JENSEN PROOF OF A CURIOUS BINOMIAL IDENTITY

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Abstract

By means of the Jensen formulae on binomial convolutions, a new proof is presented for a curious identity due to Z.-W. Sun.

Based on double recurrence relations, Sun [6] discovered the following binomial identity

$$S_m \;:=\; (x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i} - \sum_{i=0}^{m}(-4)^i\binom{x+i}{m-i} \;=\; (x-m)\binom{x}{m}.$$

Recently, three alternative proofs have been provided by Panholzer and Prodinger [5] via the generating function method, by Merlini and Sprugnoli [4] through Riordan arrays, and by Ekhad and Mohammed [2] based on the "WZ" method. Combining Jensen's identity and Chu-Vandermonde convolution formulae on binomial coefficients, we present yet another proof for this result which provides a shortcut.

By means of the Jensen formulae (cf. [1, Eq 8] for example) on binomial convolutions

$$\sum_{i=0}^{m}\binom{a+bi}{i}\binom{c-bi}{m-i} \;=\; \sum_{k=0}^{m}\binom{a+c-k}{m-k}b^k$$

the first binomial sum displayed in S_m can be reformulated as

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i} &= (-1)^m\sum_{i=0}^{m}\binom{y+2i}{i}\binom{-1-x-y+m-2i}{m-i}\\ &= (-1)^m\sum_{k=0}^{m}\binom{-1-x+m-k}{m-k}2^k \;=\; \sum_{k=0}^{m}\binom{x}{m-k}(-2)^k.\end{aligned}$$

For a complex x and a natural number n, denote the shifted factorial of x of order n by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

In accordance with the parity of k, writing $k := \delta + 2k'$ with $\delta := 0, 1$ and then performing the replacement $j := i - \delta - k'$, we can derive the following binomial coefficient identity:

$$\begin{aligned}
\sum_{\frac{k}{2} \le i \le k} \binom{i}{k-i}(-4)^i &= (-4)^{\delta+k'} \sum_{j=0}^{k'} \binom{\delta+k'+j}{k'-j}(-4)^j \\
&= (-4)^{\delta+k'} \sum_{j=0}^{k'} \frac{(-1)^j(\delta+k'+j)!}{j!(k'-j)!(1/2+\delta)_j} \\
&= (-4)^{\delta+k'} \frac{(1+\delta)_{k'}}{(1/2+\delta)_{k'}} \sum_{j=0}^{k'} \binom{-1-\delta-k'}{j}\binom{\delta-1/2+k'}{k'-j} \\
&= (-4)^{\delta+k'} \frac{(1+\delta)_{k'}}{(1/2+\delta)_{k'}} \binom{-3/2}{k'} = (-1)^k 2^k (1+k)
\end{aligned}$$

where the Chu-Vandermonde convolution formulae has been applied.

This binomial identity allows us to express the second sum displayed in S_m as

$$\begin{aligned}
\sum_{i=0}^{m} \binom{x+i}{m-i}(-4)^i &= \sum_{i=0}^{m}(-4)^i \sum_{k=i}^{m} \binom{x}{m-k}\binom{i}{k-i} \\
&= \sum_{k=0}^{m} \binom{x}{m-k} \sum_{\frac{k}{2} \le i \le k} \binom{i}{k-i}(-4)^i \\
&= \sum_{k=0}^{m} \binom{x}{m-k}(-2)^k(k+1).
\end{aligned}$$

Therefore the linear combination of the two binomial sums in S_m results in

$$\begin{aligned}
S_m &= \sum_{k=0}^{m}(x+m-k)\binom{x}{m-k}(-2)^k = \sum_{k=0}^{m}\big\{(x-m+k)+2(m-k)\big\}\binom{x}{m-k}(-2)^k \\
&= \sum_{k=0}^{m}(1+m-k)\binom{x}{1+m-k}(-2)^k - \sum_{k=0}^{m}(m-k)\binom{x}{m-k}(-2)^{k+1} \\
&= (m+1)\binom{x}{m+1}
\end{aligned}$$

where the two sums with respect to k in the last line have been telescoped. This completes the proof of the identity originally due to Sun.

References

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