k(z))$ at $i\infty$ has the same order in the variable q, as the expansion of $f(z)$ at zero has in the variable $q^{\frac{1}{k}}$). For the rest of section 1, let $k = pQ$, where p and Q are two distinct primes strictly greater than 3. Then $S_k(z)$ interchanges the polar orders of f at the cusps $\frac{1}{p}$ and $\frac{1}{Q}$ as well.

1.2 Evaluation at 0

Pursuant to our task, we need to determine the order of L_k at each of the cusps. We determine the expansion about 0. To do so we subject L_k to the transformation S_k and then compute the expansion as usual.

$$L_k(S_k z) = \eta\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z\right] \sum_{\mu=0}^{k-1} \left\{\eta\left[\begin{pmatrix} 24\mu k & -1 \\ k^2 & 0 \end{pmatrix} z\right]\right\}^{-1}.$$

We need to factor the inner matrix as the product of a unimodular matrix and an upper triangular matrix. Upon factoring and applying the transformation identities we obtain:

$$L_k(S_k z) = \frac{1}{\sqrt{k}}\eta(z) \sum_{\mu=0}^{k-1} \frac{1}{\varepsilon(\frac{24\mu}{\delta}, b, \frac{k}{\delta}, d)\, \delta^{1/2}\, \eta\left(\delta^2 z + \frac{\delta y}{k}\right)},$$

where $\delta = (\mu, k)$, and the matrix inside factors as:

$$\begin{pmatrix} 24\mu/\delta & b \\ k/\delta & d \end{pmatrix} \begin{pmatrix} \delta k & y \\ 0 & k/\delta \end{pmatrix},$$

for some integers b, d and y such that

$$\frac{24\mu}{\delta} y + \frac{bk}{\delta} = -1,$$

$$\frac{24\mu}{\delta}d - \frac{bk}{\delta} = 1.$$

Together, the two equations yield that $y = -d$ (in the case $\mu = 0$ this is clear). Thus we obtain the following expression:

$$L_k(S_k z) = \frac{1}{\sqrt{k}} \prod_{m=1}^{\infty} (1-q^m) \sum_{\mu=0}^{k-1} \left\{ \varepsilon\, \delta^{\frac{1}{2}}\, e^{\frac{\pi i y \delta}{12k}}\, q^{\frac{\delta^2-1}{24}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i \delta y n} q^{\delta^2 n}\right) \right\}^{-1}. \tag{3}$$

Now, since $\mu = 0$ is the only term of the sum in which $\delta = k$, and this yields the largest negative power of q, we see that $\mathrm{ord}_0(L_k) \geq (1-k^2)/24$. We will return to equation 3 later.

1.3 Evaluation at all other cusps

Since k is square-free, all the other cusps of $\Gamma_0(k)$ occur at the points $\{1/l|\, 1 < l < k,\ l|k\}$, in other words, at $\frac{1}{p}$ and $\frac{1}{Q}$. To determine the order of L_k at $1/l$, we subject it to the transformation W^{-l} and then evaluate. Thus, we obtain

$$L_k\left(W^{-l}z\right) = \eta\left(kW^{-l}z\right) \sum_{\mu=0}^{k-1} \left\{ \eta\left(\frac{W^{-l}z + 24\mu}{k}\right) \right\}^{-1}$$
$$= \eta\left[\begin{pmatrix} k/l & 1-k/l \\ -1 & 1 \end{pmatrix} \begin{pmatrix} l & k/l-1 \\ 0 & k/l \end{pmatrix} z\right] \sum_{\mu=0}^{k-1} \left\{ \eta\left[\begin{pmatrix} 1-24l\mu & 24\mu \\ -kl & k \end{pmatrix} z\right] \right\}^{-1}.$$

Now, let $\delta = (1-24\mu l, k/l)$. Then the matrix inside factors as

$$\begin{pmatrix} (1-24\mu l)/\delta & b \\ (-kl)/\delta & d \end{pmatrix} \begin{pmatrix} \delta & y \\ 0 & k/\delta \end{pmatrix},$$

for integers b, d and y such that

$$\frac{1-24\mu l}{\delta}d + \frac{bk}{\delta}l = 1,$$
$$\frac{1-24\mu l}{\delta}y + \frac{bk}{\delta} = 24\mu,$$
$$\frac{-kl}{\delta}y + \frac{kd}{\delta} = k.$$

The final equation above yields $-ly + d = \delta$. Using this fact we see that

$$L_k(W^{-l}(z)) = e^{-\frac{k+l}{12l}\pi i} \sqrt{l/k}\; \eta\left(\frac{l^2 z - l + k}{k}\right) \sum_{\mu=0}^{k-1} \left\{ \varepsilon\, \delta^{1/2}\, \eta\left(\frac{\delta^2 z + \delta y}{k}\right) \right\}^{-1}$$

$$= e^{-\frac{k+l}{12l}\pi i}\sqrt{l/k}\prod_{m=1}^{\infty}\left(1-\rho^{(k-l)n}q^{\frac{l^2}{k}n}\right)\times$$

$$\sum_{\mu=0}^{k-1}\left\{\varepsilon\,\delta^{1/2}\rho^{\frac{dy+k-l}{24}}q^{\frac{\delta^2-l^2}{24k}}\prod_{n=1}^{\infty}\left(1-\rho^{\delta yn}q^{\frac{\delta^2}{k}n}\right)\right\}^{-1}, \quad \left(\rho=e^{\frac{2\pi i}{k}}\right).$$

2. Specialization to k = 35

2.1 Obtaining an explicit polynomial identity at $i\infty$

Now, we take $k = 35$ for the remainder of this paper. We see from the results above that $L_{35}(z)$ has a zero of order 2 at $i\infty$ in the variable q, a pole of order 51 at 0 in the variable $q^{\frac{1}{35}}$, is holomorphic at 1/5 in $q^{\frac{1}{7}}$, and has a zero of order at least 1 at 1/7 in the variable $q^{\frac{1}{5}}$. We refer the reader to [New57] for an explicit construction of the basis functions with which we will work, and for an account of the functions A_9 and B_8 (as well as other similar functions used in deriving the basis of functions at infinity). In summary, the basis functions B_4, B_5, B_6, and B_7 all have poles only at $i\infty$ in the same order as the value of their respective subscript. Similarly, A_9 and B_8 have poles of order 9 and 8, respectively, at $i\infty$, and A_9 has zeroes of orders 5, 1, and 3 at $0, 1/5$ and $1/7$ respectively, while B_8 has zeros of orders 6 and 2 at 0 and 1/7 respectively.

Suffice it to say that $G(z) = A_9(z)B_8^8(z)L_{35}(z)$ is pole-free in $\mathcal{H}$ and has order -71 at $i\infty$, 2 at 0, 0 (or possibly greater) at 1/5, and 20 at 1/7. So we have:

Theorem 1. $G(z) \in \mathbb{C}[B_4, B_5, B_6, B_7]$.

The proof follows from the work in the previous section and the fact that the B_i do form a polynomial basis for the set of modular functions in $H_{i\infty}$.

Since the order at $i\infty$ is known, the polynomial is determined, using Mathematica to calculate the coefficients. Among the possible ways of choosing an appropriate form, it was determined to use B_7 to keep the total degree (as a polynomial in the B_i) as small as possible. For instance the terms matching q^{-65}, q^{-64}, q^{-63} are $B_7^8B_5B_4$, $B_7^8B_4^2$, B_7^9 respectively. Call the polynomial that we obtain P_∞. Then $L_{35}(z) = A_9^{-1}B_8^{-8}P_\infty$.

Theorem 2. *Let c_i be the coefficient on the term of P_∞ corresponding to q^i. Then the polynomial P_∞ is determined by the coefficients in the appendix in table 1, where c_i is the given i^{th} coefficient multiplied by* 1225 $(= 35^2)$.

Corollary 3. $p(35n + 19) \equiv 0 \bmod 35$.

Remark 1. This identity also shows how the genus dramatically affects a polynomial identity associated with such an identity for the partition function. For the cases $k = 5$

and $k = 7$, where the genus is 0, Rademacher obtains a linear polynomial and a quadratic polynomial, respectively, in the respective basis function. Here, the genus is 3 and a polynomial of degree 11 in the *four* basis functions is obtained. In a similar manner, modular identities for the congruences for 55 and 77 may surely be constructed. It is expected that more basis functions and a higher-degree polynomial will be necessary.

2.2 An Identity at 0

We return to equation 3. Recalling the connection with the partition function p, and separating into cases depending on δ, we have :

$$L_k(S_k(z)) = \sum_{\delta|35} \frac{q^{\frac{1-\delta^2}{24}}}{\sqrt{k}} \sum_{\mu=0}^{34} \varepsilon^{-1}\delta^{-\frac{1}{2}} \sum_{n=0}^{\infty} e^{\frac{(24n-1)\delta y\pi i}{12k}} p(n) q^{\delta^2 n} \prod_{m=1}^{\infty} (1-q^m).$$

When we group the outer sum into terms by δ, we obtain Gauss sums for the Jacobi-Legendre symbol of the appropriate modulus. We have the following terms:

$$\begin{aligned}
\delta = 35: &\quad \frac{1}{35q^{51}} \prod_{m=1}^{\infty}(1-q^m) \sum_{n=0}^{\infty} p(n) q^{1225n}, \\
\delta = 7: &\quad -\frac{1}{7q^2} \prod_{m=1}^{\infty}(1-q^m) \sum_{n=0}^{\infty} \left(\frac{n-4}{5}\right) p(n) q^{49n}, \\
\delta = 5: &\quad -\frac{1}{5q} \prod_{m=1}^{\infty}(1-q^m) \sum_{n=0}^{\infty} \left(\frac{n-5}{7}\right) p(n) q^{25n}, \\
\delta = 1: &\quad \prod_{m=1}^{\infty}(1-q^m) \sum_{n=0}^{\infty} \left(\frac{n-19}{35}\right) p(n) q^{n},
\end{aligned}$$

where (-) indicates the Jacobi-Legendre symbol.

Now, $L_{35}(S_{35}z)$ has order -51 (resp. $2, 1, -1$) at $i\infty$ (resp. $0, 1/5, 1/7$). For the last two points we really only know that the order at each of these points is greater than or equal to the given value. Nevertheless, this is sufficient to determine that the function $h(z) = B_8(z)L_{35}(S_{35}z)$ has its only pole at $i\infty$, and the order of the pole there is 59. Thus, $h(z) \in \mathbb{C}[B_4, B_5, B_6, B_7]$.

To clear fractions, multiply by 35 and then note $35L_{35}(S_{35}z) = B_8^{-1}P_0$, where P_0 is a polynomial to be determined in the four basis functions. So

$$q^{-51} - 5q^{-2} - 7q^{-1} + 35 + ... = P_0 B_8^{-1} \sum_{n=0}^{\infty} p(n) q^n.$$

This yields the following:

Theorem 4. *$35L_{35}(S_{35}z)$ is a polynomial in the basis functions B_4, B_5, B_6, B_7, with integral coefficients given in table 2 of the appendix.*

Corollary 5.

$$\sum_{n=0}^{\infty}\left\{p(n)q^{1225n}-5\left(\frac{n-4}{5}\right)p(n)q^{49(n+1)}\right.$$
$$\left.-7\left(\frac{n-5}{7}\right)p(n)q^{25(n+2)}+35\left(\frac{n-19}{35}\right)p(n)q^{n+51}\right\}$$
$$= \; q^{51}P_0B_8^{-1}\sum_{n=0}^{\infty}p(n)q^n.$$

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3. Appendix: tables

Table 1: Coefficients for P_∞

i	coefficient	i	coefficient
-69	14	-33	-1056922323822851391340 5
-68	10487	-34	-29246449555394872646896
-67	1050000	-33	-30752043774402282243297
-66	37305717	-32	-10500761656485195685244
-65	689742738	-31	-6399458451803032312 5
-64	6710739396	-30	7473966103683716867 39
-63	66723377634	-29	41866861538739825786329
-62	306529775888	-28	-154825275874629072148111
-61	944992220254	-27	-330675045299039974754087
-60	8032835276603	-26	-330675045299039974754087
-59	41786228768706	-25	-195434688782712582767044
-58	163674979606784	-24	106519373950966952221
-57	201541582793635	-23	-4489388038461810926594
-56	1192164476376773	-22	93815803283069768407781
-55	2133885512377322	-21	-868861692593019875600320
-54	-224422822302317	-20	-780240652273282530768564
-53	9311212319487067	-19	-903341606561540466301311
-52	53104246090438488	-18	-575916410654649291605538
-51	174875061214811942	-17	-7238738733427171422 5
-50	597905135797416 86	-16	5279005749531844180716
-49	4183638907336056 49	-15	-329709784193546752041430
-48	1123804424169272601	-14	-754715959977583205060136
-47	-1849756259047888932	-13	2673409023878484316383343
-46	-3961615256484350591	-12	3038512747510492176975342

continued on next page

Table 1: *c*ontinued

i	coefficient	i	coefficient
-45	1372328949965814202	-11	1778681601887609961535840
-44	19742808525931603996	-10	16760790778989818745
-43	2177933533113105157	-9	-1894749705906830722098
-42	23516820211582241283	-8	189938163107999640927156
-41	123390318981474451231	-7	4789820626890790384933368
-40	-271841378745461882544	-6	-1532454919226226734377599
-39	-777621797568622505476	-5	-1735248899286586237586336
-38	-673143178730309518022	-4	-997109474512591275870705
-37	268511875093390021788	-3	0
-36	10323721732692166606	-2	0
-35	334287613007887666292	-1	0
-34	3963341600197861903691	0	-3000539263195134467759624

Table 2: Coefficients for P_0

i	coefficient	i	coefficient
-59	1	-29	101856
-58	-5	-28	-1088550
-57	13	-27	-779536
-56	-24	-26	-868518
-55	-124	-25	-638720
-54	-167	-24	183785
-53	-100	-23	310027
-52	186	-22	-1009582
-51	82	-21	-5533
-50	-784	-20	8917982
-49	1899	-19	10010374
-48	7839	-18	6789719
-47	9103	-17	-21976
-46	5400	-16	113545
-45	-531	-15	35220
-44	2196	-14	21729982
-43	-786	-13	-703140
-42	13705	-12	-920456
-41	-1006	-11	-843856
-40	-4690	-10	-236844
-39	1882	-9	-376949
-38	-6611	-8	1365595
-37	-16039	-7	-4046123
-36	44380	-6	-12312578
-35	-84846	-5	-13942378

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Table 2: *c*ontinued

i	coefficient	i	coefficient
-34	-393717	-4	-9425016
-33	-427964	-3	0
-32	-282747	-2	0
-31	-17208	-1	0
-30	-53917	0	-30151150

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