A NEGATIVE ANSWER TO TWO QUESTIONS ABOUT THE SMALLEST PRIME NUMBERS HAVING GIVEN DIGITAL SUMS

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Abstract

Denote by $\rho(k)$ the smallest prime number with digital sum k (not a multiple of 3). Richard K. Guy asked whether the congruences $\rho(k) \equiv 99 \pmod{100}$ and $\rho(k) \equiv 999 \pmod{1000}$ hold for all $k > 38$, respectively $k > 59$. Counterexamples to this are given, inter alia, for $k = 86$ and $k = 104$. Moreover, several open problems and conjectures about $\rho(k)$ are discussed.

1. Introduction

In his famous book about unsolved problems in Number Theory, Richard K. Guy [1] asks in section A3 the following three questions:

> "If $\rho(k)$ is the smallest prime with digital sum k, is $\rho(k) \equiv 9 \pmod{10}$ for $k > 25$? Is it $\equiv 99 \pmod{100}$ for $k > 38$? And $\equiv 999 \pmod{1000}$ for $k > 59$?"

We intend to show that the two latter congruences do not hold by constructing a counterexample in the following paragraph. The final section is concerned with some open problems and conjectures concerning the prime numbers $\rho(k)$ for k not a multiple of 3.

2. Counterexamples

A first counterexample to two of the claims above is given for $k = 86$. Using some simple combinatorial methods, we find that the smallest number with digital sum 86 is

$$5\,999\,999\,999 \tag{1}$$

followed by the numbers

$$6\,899\,999\,999, \quad (2)$$
$$6\,989\,999\,999, \quad (3)$$
$$6\,998\,999\,999, \quad (4)$$
$$6\,999\,899\,999, \quad (5)$$
$$6\,999\,989\,999, \quad (6)$$
$$6\,999\,998\,999, \quad (7)$$
$$6\,999\,999\,899, \quad (8)$$
$$6\,999\,999\,989. \quad (9)$$

The integers (1) to (8) turn out to be composites, as they are divisible respectively by 7, 6089, 827, 19, 61, 31, 7 and 3011.
The number (9) is prime. So $\rho(86) = 6999999989 \equiv 989 \pmod{1000} \equiv 89 \pmod{100}$. Therefore the two latter questions above have to be answered with "no", while the first one remains open. It might moreover be interesting to note that $k = 86$ is not the only known case for which the said congruences do not hold. Other examples are $\rho(104) = 699999999989$, $\rho(137) = 4 \cdot 10^{15} - 11$, $\rho(317) = 4 \cdot 10^{35} - 11$, $\rho(400) = 6 \cdot 10^{44} - 11$ and $\rho(580) = 6 \cdot 10^{64} - 11$. Perhaps there are many other primes of the form $a \cdot 10^n - 11$ with $a \in \{1, 3, 4, 6, 7, 9\}$ (the parameters $a \in \{2, 5, 8\}$ give only multiples of 3), which lead to further examples. Lists of such primes can be found in the Online Encyclopedia of Integer Sequences [2] (the reference codes are respectively A092767, A102737, A102738, A102739, A102740 and A102741). Additionally, we have $\rho(260) = 10^{29} - 101$.

3. Open problems and conjectures

A number of open problems dealing with $\rho(k)$ are waiting to be solved. Is there a $k > 25$ with $\rho(k) \equiv 79$, or $\equiv 97 \pmod{100}$? Moreover, one might ask whether $\rho(k)$ exists for every k not a multiple of 3. The answer to this is very probably positive, and therefore the above question could be changed into the following one (Question 1):

> Does an integer m exist, such that the congruences $\rho(k) \equiv 99 \pmod{100}$ and $\rho(k) \equiv 999 \pmod{1000}$ hold for all $k > m$?

Let $S(k)$ be the set consisting of the k smallest integers with digital sum k, and let $N(k)$ be the number of integers in $S(k)$ that end in "999". We know that as $k \to \infty$, $\frac{N(k)}{k} \to 1$; however, as $N(k) < k$ for all $k > 10$, we conjecture that there exist infinitely many values of k for which $\rho(k)$ does not satisfy the congruences in Question 1. If this is true, the integer m does not exist.

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References

[1] Richard K. Guy, *Unsolved problems in number theory*, Springer Verlag, New York 2004 (3th edition), 15

[2] Neil J. A. Sloane, *Online Encyclopedia of Integer Sequences*, published electronically at http://www.research.att.com/njas/sequences/