A CHARACTERIZATION OF MINIMAL ZERO-SEQUENCES OF INDEX ONE IN FINITE CYCLIC GROUPS

Scott T. Chapman[1]
Trinity University, Department of Mathematics, One Trinity Place, San Antonio, TX 78212-7200, USA
schapman@trinity.edu

William W. Smith
The University of North Carolina at Chapel Hill, Department of Mathematics, Phillips Hall, Chapel Hill, NC 27599-3250, USA
wwsmith@email.unc.edu

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Abstract

Let $G \cong \mathbb{Z}_n$ where n is a positive integer. A finite sequence $S = \{g_1, \ldots, g_k\}$ of not necessarily distinct elements from G for which $\sum_{i=1}^{k} g_i = 0$ is called a zero-sequence. If a zero-sequence S contains no proper subzero-sequence, then it is called a *minimal zero-sequence*. The notion of the *index* of a minimal zero-sequence (see Definition 1) in $\mathbb{Z}_n$ has been recently addressed in the mathematical literature. In this note, we offer a characterization of minimal zero-sequences in $\mathbb{Z}_n$ with index 1.

Let G be an additive abelian group and $S = \{g_1, \ldots, g_k\}$ a finite sequence of not necessarily distinct elements from G. Denote by $\mid S \mid = k$ the number of elements in S (or the *length* of S) and let $\text{supp}(S) = \{g \mid g \in G \text{ with } g = g_i \text{ for some } i\}$ be the *support* of S. Various properties of the sequence S have been considered over the last several years in the mathematical literature. Some of these properties are among the following.

1. S is *zero-free* if $\sum_{i \in \mathcal{I}} g_i \neq 0$ for any nonempty subset $\mathcal{I} \subseteq \{1, 2, \ldots, k\}$.

2. S is a *zero-sequence* if $\sum_{i=1}^{k} g_i = 0$.

3. A zero-sequence S is a *minimal zero-sequence (or MZS)* if for every nonempty $\mathcal{I} \subsetneq \{1, 2, \ldots, k\}$, the sequence $\{g_i\}_{i \in \mathcal{I}}$ is zero-free.

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4. A zero-sequence S which is not an MZS is an *almost minimal zero-sequence (or AMZS)* if for every nonempty $\mathcal{I} \subsetneq \{1, 2, \ldots, k\}$ where the sequence $\{g_i\}_{i\in\mathcal{I}}$ is a zero-sequence, then $\{g_i\}_{i\in\mathcal{I}}$ is a minimal zero-sequence.

In this article, we will consider a property of minimal zero-sequences in finite cyclic groups which was introduced in the literature in [2] and consequently considered in greater detail in [4] and [7]. Some notation will be necessary before giving a formal statement describing this property. Since the ordering of the elements in a sequence S does not matter, we will view sequences as elements of $\mathcal{F}(G)$, the free abelian monoid on G. Hence, we write

$$S = \prod_{g\in G} g^{n_g}$$

where only finitely many of the n_g are not zero.

Our goal is to offer a characterization of index 1 minimal zero-sequences in $\mathbb{Z}_n$. This will be done in terms of almost minimal zero-sequences (see [3, Chapter 5] for more information on AMZSs). We will find the language of block monoids useful for expressing and applying some of our arguments. For a finite abelian group G, let $\mathcal{B}(G)$ represent the set of elements in $\mathcal{F}(G)$ which are zero-sequences. Further, let $\mathcal{U}(G)$ be the subset of $\mathcal{B}(G)$ consisting of the minimal zero-sequences of G. If $S_1 = \prod_{g\in G} g^{m_g}$ and $S_2 = \prod_{g\in G} g^{s_g}$ are in $\mathcal{B}(G)$, then $\mathcal{B}(G)$ can be considered as a commutative cancellative monoid under the operation

$$S_1 * S_2 = \prod_{g\in G} g^{m_g+s_g}$$

and is commonly called a *block monoid* (more information on block monoids can be found in [6]). The irreducible elements of $\mathcal{B}(G)$ are merely the elements of $\mathcal{U}(G)$ and the *empty block* (i.e., $S = \emptyset$) acts as the identity of $\mathcal{B}(G)$. An interpretation of an almost minimal zero-sequence in terms of block monoids can be stated as follows: $B \in \mathcal{B}(G)$ is an almost minimal zero-sequence if and only if $B = B_1 \cdots B_t$ with each B_i in $\mathcal{U}(G)$ implies that $t = 2$.

Definition 1. Let G be an abelian group.

(1) Let $g \in G$ be a non-zero element with $\mathrm{ord}(g) = n > 1$. For a sequence $S = (n_1 g) \cdots (n_l g)$, where $l \in \mathbb{N}_0$ and $n_1, \ldots, n_l \in [1, n]$, we define

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n}$$

to be the g-norm of S. If $S = \emptyset$, then set $\|S\|_g = 0$.

(2) Let S be a zero-sum sequence for which $\langle \mathrm{supp}(S)\rangle \subset G$ is a nontrivial finite cyclic group. Then we call

$$\mathrm{index}(S) = \min\{\, \|S\|_g \mid g \in G \text{ with } \langle \mathrm{supp}(S)\rangle = \langle g\rangle\} \in \mathbb{N}_0$$

the *index* of S.

Notice that the index of a sequence S depends only on S and not the choice of the cyclic group G which contains $\text{supp}(S)$. Theorem 2 of [2] indicates that as n increases, there exist minimal zero-sequences of $\mathbb{Z}_n$ of arbitrarily high index. The papers [7] and [4] have both shown that for a fixed value of n, "long" minimal zero-sequences must have index 1. In particular, [4, Section 2] shows for $n \geq 10$ that a minimal zero-sequence S in $\mathbb{Z}_n$ with $\mid S \mid > \frac{2n}{3}$ must have index 1.

When restricting our attention to cyclic groups, the g-norm of an zero-sequence can be used to draw some helpful conclusions. We determine some basic properties of the g-norm in the next proposition.

Proposition 2. *Let G be an abelian group, $g \in G$ a nonzero element and S, $T \in \mathcal{B}(\langle g \rangle)$.*

(1) $\| \cdot \|_g : \mathcal{B}(\langle g \rangle) \to \mathbb{N}_0$ is a monoid homomorphism (i.e., $\|S * T\|_g = \|S\|_g + \|T\|_g$).

(2) $\|S\|_g = 0$ if and only if $S = \emptyset$.

(3) $\|0\|_g = 1$.

(4) If $\|S\|_g = 1$, then S is a MZS.

(5) If $\|S\|_g = 2$, then S is an AMZS.

Proof. The proofs of (1)-(3) are clear. For (4), if $S = S_1 * S_2$ with S_1 and S_2 in $\mathcal{B}(\langle g \rangle)$, then $1 = \|S\|_g = \|S_1\|_g + \|S_2\|_g \geq 2$, a contradiction. For (5), if S is neither an MZS or an AMZS, then $S = S_1 * S_2 * S_3$ for S_1, S_2 and S_3 in $\mathcal{B}(\langle g \rangle)$. The argument now follows as in (4). □

We note that index one MZSs satisfy several interesting properties. Two of these properties follow. Recall that if $S = \prod_{g \in G} g^{n_g}$ is an MZS in $\mathbb{Z}_n$, then the *cross number of* S is defined as $\Bbbk(S) = \sum_{g \in G} \frac{n_g}{\text{ord}(g)}$ where $\text{ord}(g)$ represents the order of g in G (more information on the cross number can be found in [1]). For $S \in \mathcal{B}(G)$ consider these properties.

(P1) $S * S$ is an AMZS in $\mathbb{Z}_n$.

(P2) $\Bbbk(S) \leq 1$.

It follows directly from Proposition 2 that $S = \prod_{g \in G} g^{n_g}$ an MZS in $\mathbb{Z}_n$ with $\|S\|_g = 1$ satisfies **(P1)**. That $\|S\|_g = 1$ implies $\Bbbk(S) \leq 1$ can be seen as follows. Suppose $S = (n_1 g) \cdots (n_l g)$ is written as in Definition 1 with $n = \text{ord}(g)$. Then

$$\Bbbk(S) = \sum_{i=1}^{l} \frac{1}{\text{ord}(n_i g)} = \sum_{i=1}^{l} \frac{1}{\frac{n}{\gcd(n_i, n)}} \leq \sum_{i=1}^{k} \frac{n_i}{n} = \|S\|_g = 1.$$

Hence we have the following.

Proposition 3. *If S is a MZS of $\mathbb{Z}_n$ with* $\text{index}(S) = 1$, *then S satisfies properties* (**P1**) *and* (**P2**).

Example 4. Properties **(P1)** and **(P2)** do not characterize MZSs of index 1. Notice that all of the index 2 MZSs in [2] do not satisfy **(P1)** (see in particular the proof of [2, Theorem 2]). A slight modification of the construction used in [2] yields the following example. Let $G = \mathbb{Z}_{23}$ and set $S = 2 \cdot 7 \cdot 9 \cdot 11 \cdot 17$. It is a routine calculation to check the 22 possible values of $\|S\|_g$ and determine that $\text{index}(S) = 2$. Since $\Bbbk(S) \leq 1$, S satisfies **(P2)**. For considering property **(P1)**, note that $\|S\|_1 = 2$ and so $\|S * S\|_1 = 4$. To establish that $S * S$ is an AMZS, one needs only observe that if it were not, then $S * S = A * B * C$ for some zero sequences A, B, and C. It follows that this has to be done (with the proper choice of g) so that $\|A\|_g = \|B\|_g = 1$ and $\|C\|_g = 2$. The key then to observing such a decomposition is impossible is to note that $7^2 \cdot 9$ is the only subsequence of $S * S$ that sums to 23.

While **(P1)** and **(P2)** do not offer the characterization of index 1 MZSs we desire, a relatively simple condition involving the AMZS's which contain an MZS S does provide a characterization.

Theorem 5. *Let G be an abelian group and S a minimal zero-sequence over G such that* $\text{supp}(S)$ *generates a cyclic group H of order $n \geq 2$. Then the following statements are equivalent:*

- *(a) There exists some AMZS $A \in \mathcal{F}(H)$ of length $|A| = |S| + n$ where S divides A in $\mathcal{B}(G)$.*
- *(b) There exists some $g \in H$ such that $g^n S$ is an AMZS.*
- *(c)* $\text{index}(S) = 1$.

Proof. (a) ⇒ (b) Let $A = ST$ be an AMZS of length $|S| + n$ for some $T \in \mathcal{F}(H)$. Then T is a minimal zero-sum sequence of length n. Thus, for example by [5, Lemma 13], there exists some $g \in H$ such that $T = g^n$.

(b) ⇒ (c) Let $g \in H$ and $A = g^n S$ an AMZS. Then there are $m_1, \ldots, m_l \in [1, n-1]$ with $m_1 \leq \ldots \leq m_l$ such that $S = \prod_{i=1}^{l}(m_i g)$. We assert that $\|S\|_g = 1$. Assume to the contrary that

$$\|S\|_g = \frac{m_1 + \ldots + m_l}{n} = k \text{ with } k \geq 2.$$

Since S is a minimal zero-sum sequence, there exist u, $v \in [1, l-1]$ such that

$$(k-2)n < m_1 + \ldots + m_u < (k-1)n < m_1 + \ldots + m_u + m_{u+1}$$

and

$$m_{u+1} + \ldots + m_v < n < m_{u+1} + \ldots + m_v + m_{v+1}.$$

We set

$$r = (k-1)n - (m_1 + \ldots + m_u),$$

$$s = n - (m_{u+1} + \ldots + m_v)$$

and we define

$$N_1 = g^r \prod_{i=1}^{u} (m_i g), \; N_2 = g^s \prod_{i=u+1}^{v} (m_i g) \text{ and } N_3 = g^{n-(r+s)} \prod_{i=v+1}^{l} (m_i g).$$

By construction, N_1, N_2 and N_3 are zero-sum sequences with $A = N_1 N_2 N_3$, a contradiction to the fact that A is an AMZS.

(c) $\Rightarrow$ (a) Let $g \in H$ such that $\|S\|_g = 1$. We set $A = g^n S$, and since $\|A\|_g = 2$, it follows that A is an AMZS. □

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