NIM WITH A MODULAR MULLER TWIST

Hillevi Gavel
Department of Mathematics and Physics, Mälardalen University, Västerås, Sweden
hillevi.gavel@mdh.se

Pontus Strimling
Department of Mathematics and Physics, Mälardalen University, Västerås, Sweden
pontus.strimling@mdh.se

Received: 11/20/03, Revised: 8/11/04, Accepted: 9/24/04, Published: 9/28/04

Abstract

We study Nim with the additional rule that each player can put a constraint on the next player's move. This is called a *Muller twist*. The constraints are of the form "the number taken must not be equivalent to *some numbers* modulo n". A solution for a large group of such games is given.

1 Introduction

An interesting twist on a combinatorial game can be obtained if one allows the players to put constraints on each other's moves. This is called "introducing a *Muller Twist*", after Blaise Muller who invented the game *Quarto*®, a game where each player chooses the piece the next player should place. This game and several others are described in an article [1] by F. Smith and P. Stănică.

A move in a game with a Muller twist consists of a "physical" move and a choice of a restriction. A position consists of a "physical" position and a restriction. Imagine a chess game where the other player determines which kind of piece you are allowed to move or points out a square that you must not move to, or a bridge game where your opponent determines which suite you should play. These would all be Muller twist games.

Adding a Muller twist to a single game may obviously make it more entertaining to play. It may also make the mathematical analysis of the game more challenging. In this article we will study the game of Nim with restrictions on the number of sticks that can be taken, aiming to solve as large a group of such games as possible.

2 The game Nim and twists thereupon

Nim is an important and well understood combinatorial game, see [2] and [3]. It is commonly played with sticks divided into piles. In each move a player may pick as many sticks as he wants to, but only from one pile. The two players take turns and when a player cannot make a move (because there are no sticks left) he loses. It can be shown [2, 3] that a position is a winning position for the player leaving it (a *$\mathcal{P}$-position*) if the Nimsum of the pile sizes is 0; otherwise, the position is a winning position for the person receiving it (an *$\mathcal{N}$-position*). The Nimsum $\oplus$ of a set of numbers is computed by bitwise Xor on the binary representation of the numbers, that is, addition modulo 2 without carry. So

$$9 \oplus 3 = (2^3 + 2^0) \oplus (2^1 + 2^0) = 2^3 + 2^1 + (\underbrace{2^0 \oplus 2^0}_{0}) = 10$$

The key observation is (see [3]) the following lemma, which we present without proof:

Lemma 1

- If a set of numbers have Nimsum 0 and you change one of them, then the new set of numbers will have a Nimsum differing from 0.
- If a set of numbers have Nimsum differing from 0, then there is a way to decrease at least one of the numbers so that the new Nimsum is 0. □

2.1 Odd-or-even-Nim

Odd-or-even-Nim is analyzed in F. Smith and P. Stănică's article [1]. In this twisted form, the previous player tells the next player if she should take an odd or an even number of sticks. Smith and Stănică show that the $\mathcal{P}$-positions are the positions satisfying one of three properties:

- Nimsum 0, restriction even
- Nimsum 1, restriction even
- Nimsum 0, all piles are even and restriction odd.

The proof is elegant, but provides no intuition for why the results look like they do. In order to gain better understanding, we looked closer at the game structure and generalized the Muller twist.

"Odd" or "even" tells if a number is equivalent to 1 or to 2 modulo 2. A related twist is to play modulo 3 instead. Then there are three classes of numbers, equivalent to 1,

$$(4,2,1)^{\not\equiv 1}$$

$$(2,2,1)^{?} \qquad (1,2,1)^{?} \qquad (4,1)^{?}$$

$$(4,2,1)^{\not\equiv 1}$$

$$(2,2,1)^{\not\equiv 1}$$

$$(2,1)^{\not\equiv 1} \qquad (2,1)^{\not\equiv 2}$$

$$(1)^{\not\equiv 1} \qquad (1,1)^{\not\equiv 1}$$

$$\dagger \qquad \dagger$$

Figure 1: A player is passed three piles containing 4, 2 and 1 sticks together with the restriction "the number taken must not be equivalent to 1 modulo 3". That means that taking 1 or 4 sticks is forbidden, so the player has to take 2 or 3 sticks from one of the piles. There are three ways to do this. If the player chooses to take 2 from the largest pile, and hands over the position with the restriction "not 1", the other player has to take all the sticks in one of the two 2-stick piles. Then, there are two "sensible" constraints that can be put. (Forbidding 3, which is not possible anyway, seems pointless.) Both constraints make it possible for the first player to reduce the game to a set of one-stick piles, and setting the restriction "not 1" makes this a terminal position, a losing position for the other player. Thus, $(4,2,1)$ with the restriction "not 1" is a winning position for the player being handed that position; an $\mathcal{N}$-position. Furthermore, going to $(2,2,1)$ and setting the restriction "not 1" is the *only* way to win. One can check this by investigating the other options in the same way as used with this one.

2 or 3, and the player can disallow one of them. So the player leaving the position says "the number of sticks taken must *not* be equivalent to x modulo 3". An example of a game of this form is given in Figure 1.

We will study several variations of this game, varying the modulo used and the number of forbidden moves. We will solve those games where all combinations of up to a given number of options are allowed as restrictions, and also some games where not all combinations may be used.

2.2 Modular one-blocking Nim

In our first generalized version of odd-or-even-Nim, which we will call *modular one-blocking Nim*, the restriction is of the form "the number of sticks taken must not be equivalent to x modulo n".

Our analysis of this game turned up a surprise: The modulo used actually does not matter! The only important thing is whether the restriction allows you to take *exactly* one stick or not.

Theorem 1 In modular one-blocking Nim the $\mathcal{P}$-positions are characterized by having either of two properties:

1. Positions where the Nimsum of the quotients of the pile sizes divided by 2 is 0, and where the restriction is “the number of sticks taken must not be equivalent to 1 modulo n”.

2. Positions where the Nimsum of the quotients is 0 and where all the remainders of the pile sizes divided by 2 are 0 (that is, all the pile sizes are even), any restriction.

Proof First we must show that if we are given a position of type 1 or 2, then we have to return a position from the complement set (if any move at all is possible):

1. If the Nimsum of the quotients is 0 and the restriction is “not 1”, then we have to take at least *two* sticks. Taking two or more sticks from a pile changes the quotient, which means that the Nimsum of the quotients will change too (Lemma 1), so we will hand over a position where the Nimsum is not 0.

2. If the Nimsum is 0 and all the remainders are 0 too, then we have to change a quotient when we take some sticks, even if we take just one and thus cause the Nimsum to change.

Second, we must show that if we are given a position from the complement set, then it is always possible to make a move and return a position of type 1 or 2:

1. If the Nimsum of the quotients is not 0, then there is at least one way to change the quotients to make it so. There are two different ways to give the chosen pile the desired quotient – one where the remainder is 0 and one where it is 1. The restriction is only able to block one of these alternatives. We will certainly be able to hand over a position of type 1. If none of the remainders are 1, we can hand over type 2 instead.

2. If the Nimsum is 0, at least one remainder is 1 and we are allowed to take just one stick, then we can take this remainder and keep the Nimsum 0. We can either hand over a position of type 1 or 2 (the latter if just one of the remainders in the original position was 1). □

If we look at the game in Figure 1, we find that the position $(4, 2, 1)^{\neq 1}$ has the quotients $(2, 1, 0)$. The Nimsum of the quotients is $2 \oplus 1 \oplus 0 = 3 \neq 0$, so this position is indeed an $\mathcal{N}$-position. When the player moves to $(2, 2, 1)^{\neq 1}$ the quotients are changed to $(1, 1, 0)$, and $1 \oplus 1 \oplus 0 = 0$. The remainders are $(0, 0, 1)$, so the player has to block 1 to make this a $\mathcal{P}$-position. The given move is the only way for the first player to win the game, as stated.

The $\mathcal{P}$-positions in modular one-blocking Nim are the same as in odd-or-even-Nim (if we regard “not 1” as the same thing as “even” and everything else as “odd”). An

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(a) Constraint “not 1”. (In odd-or-even-Nim: constraint even.)

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(b) Constraint anything except “not 1”. (In odd-or-even-Nim: constraint odd.)

Figure 2: $\mathcal{N}$- and $\mathcal{P}$-positions when playing modular one-blocking Nim with two piles. On the matrix we have superimposed a grid. Positions within the same “box” of the grid have the same quotients when divided by 2. The remainders give the position within the box. In (b) we can note that the $\mathcal{P}$-positions are the ones where both numbers are even, that is, where both remainders are 0.

illustration is given in Figure 2. The equivalence of the two sets follows from the following observations:

The quotient when dividing a number by 2 is what you get if you cut out the least significant bit, while the remainder is that bit. If the Nimsum of the quotients is 0, then the Nimsum of the original numbers will have 0s in all positions except the least significant one, that may be 0 or 1. That is, the Nimsum of the original numbers will be 1 or 0. The $\mathcal{P}$-positions of odd-or-even-Nim were described as

- Nimsum 0, restriction *even*
- Nimsum 1, restriction *even*
- Nimsum 0, all numbers even, restriction *odd*

The first two kinds are the same as type 1, and the last one is type 2. (If all the numbers are even, the least significant bits will be 0, so if the Nimsum of the quotients is 0, so is the Nimsum of the original numbers.)

3 Further generalization

Even further generalizations of odd-or-even-Nim can be made. In the previous section, we blocked *one* modular option. We will now consider a game where a greater number of options can be blocked. As a start, we will express Theorem 1 in a more general way:

Theorem 1 (rephrased) In modular one-blocking Nim the $\mathcal{P}$-positions are the positions where the Nimsum of the quotients of the pile sizes divided by 2 is 0 and where all the remainders are smaller than the smallest amount that can be taken according to the constraint. □

The reason that the theorem involves division by *two* is that we can force the adversary to take at least two sticks, but not more. If we block 1, taking 2 is an option. If we block 2, taking just one is possible. We can call 2 "the smallest unavoidable number".

3.1 Blocking a fixed number of options

Our next generalized set of games will be called *k-blocking modular Nim*. Here, a constraint consists of exactly k different subconstraints of the kind "you must not take a number equivalent to x modulo n". All the $\binom{n}{k}$ possible combinations of subconstraints may be used. So in 4-blocking modular Nim with modulo 10, one possible restriction is "you can not take a number equivalent to 1, 3, 6 or 8 modulo 10".

Lemma 2 If the constraint consists of k subconstraints, then you can force the adversary to take at least $k+1$ sticks. (That is when the set of subconstraints is "not 1", "not 2", ... , "not k".) We will call this the *strictest constraint*. □

Theorem 2 In k-blocking modular Nim the $\mathcal{P}$-positions are the positions where

- the Nimsum of the quotients of the pile sizes when divided by $k+1$ is 0

and

- all remainders are smaller than the smallest number of sticks that can be taken according to the given constraint.

Proof Starting with the $\mathcal{P}$-positions:

- If all the remainders are smaller than the smallest number of sticks that can be taken, then the move will change one of the quotients, so the Nimsum of the quotients will not be 0 (Lemma 1).

Continuing with the $\mathcal{N}$-positions:

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(a) The strictest constraint: "not 1 and not 2".

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(b) "Not 1, not 3" and "not 1, not 4". (That is, constraints blocking 1 but not 2.)

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(c) Constraints "not 2, not 3", "not 2, not 4" and "not 3, not 4". (That is, constraints not blocking 1.)

Figure 3: Playing 2-blocking Nim modulo 4, and using two piles. "2-blocking" means that it is the quotient and remainder when dividing by three that is important. When using just two piles, the Nimsum equals 0 if and only if the piles are equal, which means that the $\mathcal{P}$-positions will be placed along the main diagonal.

- If the Nimsum of the quotients is not 0, then one can make it 0 by changing one of the quotients (Lemma 1). There are $k+1$ different pile sizes that have the desired quotient, and the constraint is only able to block k of them. Having made the Nimsum 0, we can choose a constraint that is bigger than all the remainders. (If we choose the strictest constraint, the one that forces the adversary to take at least $k+1$ sticks, all the remainders will be smaller than that, as they can not exceed k.)

  If the Nimsum of the quotients is 0, but at least one of the remainders is at least as big as the smallest number of sticks we may take, then we can decrease that remainder keeping the quotient constant. The Nimsum will still be 0, and we can choose a constraint strict enough to leave a $\mathcal{P}$-position.

  (Note: This situation only occurs after suboptimal play, as the player giving the constraint could have chosen a stricter one, making the position a $\mathcal{P}$-position instead.) □

An illustration of $\mathcal{N}$- and $\mathcal{P}$-positions in a game is given in Figure 3.

Corollary 1 (push-them-upwards) It is never an advantage to give any other constraint than the strictest one, and sometimes it is a disadvantage. □

3.2 Blocking a variable number of options

A further generalizations of the game allows blocking of any number of options, up to a certain limit k. This does not change the analysis, since it is never disadvantageous to

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$
1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
3	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$
4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
6	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(a) Constraint "not 2, not 3" (that is: the number taken must be equal to 1 modulo 3).

	0	1	2	3	4	5	6
0	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
1	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$
4	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$
6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

(b) Constraint "not 1, not 3" (the number taken must be equal to 2 modulo 3).

Figure 4: Playing blocking Nim modulo 3, each constraint blocking two options, but the strictest combination "not 1, not 2" disallowed. The pattern is completely different from when the strictest combination is allowed. If "not 1, not 2" is an option, then we get the same pattern as in Figure 3.

use the maximum number k of blockings.

3.3 Blocking certain combinations of options

We propose as an open problem to find the solution of the general modular blocking Nim game, that is, when the set of allowed restrictions is *any* subset of the set of all possible combinations of subrestrictions.

The following observations tell how far we have come with this problem:

Observation 1 The characterization of $\mathcal{P}$-positions in Theorem 2 holds even if we do not allow all combinations of at most k subconstraints, as long as the strictest constraint, that is, the combination "not 1, not 2, ... , not k" is an option. The same proof goes through. The important thing is that we are able to use the strictest constraint, and thus force the adversary to change a quotient. □

Observation 2 The characterization of $\mathcal{P}$-positions in Theorem 2 does *not* hold if we *do not* include the strictest constraint "not 1, not 2, ... , not k" as an option. The parts "the constraint is only able to block k of them" and "If we choose the strictest constraint" of the proof will not work anymore.

If the grid size is determined by the number of constraints, so that we divide by $k+1$, then we are not always able to force the adversary to change a quotient, since we need the strictest constraint to ensure that.

If the grid size is determined by the lowest unavoidable number (less than $k+1$) the k subconstraints may be able to block every way to get the desired quotient. □

Figure 4 shows an example of a game where the strictest constraint is excluded.

4 Conclusion

There are surprisingly few Muller twist games solved. Some, besides Odd-or-even-Nim, are described in [1], some others in [4]. We hope that we have given inspiration for further works in this field. We thank an anonymous referee and Kimmo Eriksson for many useful comments.

References

[1] F. Smith, P. Stănică, *Comply/Constrain Games or Games with a Muller Twist*, Integers 2, art. G3, 2002.

[2] E.R. Berlekamp, J.H. Conway, R.K. Guy, *Winning Ways*, A K Peters, Ltd., 2001.

[3] J.H. Conway, *On Numbers and Games*, A K Peters, Ltd., 2001.

[4] A. Flammenkamp, A. Holshouser, H. Reiter, *Dynamic One-Pile Blocking Nim*, Electron. J. Combin., Volume 10(1), art. N4, 2003.