ON SYMMETRIC AND ANTISYMMETRIC BALANCED BINARY SEQUENCES

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Abstract

Let $X = (x_1, \dots, x_n)$ be a finite binary sequence of length n, *i.e.*, $x_i = \pm 1$ for all i. The *derived sequence* of X is the binary sequence $\partial X = (x_1x_2, \dots, x_{n-1}x_n)$ of length $n-1$, and the *derived triangle* of X is the collection ΔX of all derived sequences $\partial^i X$ for $0 \le i \le n-1$. We say that X is *balanced* if its derived triangle ΔX contains as many $+1$'s as -1's. This concept was introduced by Steinhaus in 1963. It is known that balanced binary sequences occur in every length $n \equiv 0$ or $3 \bmod 4$, and in none other. In this paper, we solve the problem of determining all possible lengths of symmetric and of antisymmetric balanced binary sequences. We prove that (1) there exists a symmetric balanced binary sequence of length n if and only if $n \equiv 0$, 3 or $7 \bmod 8$, and (2) there exists an antisymmetric balanced binary sequence of length n if and only if $n \equiv 4 \bmod 8$.

1. Introduction

Let $X = (x_1, x_2, \dots, x_n)$ be a *binary sequence* of length n, i.e., a sequence with $x_i = \pm 1$ for all i. We define the *derived sequence* ∂X of X as $\partial X = (y_1, \dots, y_{n-1})$ where $y_i = x_i x_{i+1}$ for all i. By convention, we agree that $\partial X = \emptyset$ whenever $n = 0$ or 1, where $\emptyset$ stands for the empty binary sequence of length $n = 0$. More generally, for $k \ge 0$, we denote by $\partial^k X$ the kth derived sequence of X, defined recursively as usual by $\partial^0 X = X$ and $\partial^k X = \partial\big(\partial^{k-1} X\big)$ for $k \ge 1$.

We denote by ΔX the collection $X, \partial X, \dots, \partial^{n-1} X$ of the iterated derived sequences of X. This collection may be pictured as a triangle, as in the following example: if $X =$

$(1, 1, -1, 1, -1, 1, 1)$, abbreviated as $++-+-++$, then $\Delta X =$

$$\begin{array}{c} + + - + - + + \\ + - - - - + \\ - + + + - \\ - + + - \\ - + - \\ - - \\ + \end{array}$$

Notation 1.1 Let $X = (x_1, \ldots, x_n)$ be a binary sequence of length n. We denote by $\sigma(X)$ the sum of the x_i, i.e., $\sigma(X) = \sum_{i=1}^{n} x_i$. We also denote by $\sigma\Delta(X)$ the sum of the elements in $\Delta(X)$, i.e., $\sigma\Delta(X) = \sum_{i=0}^{n-1} \sigma\big(\partial^i X\big)$.

Definition 1.2 The binary sequence $X = (x_1, \ldots, x_n)$ is said to be *balanced* if its derived triangle $\Delta(X)$ contains as many $+1$'s as -1's. In other words, X is balanced if $\sigma\Delta(X) = 0$.

This concept was introduced by Steinhaus in [3]. The author observed there that no binary sequence X of length $n \equiv 1$ or $2 \bmod 4$ may be balanced. Indeed, in that case the total number of terms in $\Delta(X)$, namely $\binom{n+1}{2}$, is odd, and so is $\sigma\Delta(X)$.

Steinhaus asked in [3] whether balanced binary sequences occurred in every length $n \equiv 0$ or $3 \bmod 4$. This was answered positively by Harborth in 1972 [2]. More recently, a new solution was proposed in [1], through the construction of *strongly balanced* binary sequences in every length $n \equiv 0$ or $3 \bmod 4$. (A binary sequence of length n is *strongly balanced* if its initial segment of length k is balanced for all $k \equiv n \bmod 4$.)

In this paper, we shall deal with balanced binary sequences having one of the special properties defined below. First we introduce a

Notation 1.3 Let $X = (x_1, \ldots, x_n)$ be a binary sequence. We denote by $\overline{X}$ the reversed sequence $\overline{X} = (x_n, \ldots, x_1)$.

Definition 1.4 The sequence $X = (x_1, \ldots, x_n)$ is said to be *symmetric* if $\overline{X} = X$, *i.e.*, if $x_{n+1-i} = x_i$ for all $1 \le i \le n$. It is said to be *antisymmetric* if $\overline{X} = -X$, *i.e.*, if $x_{n+1-i} = -x_i$ for all $1 \le i \le n$. Finally, X is said to be *zero-sum* if $\sigma(X) = \sum_{i=1}^{n} x_i = 0$.

In Section 5 of [1], we stated a couple of problems related to balanced sequences. Here, we shall address two of them:

1. Do there exist infinitely many **symmetric** balanced binary sequences?
2. Do there exist infinitely many **zero-sum** balanced binary sequences?

The second problem is due to M. Kervaire. The interest for zero-sum balanced binary sequences X lies in the fact that their derived sequences ∂X are also balanced. We shall show here that such sequences occur in every length $n \equiv 4 \bmod 8$, by constructing suitable antisymmetric ones. In a forthcoming paper, an independent construction will provide zero-sum balanced binary sequences in every length $n \equiv 0 \bmod 4$, thus giving one more solution to Steinhaus' original problem.

2. Statements of results

The purpose of the present paper is to answer positively both problems above. Our answer to the first one is embodied in the following result.

Theorem 2.1 *There exists a symmetric balanced binary sequence of length $n \geq 1$ if and only if $n \equiv 0, 3$ or $7 \bmod 8$.*

As for the second question, we shall answer it by producing infinitely many antisymmetric balanced binary sequences. Clearly, any antisymmetric binary sequence X is zero-sum. In fact, as in the symmetric case, we shall determine all their possible lengths.

Theorem 2.2 *There exists an antisymmetric balanced binary sequence of length $n \geq 1$ if and only if $n \equiv 4 \bmod 8$.*

Corollary 2.3 *There exists a zero-sum balanced binary sequence of every length $n \equiv 4 \bmod 8$.*

As shown in Section 4, there is a strong relationship between the symmetric and the antisymmetric case. This relationship allows us to deduce Theorem 2.2 from Theorem 2.1, and reads as follows.

Proposition 2.4 *Let X be a binary sequence of length n. Then, X is antisymmetric and balanced if and only if $n \equiv 4 \bmod 8$ and $\partial(X)$ is symmetric and balanced.*

The existence statement of Theorem 2.1 is established by suitable constructions, described in Section 5 and proved valid in Section 6.

Finally, observe that Theorems 2.1 and 2.2 together produce balanced binary sequences in every length $n \equiv 0$ or $3 \bmod 4$, thereby solving again Steinhaus' original problem.

3. The symmetric case

In this section, we establish one part of Theorem 2.1, namely the necessity of the condition $n \equiv 0$, 3 or $7 \bmod 8$ for the length n of a symmetric balanced binary sequence. We start with a lemma on the value mod 4 of the sum of a symmetric binary sequence.

Lemma 3.1 *Let $Z = (z_1, \dots, z_m)$ be a symmetric binary sequence of length m.*

(1) *If m is even, then $\sigma(Z) \equiv m \bmod 4$.*

(2) *If m is odd, then $\sigma(Z) \equiv m + z_h - 1 \bmod 4$, where $h = \lceil m/2 \rceil = (m+1)/2$.*

Proof. (1) Assume $m = 2h$. Then $Z = (z_1, \dots, z_h, z_h, \dots, z_1)$. Thus, $\sigma(Z) = 2\sum_{i=1}^{h} z_i$. As $z_i \equiv 1 \bmod 2$ for all i, it follows that $\sum_{i=1}^{h} z_i \equiv h \bmod 2$, whence $\sigma(Z) \equiv 2h \equiv m \bmod 4$.

(2) Assume $m = 2h - 1$. Then $Z = (z_1, \dots, z_{h-1}, z_h, z_{h-1}, \dots, z_1)$. Thus, $\sigma(Z) = z_h + 2\sum_{i=1}^{h-1} z_i$. We have $\sum_{i=1}^{h-1} z_i \equiv h - 1 \bmod 2$, whence $\sigma(Z) \equiv z_h + 2(h-1) \bmod 4$. It follows that $\sigma(Z) \equiv m + z_h - 1 \bmod 4$, as claimed. □

We are now ready to state and prove the main result of this section.

Proposition 3.2 *Let $X = (x_1, \dots, x_n)$ be a symmetric binary sequence of length n. Assume that X is balanced.*

(1) *If n is even, then $n \equiv 0 \bmod 8$.*

(2) *If n is odd, then $n \equiv -2x_h + 1 \bmod 8$, where $h = \lceil n/2 \rceil$. In other words, $n \equiv -1$ or $3 \bmod 8$, depending on whether $x_h = 1$ or -1, respectively.*

Proof. As X is balanced, we have $0 = \sigma\Delta(X) = \sum_{i=0}^{n-1} \sigma\big(\partial^i X\big)$. We start by evaluating the individual summands $\sigma\big(\partial^i X\big) \bmod 4$.

Claim $\sigma\big(\partial^i X\big) \equiv n - i \bmod 4$, for all $1 \leq i \leq n-1$.

Clearly, $\partial^i X$ is symmetric of length $n - i$, for all $1 \leq i \leq n - 1$.

- The case n even: if i is even, then $n - i$ is also even and the claim follows from (1) of Lemma 3.1. If i is odd, then $n - i$ is also odd; let then y denote the middle term of $\partial^i X$, at position $\lceil (n-i)/2 \rceil$. Since $i \geq 1$, $\partial^i X$ is the derived sequence of $\partial^{i-1} X$. Given that $\partial^{i-1} X$ is symmetric, it follows that $y = 1$, as y is the product of the two equal middle terms in $\partial^{i-1} X$. Thus, by (2) of Lemma 3.1 we have $\sigma\big(\partial^i X\big) \equiv n - i + y - 1 \equiv n - i \bmod 4$, as claimed.
- The case n odd: if i is odd, then $n - i$ is even, and the claim directly follows from (1) of Lemma 3.1. If i is even, then $i \geq 2$, and it follows from (2) of Lemma 3.1 that $\sigma\big(\partial^i X\big) \equiv n - i + a_i - 1 \bmod 4$, where a_i denotes the middle term of $\partial^i X$. But again, $a_i = 1$, as $\partial^i X$ is the derived sequence of $\partial^{i-1} X$. The claim follows in this case as well.

We are ready to conclude the proof of the Proposition. Note that the claim implies that $\sigma\Delta(X) \equiv \sigma(X) + (n-1)n/2 \bmod 4$.

(1) Assume n is even. We have $\sigma(X) \equiv n \bmod 4$ by (1) of Lemma 3.1, and thus $\sigma\Delta(X) \equiv n(n+1)/2 \bmod 4$. Since $\sigma\Delta(X) = 0$ by assumption, it follows that $n \equiv 0 \bmod 8$, as desired.

(2) Assume n is odd. Let $n = 2h - 1$. By (2) of Lemma 3.1, we have $\sigma(X) \equiv n + x_h - 1 \bmod 4$, where x_h is the middle term of X. It then follows from the claim that $\sigma\Delta(X) \equiv n(n+1)/2 + x_h - 1 \bmod 4$. Since $\sigma\Delta(X) = 0$, we get that $n(n+1) \equiv 2(1-x_h) \bmod 8$. As n is odd, we have $n^2 \equiv 1 \bmod 8$, whence $1 + n \equiv 2(1 - x_h) \bmod 8$. The desired statement follows. □

4. From the symmetric to the antisymmetric case

There is a strong relationship between symmetric and antisymmetric balanced binary sequences, as we show here.

Proposition 4.1 *Let X be a binary sequence of length n. Then, X is antisymmetric and balanced if and only if* $n \equiv 4 \bmod 8$ *and $\partial(X)$ is symmetric and balanced.*

Proof.

($\Rightarrow$) If X is antisymmetric, then n is even. It easily follows that $\partial(X)$ is *symmetric*, of odd length $n - 1$, and with middle term equal to -1. Now X is balanced and zero-sum, i.e., $\sigma\Delta(X) = \sigma(X) = 0$. It follows that $\partial(X)$ is also balanced, since $\sigma\Delta(\partial X) = \sigma\Delta(X) - \sigma(X) = 0$. By (2) of Proposition 3.2, it follows that the length $n-1$ of $\partial(X)$ is congruent 3 mod 8. Thus, $n \equiv 4 \bmod 8$.

($\Leftarrow$) If $n \equiv 4 \bmod 8$ and $\partial(X)$ is symmetric, balanced, of length $n - 1$, it follows from (2) of Proposition 3.2 that the middle term of $\partial(X)$ is equal to -1. Therefore, the two *primitives* of $\partial(X)$, namely X and $-X$, are antisymmetric. (Indeed, if

$$\partial(X) = (b_1, b_2, \dots , b_{h-1}, y, b_{h-1}, \dots , b_2, b_1),$$

then $X = (a, ab_1, ab_1b_2, \dots , ab_1b_2 \cdots b_{h-1}, ab_1b_2 \cdots b_{h-1}y, \dots , ab_1b_2y, ab_1y, ay)$ for some $a = \pm1$, an antisymmetric sequence if $y = -1$.) In particular, $\sigma(X) = 0$. Hence, X is also balanced. □

As a consequence, we see that Theorem 2.2 directly follows from Theorem 2.1. Indeed, Theorem 2.1 asserts in particular the existence of a symmetric balanced binary sequence of length n for every $n \equiv 3 \bmod 8$. Taking primitives of these sequences and using Proposition 4.1, it follows that there exist antisymmetric balanced binary sequences of length $n+1$ for every $n+1 \equiv 4 \bmod 8$, and in no other lengths. This is the content of Theorem 2.2.

5. Constructions

We now provide constructions of symmetric balanced binary sequences in every length $n \equiv 0$, 3 or 7 mod 8. We also describe antisymmetric balanced binary sequences of every length $n \equiv 4 \bmod 8$.

Case 1 $n \equiv 0, 7 \bmod 8$.

Let $s_0 = +-----+++++-$. Note that s_0 is antisymmetric of length 12. That is, we have $\overline{s_0} = -s_0$. Next, let $s = (s_0, \overline{s_0})$, the concatenation of s_0 and $\overline{s_0}$. Then s is symmetric, of length 24. As s plays a key role in our constructions, here it is in its full extent:

$$s = + - - - - - + + + + + - - + + + + + - - - - - + \,.$$

Writing $n = 24k + r$, we shall describe constructions depending on $r = 0$, 8, 16, and then on $r = -1$, 7, 15. This will of course cover all lengths $n \equiv 0$ or 7 mod 8.

Construction 5.1 Let $s = + - - - - - + + + + + - - + + + + + - - - - - +$. For every integer $k \geq 0$, the following binary sequences are symmetric and balanced. We denote by s^k the concatenation of s with itself k times.

$$\begin{aligned} \text{In length } 24k &\;:\; s^k. \\ \text{In length } 24k+8 &\;:\; t_k = + - + + \, s^k + + - +. \\ \text{In length } 24k+16 &\;:\; u_k = - - + + + - + + \, s^k + + - + + + - -. \end{aligned}$$

$$\begin{aligned} \text{In length } 24k-1 \; (k \geq 1) &\;:\; \partial\big(s^k\big). \\ \text{In length } 24k+7 &\;:\; + + - + \partial\big(s^k\big) + - + +. \\ \text{In length } 24k+15 &\;:\; + - + + + + - + \partial\big(s^k\big) + - + + + + - +. \end{aligned}$$

Case 2 $n \equiv 3 \bmod 8$.

Construction 5.2 We start by defining a doubly infinite periodic sequence $w = (w_i)_{i \in \mathbb{Z}}$ of period 12. Denote $\pi : \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ the canonical projection. Let

$$p = (p_0, p_1, \ldots, p_{11}) = - + + + + - + - + + + +$$

of length 12, understood as indexed over $\mathbb{Z}/12\mathbb{Z}$. Note, for later use, the following key property of p:

$$p_{\pi(i)} = p_{\pi(-i)} \quad \forall i \in \mathbb{Z}.$$

Define the sequence $w = (w_i)_{i \in \mathbb{Z}}$ by $w_i = p_{\pi(i)}$ for all $i \in \mathbb{Z}$. For any indices $i < j$ in $\mathbb{Z}$, let $w[i, j]$ denote the finite subsequence $w[i, j] = (w_i, w_{i+1}, \dots , w_j)$ of w, of length $j - i + 1$.

We are now ready to define balanced sequences with the desired properties. For all $k \in \mathbb{N}$, let

$$v_k = w[-4k - 1, 4k + 1].$$

We claim that, for every $k \geq 0$, *the binary sequence v_k is symmetric, balanced, of length $8k + 3$.*

The symmetry of v_k easily follows from the property $p_{\pi(i)} = p_{\pi(-i)}$ for all $i \in \mathbb{Z}$ noted above. The fact that v_k is balanced is proved in Section 6.

As an illustration, here are the sequences v_k for small values of k. We have $v_0 = + - +$, $v_1 = -{+}{+}{+}v_0{+}{+}{+}-$, $v_2 = {+}{+}{-}{+}v_1{+}{-}{+}{+}$, $v_3 = {+}{-}{+}{+}v_2{+}{+}{-}{+}$, $v_4 = -{+}{+}{+}v_3{+}{+}{+}-$ and so on, that is

$$v_0 = + - +,$$

$$v_1 = - + + + + - + + + + -,$$

$$v_2 = + + - + - + + + + - + + + + - + - + +,$$

$$v_3 = + - + + + + - + - + + + + - + + + + - + - + + + + - +,$$

$$v_4 = - + + + + - + + + + - + - + + + + - + + + + - + - + + + + - + + + + -.$$

Case 3 $n \equiv 4 \bmod 8$.

Finally, in order to construct antisymmetric balanced binary sequences of every length $n \equiv 4 \bmod 8$, we use Proposition 4.1 together with the sequences v_k defined in Construction 5.2 above.

Construction 5.3 For every $k \geq 0$, let w_k denote one of the two binary primitives of v_k, that is, binary sequences such that $\partial w_k = v_k$. Since v_k is symmetric and balanced, it follows from Proposition 4.1 that w_k *is antisymmetric, balanced, of length $n = 8k + 4$.* To be more explicit, observe that if $v_k = (x_1, \dots , x_n)$, where $n = 8k + 3$, then its two primitives are $w_k = (a, ax_1, ax_1x_2, \dots , ax_1 \cdots x_n)$ with $a = 1$ or -1.

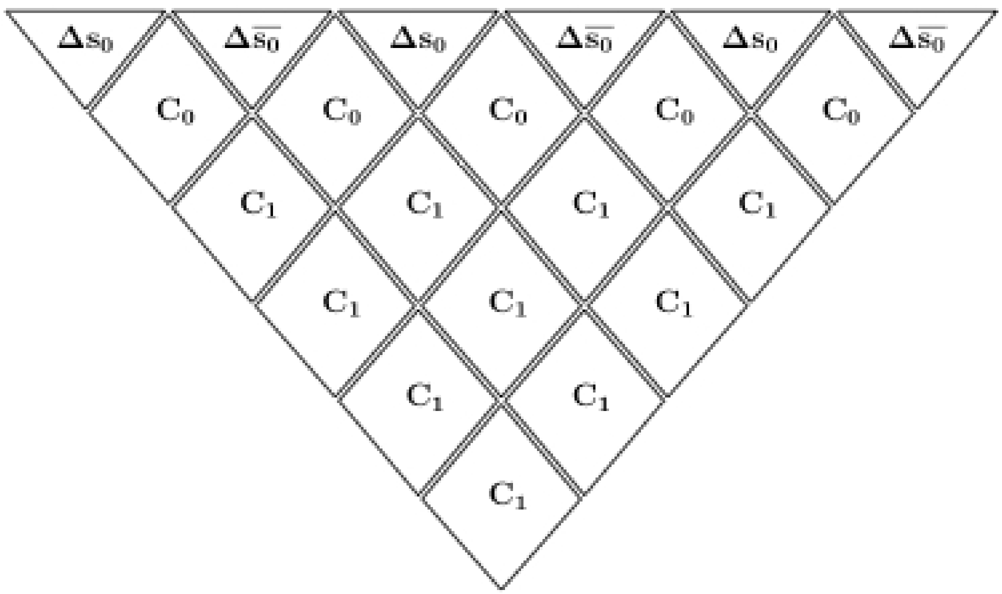
6. Proofs

We prove here that the constructions given in Section 5 are valid, in the sense that the binary sequences obtained there are indeed balanced. (Their symmetry or antisymmetry properties have already been discussed.)

The case of Construction 5.1 in length 24k.

Let us recall the notations given in Section 5. We first consider the sequence $s_0 = + - - - - - + + + + + -$, which is an antisymmetric balanced binary sequence of length 12. We then define $s = (s_0, \overline{s_0})$, which is symmetric of length 24. As easily checked, s is balanced. So Construction 5.1 is valid for $k = 1$.

Let k be a positive integer with $k \geq 2$. We want to prove that the binary sequence s^k is a symmetric balanced binary sequence of length $24k$. By construction, it is clear that s^k is symmetric of length $24k$; it remains to prove that s^k is balanced. This will come from the following remarkably simple structure of the derived triangle Δs^k:



More specifically, we will prove that there exist two squares of length 12, denoted C_0 and C_1, such that the derived triangle Δs^k is the assembly of k triangles Δs_0, k triangles $\Delta\overline{s_0}$, and the components C_0 and C_1, as depicted in the figure (representing the case $k = 3$).

It turns out that the squares C_0 and C_1 are zero-sum, i.e., $\sigma(C_0) = \sigma(C_1) = 0$. This is easy to check on the pictures of C_0 and C_1 given in the Appendix. On the other hand, s_0 and $\overline{s_0}$ are balanced and thus $\sigma(\Delta s_0) = \sigma(\Delta\overline{s_0}) = 0$. These facts together with the claimed structure of Δs^k immediately imply $\sigma(\Delta s^k) = 0$, as desired.

In order to prove that Δs^k does have this structure, we need to introduce the following notations.

Notation 6.1

- $x_{p,q}$ denotes the qth digit in the pth row of Δs^k, for all $1 \leq p \leq n-p+1$ and $1 \leq q \leq n$. In particular, the first row of Δs^k, that is s^k itself, is constituted by the elements $x_{1,1}, x_{1,2}, \ldots, x_{1,24k}$, and the left side of the triangle Δs^k consists of $x_{1,1}, x_{2,1}, \ldots, x_{24k,1}$.

The basic defining property of the triangle Δs^k thus reads $x_{p+1,q} = x_{p,q}x_{p,q+1}$ for all $p, q \geq 1$.

- For all integers i, j, m, $d^+_{i,j}(m)$ (resp. $d^-_{i,j}(m)$) represents the diagonal (resp. the antidiagonal) of length m going down from $x_{i,j}$ in Δs^k. In other words, we have: $d^+_{i,j}(m) = (x_{i,j}, x_{i+1,j}, \ldots, x_{i+m-1,j})$ and $d^-_{i,j}(m) = (x_{i,j}, x_{i+1,j-1}, \ldots, x_{i+m-1,j-m+1})$.
- For all integers i, j, m, we denote by $C_{i,j}(m)$ the square of side of length m whose four vertices are $x_{i,j}$, $x_{i+m-1,j-m+1}$, $x_{i+2(m-1),j-m+1}$ and $x_{i+m-1,j}$.
- The basic squares C_0 and C_1 are defined as $C_0 = C_{2,12}$ and $C_1 = C_{14,12}$.

The claimed structure of Δs^k is embodied in the following assertion.

Claim We have:

- $\forall p \in \{1, 2, \ldots, 2k-1\}, C_{2,12p}(12) = C_0$, and
- $\forall q \in \{1, 2, \ldots, 2k-2\}, \forall p \in \{1, 2, \ldots, 2k-q-1\}, C_{12q+2,12p}(12) = C_1$.

The first part of the Claim easily follows from

Observation 1 The square $C_{i,j}(m)$ is completely determined by the antidiagonal $d^-_{i-1,j}(m)$ and the diagonal $d^+_{i-1,j+1}(m)$.

This is a straightforward consequence of the basic property $x_{p+1,q} = x_{p,q}x_{p,q+1}$ in the derived triangle Δs^k.

In the sequel, the triangles Δs_0 and $\Delta \overline{s_0}$ will be denoted by Δ and $\overline{\Delta}$, respectively.

Let us consider the squares $C_0 = C_{2,12}(12)$ and $C_{2,24}(12)$. According to Observation 1, C_0 (resp. $C_{2,24}(12)$) is completely determined by the SE side of Δ (resp. $\overline{\Delta}$) and the SW side of $\overline{\Delta}$ (resp. Δ). But it can be easily checked in Δs^2 that the squares C_0 and $C_{2,24}(12)$ are equal. Hence, the SE side of Δ and the SW side of $\overline{\Delta}$ determine exactly the same square as the SE side of $\overline{\Delta}$ and the SW side of Δ.

In other words, by an easy induction on p using this property and the structure of s^k, we obtain: $\forall p \in \{1, 2, \ldots, 2k-1\}, C_{2,12p}(12) = C_0$. This concludes the proof of the first part of the Claim.

Remark As $\overline{s_0}$ is the sequence obtained by reversing s_0, we have $x_{1,12} = x_{1,13}$; hence the North vertex of C_0, namely $x_{2,12}$, is equal to 1. On the other hand, it is obvious that the derived triangle $\overline{\Delta}$ is the mirror image of Δ, so we derive from Observation 1 that C_0 is symmetric with respect to its vertical axis.

We now want to prove the second part of the Claim. We will first prove that every square of the form $C_{14,12p}(12)$, with $2 \leq p \leq 2k-2$, is equal to C_1.

Indeed, according to the first part of the Claim, we know that all the squares of the form $C_{14,12p}(12)$ are determined by the SE and the SW sides of two squares C_0. So, by Observation 1, we obtain that all the squares $C_{14,12p}(12)$ are equal to $C_{14,12}(12) = C_1$.

Remark As C_0 is symmetric with respect to its vertical axis, we have: $x_{14,12} = x_{13,12}x_{13,13} = (x_{13,12})^2 = 1$, and C_1 is symmetric with respect to its vertical axis, too. In particular, the sequence forming the SW side of C_1 is equal to the sequence forming its SE side.

We have proved: $\forall p \in \{2, 3, \dots, 2k-2\}, C_{14,12p}(12) = C_1$. Here is the key observation which will enable us to finish the proof of the Claim:

Observation 2 The South sides of C_1 are equal to the South sides of C_0, as sequences.

This can be checked directly in the pictures of C_0 and C_1 in the Appendix.

As a consequence, the SE and the SW sides of C_1 determine the same square as the SE and the SW sides of C_0, namely C_1.

Let us now consider the squares of the "third level" in Δs^k, i.e., the squares of the form $C_{26,12p}(12)$, with $1 \le p \le 2k-3$. As they are all determined by the SE and the SW sides of two squares C_1, we deduce from Observation 2 that they are all equal to C_1.

By an easy induction on q using the same arguments, it can be proved that all the squares of the form $C_{12q+2,12p}(12)$, with $2 \le q \le 2k-2$ and $1 \le p \le 2k-q-1$, are equal to C_1.

The Claim is now proved, giving the structure of Δs^k and concluding the validity of Construction 5.1 in length $24k$.

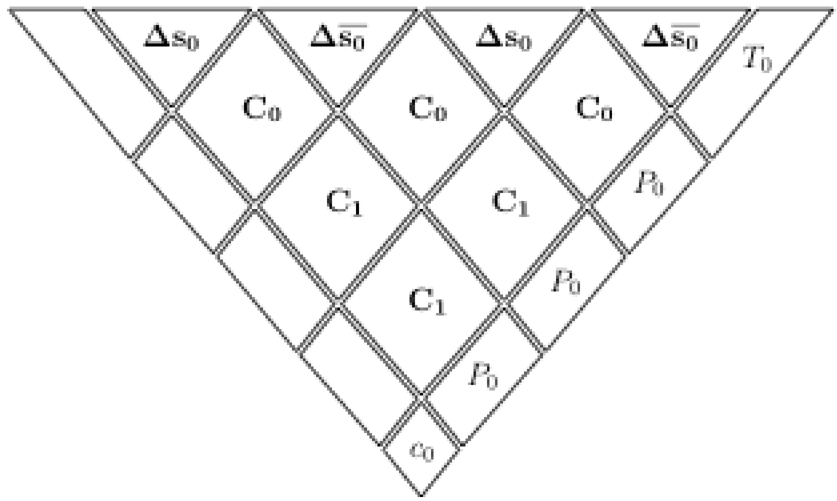
The case of Construction 5.1 in length $24k+8$.

For every $k \in \mathbb{N}$, we consider the following symmetric binary sequence of length $24k+8$: $t_k = +-++\, s^k ++-+$. We want to prove that, for every $k \in \mathbb{N}$, t_k is balanced. The case $k = 0$ can easily be proved by inspection: it suffices to build the derived triangle of the corresponding sequence of length 8 and check that its sum is 0.

Suppose now that k is positive. Schematically, enlarging Δs^k into Δt_k can be seen as adding two diagonal strips of width 4 on both sides of Δs^k: one NW/SE diagonal strip on its left, denoted D_l^k, and one NE/SW diagonal strip on its right, denoted D_r^k. As we already have proved that Δs^k has sum 0, it just remains to check that D_l^k and D_r^k both have sum 0. The fact that t_k is balanced will follow.

First of all, t_k is a symmetric sequence and Δs^k is symmetric with respect to its vertical axis. So Δt_k is also symmetric with respect to its vertical axis. In particular, the strip D_l^k is the mirror image of the strip D_r^k. Hence it suffices to prove that D_r^k has sum 0, and it will follow immediately that the sequence t_k is balanced.

Actually, we want to prove that Δt_k has the following periodic structure:



More specifically, we will prove that the diagonal strip D_r^k is made of one trapezoid T_0, of $2k-1$ parallelograms P_0 and of one square c_0 of side of length 4. Let us recall the notions introduced in [1]:

Notation 6.2

- For $i \equiv 1 \bmod 4$ and $j \equiv 1 \bmod 4, 1 \leq i \leq j$, $T_{i,j}$ denotes the trapezoid of digits whose four vertices are $x_{1,j}$, $x_{1,j+3}$, $x_{i+3,j+1-i}$ and $x_{i,j+1-i}$.
- For all integers i, j such that $1 \leq i \leq j$, $P_{i,j}$ denotes the parallelogram of digits of width 4 and length 12 whose four vertices are $x_{i,j}$, $x_{i+3,j}$, $x_{i+14,j-11}$ and $x_{i+11,j-11}$.

Let us now consider the derived triangle of t_1 and define the trapezoid $T_0 = T_{13,29}$, the parallelogram $P_0 = P_{14,16}$, and the square $c_0 = C_{26,4}(4)$.

$* \; T_0 := T_{13,29} =$

+ + − +
+ + − −
− + − +
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+ + − +
− + − −
− − − +
+ + + −
+ + + −
− + + −
+ − + −
+ − − −
− + +
− +
−

$* \; P_0 := P_{14,16} =$

+
+ −
− − +
+ + − −
+ + − +
− + − −
− − − +
+ + + −
+ + + −
− + + −
+ − + −
+ − − −
− + +
− +
−

$$* \, c_0 := C_{26,4}(4) = \begin{array}{ccccccc} & & & + & & & \\ & & - & & - & & \\ & + & & + & & + & \\ - & & + & & + & & - \\ & - & & + & & - & \\ & & - & & - & & \\ & & & + & & & \end{array}$$

Denoting by the symbol + the NE/SW concatenation, here is the key formula we shall prove:

Claim $\forall k \geq 1, D_r^k = T_0 + (2k-1)P_0 + c_0$.

As it can be checked in the definitions of these three quadrilaterals, each one is of sum 0. Hence, from the Claim and the symmetric structure of Δt_k, we are done since, for all $k \geq 1$, we have

$$\begin{aligned} \sigma\Delta(t_k) &= \sigma(D_l^k) + \sigma(\Delta s^k) + \sigma(D_r^k) - \sigma(c_0) \\ &= \sigma(\Delta s^k) + 2\sigma(D_r^k) - \sigma(c_0) \\ &= \sigma(\Delta s^k) + 2\sigma(T_0) + 2(2k-1)\sigma(P_0) + \sigma(c_0) \\ &= 0. \end{aligned}$$

Let us now prove the Claim. To this end, the next observation (see also [1]) will be useful.

Observation 3 The trapezoid $T_{i,j}$ is completely determined by its North side and by the antidiagonal $d^-_{1,j-1}(i)$, which consists of the 12 digits adjacent to its West side. In the same way, the parallelogram $P_{i,j}$ is completely determined by the diagonal of length 4 going down from $x_{i-1,j-1}$ and by the antidiagonal of length 12 going down from $x_{i-1,j}$, namely by $d^+_{i-1,j+1}(4)$ and $d^-_{i-1,j}(12)$.

Let $k \geq 2$. Observation 3 applied to the trapezoid $T_{13,24k+5}$ in Δt_k gives that $T_{13,24k+5}$ is completely determined by its North side, i.e., the digits $++-+$, and the antidiagonal $d^-_{1,24k+4}(13)$, namely the SE side of $\overline{\Delta}$. But this is also the case of the trapezoid T_0 in Δt_1, so we deduce that $T_{13,24k+5} = T_0$.

We now apply the second part of Observation 3 to the parallelogram $P_{14,24k-8}$ in Δt_k: it is completely determined by the South side of the trapezoid $T_{13,24k+5} = T_0$ and by the SE side of the square C_0 (see the proof for Construction 5.1 in length $24k$). As it is also the case of the parallelogram P_0 in Δt_1, we obtain the following equality: $P_{14,24k-8} = P_0$.

We need one last observation to conclude.

Observation 4 The South side of P_0 is equal to the South side of T_0.

This can be checked directly in the definitions of T_0 and P_0.

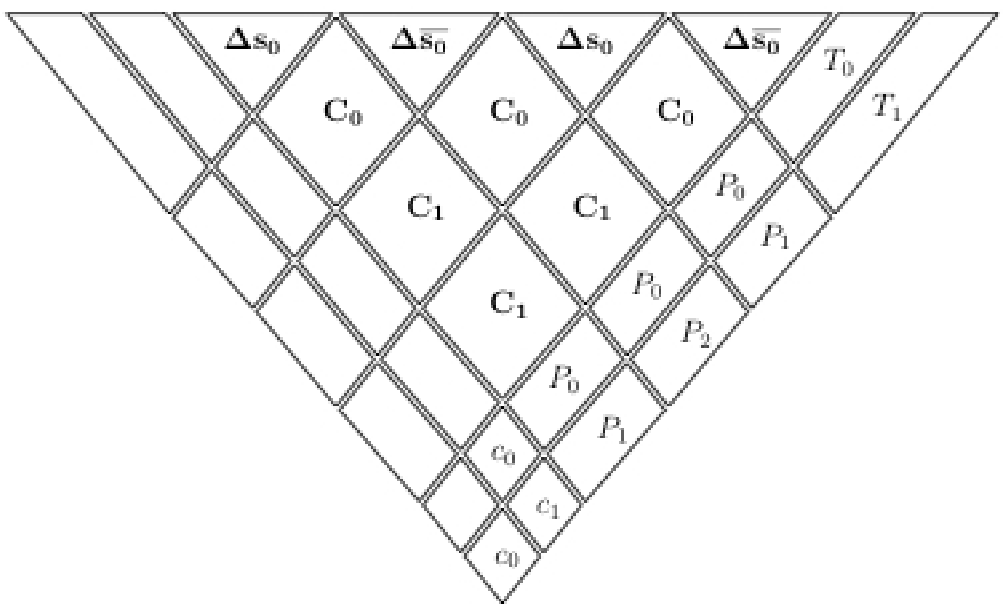
The parallelogram under $P_{14,24k-8} = P_0$ in the diagonal strip D_r^k, namely $P_{26,24k-20}$, is completely determined by the South side of P_0 and the SE side of the square C_1. By Observation 2, the South sides of C_1 are equal to the South sides of C_0. Hence, using Observation 4, we derive that $P_{26,24k-20}$ is completely determined by the South side of T_0 and the SE side of the square C_0, which means that we have: $P_{26,24k-20} = P_0$.

Using this last argument repeatedly, we prove that the strip D_r^k contains $2k-1$ parallelograms P_0. Finally, we obtain that the last square of side 4 in the strip D_r^k is completely determined by the South side of a parallelogram P_0 and the South side of its mirror image; hence it is equal to c_0.

We have now proved the Claim, concluding the validity of Construction 5.1 in length $24k+8$.

The case of Construction 5.1 in length $24k+16$.

We shall prove that, for every $k \in \mathbb{N}$, the derived triangle of the sequence $u_k = --++t_k++--$ has the following periodic structure:



Let $\tilde{D}_r^k$ (resp. $\tilde{D}_l^k$) denote the NE/SW diagonal strip (resp. the NW/SE diagonal strip) we add to the right (resp. to the left) of Δt_k to enlarge Δt_k into Δv_k. By considerations of symmetry, we know that $\tilde{D}_l^k$ is the mirror image of $\tilde{D}_r^k$.

We want to prove that the strip $\tilde{D}_r^k$ is made of one trapezoid T_1, of k parallelograms P_1 and $k-1$ parallelograms P_2, and of two squares of side 4, namely c_0 and another square c_1. In symbols, we want to prove this equality:

$$\forall k \geq 1,\ \tilde{D}_r^k = T_1 + P_1 + \underbrace{(P_2 + P_1) + \ldots + (P_2 + P_1)}_{(k-1)\ times} + c_1 + c_0.$$

This can be proved using Observations 1 and 3, and the special form of the strip D_r^k. The method is exactly the same as in length $24k+8$, except that two different parallelograms alternate to form the strip $\tilde{D}_r^k$, namely P_1 and P_2. We shall not give all the details, only the properties of the quadrilaterals forming the strip $\tilde{D}_r^k$ which enable us to prove its periodic structure.

Writing explicitly T_1 and P_2, one notes that their South sides are equal. But the South side of T_1 and the East side of P_0 determine P_1, and the South side of P_1 and the East side of P_0 determine P_2. So the third parallelogram in the strip $\tilde{D}_r^k$ will be equal to P_1, and so on. Finally, the last parallelogram of the strip will be P_1.

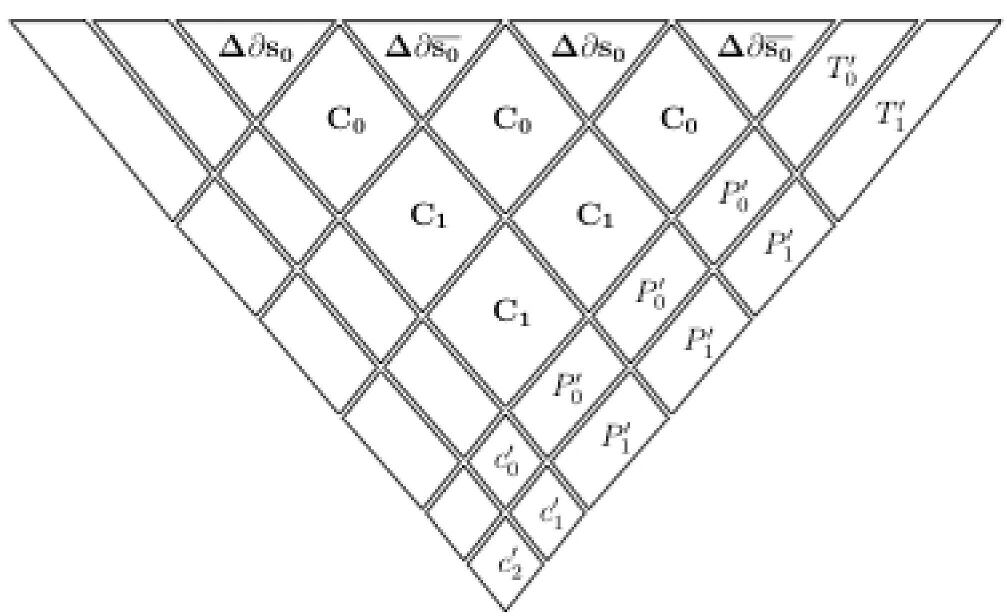
Hence the first square of $\tilde{D}_r^k$ is determined by the South side of P_1 and the SE side of c_0; it is equal to c_1. But the SW side of c_1 is equal to the South side of P_0, so the last square of $\tilde{D}_r^k$ is determined by the South sides of c_1 and it is equal to c_0 (see the proof for Construction 5.1 in length $24k+8$).

It remains to check that the quadrilaterals T_1, P_1, P_2 and c_1 all have sum 0, and this is straightforward. As we already proved that, for every $k \in \mathbb{N}$, the sequence t_k is balanced, it follows immediately from the structure of Δu_k that the sequence u_k is also balanced, for every $k \in \mathbb{N}$.

The case of Construction 5.1 in length $n \equiv 7 \bmod 8$.

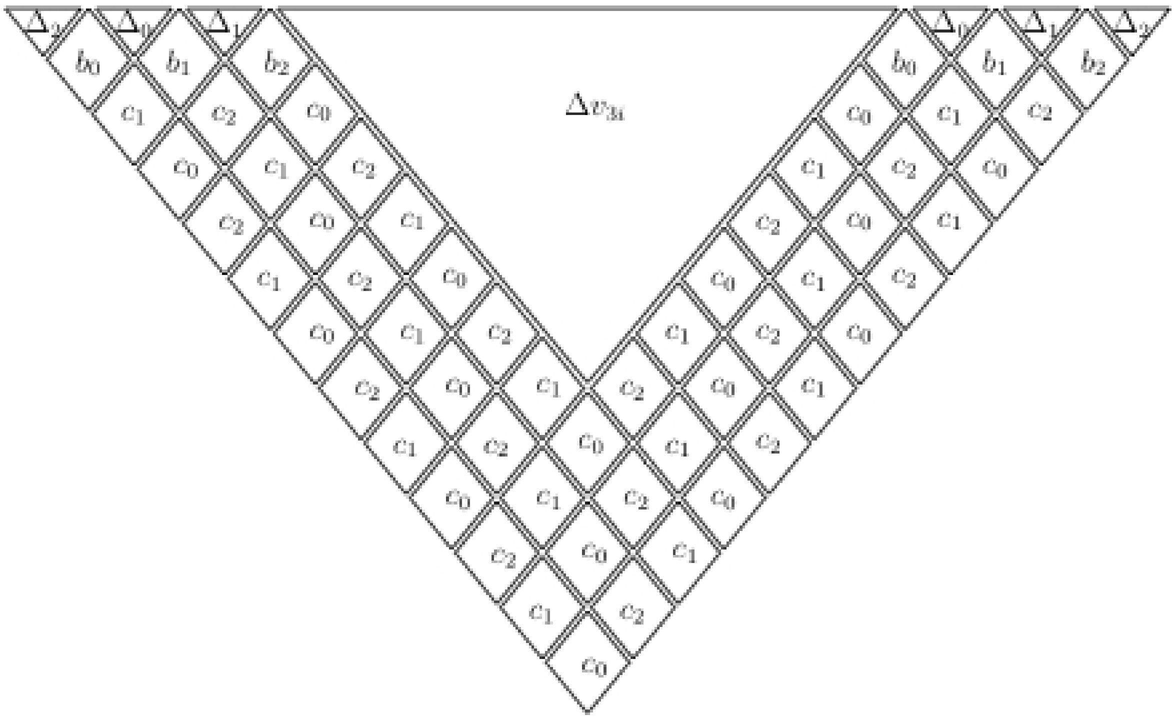
Let k be a positive integer. We know that the sequence s^k is balanced and zero-sum, i.e., $\sigma\Delta(s^k) = \sigma(s^k) = 0$. It follows that $\partial(s^k)$ is also balanced, since $\sigma\Delta(\partial s^k) = \sigma\Delta(s^k) - \sigma(s^k) = 0$. Moreover, the derived triangle of ∂s^k is obtained from Δs^k by removing its first line, and thus has a similar structure as Δs^k.

The proof in length $n \equiv 7 \bmod 8$ is similar to the proof in length $n \equiv 0 \bmod 8$. We only give the global structure in length $24k+15$, which also displays the structure of the derived triangles in lengths $24k-1$ and $24k+7$:



The case of Construction 5.2 in length $n \equiv 3 \bmod 8$.

By an easy yet tedious induction on i, one can prove that the strips we add to Δv_{3i} to obtain Δv_{3i+1}, Δv_{3i+2} and then $\Delta v_{3(i+1)}$, have the following periodic form:



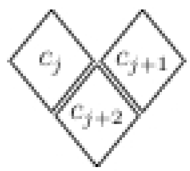
In this picture, Δ_0, Δ_1 and Δ_2 represent triangles of side of length 3, whereas b_0, b_1, b_2, c_0, c_1 and c_2 are squares of side 4. Here are their pictures:

$$* \; \Delta_0 = \overline{\Delta_1} = \begin{array}{ccccc} + & & + & & - \\ & + & & - & \\ & & - & & \end{array} \qquad * \; \Delta_2 = \overline{\Delta_2} = \begin{array}{ccccc} + & & - & & + \\ & - & & - & \\ & & + & & \end{array}$$

$$* \; b_0 = \overline{b_2} = \begin{array}{ccccccc} & & & + & & & \\ & & + & & + & & \\ & - & & + & & + & \\ - & & - & & + & & - \\ & + & & - & & - & \\ & & - & & + & & \\ & & & - & & & \end{array} \qquad * \; b_1 = \overline{b_1} = \begin{array}{ccccccc} & & & + & & & \\ & & - & & - & & \\ & + & & + & & + & \\ - & & + & & + & & - \\ & - & & + & & - & \\ & & - & & - & & \\ & & & + & & & \end{array}$$

$$* \, c_0 = \overline{c_0} = \begin{array}{ccccccc} & & & + & & & \\ & & + & & + & & \\ & - & & + & & - & \\ + & & - & & - & & + \\ & - & & + & & - & \\ & & - & & - & & \\ & & & + & & & \end{array} \qquad * \, c_1 = \overline{c_2} = \begin{array}{ccccccc} & & & + & & & \\ & & - & & - & & \\ & - & & + & & + & \\ + & & - & & + & & + \\ & - & & - & & + & \\ & & + & & - & & \\ & & & - & & & \end{array}$$

The claimed structure follows from the symmetry of the triangles and of the above squares, and the following easily checked property: *for all* $j \in \mathbb{Z}/3\mathbb{Z}$, c_{j+2} *is completely determined by the SE side of* c_j *and the SW side of* c_{j+1}. Graphically, this property may be pictured as follows:



This concludes the proof of the validity of the constructions in Section 5. □

Appendix: Pictures of Δ, C_0 and C_1 in the proof of Construction 5.1.

$* \; \Delta := \Delta s_0 =$

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+

$* \; C_0 := \; C_{2,12}(12) =$

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− −
+

$* \, C_1 := C_{14,12}(12) =$

$$\begin{array}{c}
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+ \quad + \\
- \quad + \quad - \\
+ \quad - \quad - \quad + \\
+ \quad - \quad + \quad - \quad + \\
- \quad - \quad - \quad - \quad - \quad - \\
- \quad + \quad + \quad + \quad + \quad + \quad - \\
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+
\end{array}$$

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References

[1] Eliahou, Shalom and Hachez, Delphine, On a problem of Steinhaus concerning binary sequences, Experimental Mathematics 13, No.2 (2004) 215-229.

[2] Harborth, Heiko, Solution of Steinhaus's Problem with Plus and Minus Signs, J. Comb. Th. (A) 12 (1972) 253-259.

[3] Steinhaus, Hugo, "One Hundred Problems in Elementary Mathematics", Pergamon, Elinsford, N.Y. (1963) 47-48.