SOME RESULTS FOR SUMS OF THE INVERSES OF BINOMIAL COEFFICIENTS

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Received: 7/15/04, Revised: 8/11/05, Accepted: 9/26/05, Published: 10/11/05

Abstract

In this paper, the authors establish some identities involving inverses of binomial coefficients and generalize an identity.

1. Introduction

For convenience, we first give some notation. The binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!}, & n \geq m, \\ 0, & n < m, \end{cases}$$

where n and m are nonnegative integers.

Binomial coefficients are classical combinatorial numbers, which play an important role in many subjects such as probability, statistics, and number theory. There are many identities related to binomial coefficients. However, computations involving the inverses of binomial coefficients are often difficult. For previous literature dealing with identities related to the inverses of binomial coefficients, see [1–7]. In this paper, we offer some new identities involving inverses of binomial coefficients. To do so, we shall make use of the following formula (see [2])

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 t^k(1-t)^{n-k}dt. \tag{1}$$

In particular, we generalize the well-known identity (due to Euler)

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}. \tag{2}$$

2. Main Results

In this section, we give the main results of this paper.

Theorem Let m be a positive integer. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2mn}{mn}} = -\frac{m}{2}\int_0^1 \frac{\ln[1-t^m(1-t)^m]dt}{t(1-t)}, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\binom{2mn}{mn}} = -\frac{m}{2}\int_0^1 \frac{\ln[1+t^m(1-t)^m]dt}{t(1-t)}, \tag{4}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)\binom{2mn}{mn}} &= -\frac{m}{2}\int_0^1 \frac{\ln[1-t^m(1-t)^m]dt}{t(1-t)} \\ &+\frac{m}{2}\int_0^1 \frac{t^m(1-t)^m+\ln[1-t^m(1-t)^m]}{t^{m+1}(1-t)^{m+1}}dt,\end{aligned} \tag{5}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2(n+1)\binom{2mn}{mn}} &= -\frac{m}{2}\int_0^1 \frac{\ln[1+t^m(1-t)^m]dt}{t(1-t)} \\ &+\frac{m}{2}\int_0^1 \frac{t^m(1-t)^m-\ln[1+t^m(1-t)^m]}{t^{m+1}(1-t)^{m+1}}dt.\end{aligned} \tag{6}$$

Proof. We first prove (3). By definition, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2mn}{mn}} &= \frac{m}{2}\sum_{n=1}^{\infty} \frac{1}{n(2mn-1)\binom{2mn-2}{mn-1}} \\ &= \frac{m}{2}\sum_{n=0}^{\infty} \frac{1}{(n+1)(2mn+2m-1)\binom{2mn+2m-2}{mn+m-1}}.\end{aligned}$$

It follows from (1) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2mn}{mn}} = \frac{m}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 t^{mn+m-1}(1-t)^{mn+m-1}dt.$$

Noticing that

$$\sum_{n=0}^{\infty} \frac{t^{mn+m}(1-t)^{mn+m}}{n+1} = -\ln[1-t^m(1-t)^m] \tag{7}$$

converges uniformly for $t \in [0,1]$, we have (3).

Now we show that (5) holds.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)\binom{2mn}{mn}} &= \frac{m}{2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)(2mn-1)\binom{2mn-2}{mn-1}} \\ &= \frac{m}{2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\int_0^1 t^{mn-1}(1-t)^{mn-1}dt \\ &= \frac{m}{2}\Big(\sum_{n=1}^{\infty}\frac{\int_0^1 t^{mn-1}(1-t)^{mn-1}dt}{n} - \sum_{n=1}^{\infty}\frac{\int_0^1 t^{mn-1}(1-t)^{mn-1}dt}{n+1}\Big).\end{aligned}$$

By using (7), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)\binom{2mn}{mn}} &= -\frac{m}{2}\int_0^1\frac{\ln[1-t^m(1-t)^m]dt}{t(1-t)} \\ &\quad +\frac{m}{2}\int_0^1\frac{t^m(1-t)^m+\ln[1-t^m(1-t)^m]}{t^{m+1}(1-t)^{m+1}}dt.\end{aligned}$$

The proofs of equalities (4) and (6) follow the same pattern and are omitted. □

To see that (3) is a generalization of (2), let $I(a) = -\frac{1}{2}\int_0^1\frac{\ln[1-at(1-t)]dt}{t(1-t)}$ $(0 \le a \le 1)$. When $a > 0$,

$$I'(a) = \frac{1}{2}\int_0^1\frac{dt}{at^2-at+1} = \frac{2}{a}\sqrt{\frac{a}{4-a}}\arctan\sqrt{\frac{a}{4-a}}.$$

Then $I(a) = 2\left(\arctan\sqrt{\frac{a}{4-a}}\right)^2 + c$, where c is a constant. Since $\lim_{a\to 0} I(a) = 0$, we get $I(a) = 2\left(\arctan\sqrt{\frac{a}{4-a}}\right)^2$. Hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}} = I(1) = \frac{\pi^2}{18}.$$

By computing the integrals in (4)-(6), we can obtain other identities involving inverses of binomial coefficients. For example, if $m = 1$ in (4), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2n}{n}} = -\frac{1}{2}\int_0^1\frac{\ln[1+t(1-t)]dt}{t(1-t)}.$$

Put $J(a) = -\frac{1}{2}\int_0^1\frac{\ln[1+at(1-t)]dt}{t(1-t)}$ $(0 \le a \le 1)$. When $a > 0$,

$$J'(a) = \frac{1}{2}\int_0^1\frac{dt}{at^2-at-1} = \frac{1}{\sqrt{a^2+4a}}\ln\Big|\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big|.$$

Using some calculus, we find that

$$J(a) = -\frac{1}{2}\Big[\ln\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big]^2 + c_1.$$

Since $\lim_{a\to 0} J(a) = 0$, we have that $c_1 = 0$. Thus

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\binom{2n}{n}} = J(1) = -2\Big(\ln\frac{\sqrt{5}-1}{2}\Big)^2. \tag{8}$$

Let

$$K(a) = \frac{1}{2}\int_0^1 \frac{at(1-t)-\ln[1+at(1-t)]}{t^2(1-t)^2}dt \quad (0 \le a \le 1).$$

Then, when $0 < a \le 1$,

$$K'(a) = -\frac{a}{2}\int_0^1 \frac{dt}{at^2-at-1}.$$

By calculus, we obtain

$$K'(a) = -\frac{\sqrt{a}}{\sqrt{a+4}}\Big[\ln\Big(\sqrt{\frac{a+4}{a}}-1\Big) - \ln\Big(\sqrt{\frac{a+4}{a}}+1\Big)\Big],$$

$$\begin{aligned} K(a) \quad = \quad & -\Big\{\frac{\sqrt{a+4}+\sqrt{a}}{\sqrt{a+4}-\sqrt{a}}\ln\Big(\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big) + \frac{\sqrt{a+4}+\sqrt{a}}{\sqrt{a+4}-\sqrt{a}} + \Big[\ln\Big(\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big)\Big]^2 \\ & -\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\ln\Big(\frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big) + \frac{\sqrt{a+4}-\sqrt{a}}{\sqrt{a+4}+\sqrt{a}}\Big\} + c_2. \end{aligned}$$

Because $\lim_{a\to 0} K(a) = 0$, we find that $c_2 = 2$. By means of (8), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(n+1)\binom{2n}{n}} = -6\Big(\ln\frac{\sqrt{5}-1}{2}\Big)^2 - \sqrt{5}\ln\Big(\frac{3-\sqrt{5}}{2}\Big) - 1.$$

In the same way, from (5) we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)\binom{2n}{n}} = \frac{\sqrt{3}\pi}{3} - \frac{\pi^2}{18} - 1.$$

Acknowledgment

The authors wish to thank the anonymous referee for his/her valuable suggestions for this paper.

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