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ON THE LARGEST k-PRIMITIVE SUBSET OF [1,n]

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Abstract

We derive bounds on the size of the largest subset of $\{1, 2, ..., n\}$ such that no element divides k others, for $k \geq 3$ and sufficiently large n.

1. Introduction

Let $S \subseteq \mathbb{N}$ be a finite set of positive integers. We say that S is k-primitive if no member of S divides k other elements in S.

Let $f_k(n)$ denote the size of the largest k-primitive subset of $[1, n]$. It is well-known that $f_1(n) = \lceil \frac{n}{2} \rceil$. Lebensold [2] showed that, if n is sufficiently large,

$$(0.672...) < \frac{f_2(n)}{n} < (0.673...).$$

In this article, we show that, for $k \geq 3$ and sufficiently large n,

$$\frac{k}{k+1} + \frac{1}{8k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{8k \ln k}.$$

Moreover, given $\epsilon > 0$, there exists $k_0(\epsilon)$ such that for $k \geq k_0(\epsilon)$ and $n \geq n_0(k)$,

$$\frac{k}{k+1} + \frac{1-\epsilon}{k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{(2e^\gamma + \epsilon)k \ln k}.$$

2. The Lower Bound

For $\alpha \in \mathbb{R}$ and $S \subseteq \mathbb{N}$, we shall write αS to denote the set $\{\alpha x : x \in S\}$. We begin by deriving a lower bound on $f_k(n)$.

Define $S_0 = \{x : (k+1)x > n\}$, with $|S_0| = \frac{nk}{k+1} + O(1)$. Clearly, S_0 is k-primitive. Let $S_1 = \{x : \frac{n}{k+3} < x < \frac{nk}{(k+1)^2},\ k(k+1)|\, x\}$. Observe that any element in S_1 has exactly $k+1$ other multiples in $[1, n]$. Let $S_2 = (k+1)S_1$, $S_3 = (k+2)S_1$ and $S' = (S_0 \cup S_1) \setminus (S_2 \cup S_3)$. Note that S' is k-primitive.

Let $S_4 = (k+1)^{-1}S_3$ and and $S_5 = k^{-1}S_2$. Any element in $S_4 \cup S_5$ has at most k other multiples in $[1, n]$. By construction, at least one of these will not occur in S'. Furthermore, no multiple of an element in S_4, except possibly itself, occurs in S_5 and vice versa. It follows that $S \doteq S' \cup S_4 \cup S_5$ is k-primitive.

Note that

$$|S_i| = \frac{n(k-1)}{k(k+1)^3(k+3)} + O(1), \text{ for } 1 \leq i \leq 5.$$

Furthermore,

$$S_i \cap S_j = \emptyset \text{ for } 1 \leq i < j \leq 5 \text{ except when } i = 4 \text{ and } j = 5.$$

Finally,

$$|S_4 \cap S_5| = \frac{n(k^3 - 4k - 1)}{k^2(k+1)^5(k+2)(k+3)} + O(1).$$

Thus we have,

$$|S| = |S_0| + |S_1| - |S_4 \cap S_5| > n\left(\frac{k}{k+1} + \frac{1}{8k^4}\right).$$

Note that for sufficiently large k,

$$|S| > n\left(\frac{k}{k+1} + \frac{1-\epsilon}{k^4}\right).$$

3. The Upper Bound

Let S be a k-primitive subset of $[1, n]$. For a positive integer $x \leq n/(k+1)$, let $C_x \doteq \{x, 2x, \ldots, (k+1)x\}$ be the chain containing x. Observe that $C_x \subseteq [1, n]$ and $|S \cap C_x| \leq k$.

Thus if $C_{x_1}, C_{x_2}, \ldots, C_{x_m}$ are pairwise disjoint, $|S| \le n - m$.

Let $X = \{x : \frac{n}{2(k+1)} < x < \frac{n}{k+1}$, x has no prime factor in $[2,k]\}$. Thus if $r \le k$ and $x \in X$, we have $(r, x) = 1$.

We claim that $\{C_{x_m}\}, x_m \in X$ is a pairwise disjoint collection.

Suppose not. Let $rx_i = sx_j$, $x_i \ne x_j$, $1 \le r < s \le k+1$. Since $r \le k$ and $x_j \in X$, we have $(r, x_j) = 1$. Thus $x_j | \, x_i$, i.e., $x_i \ge 2x_j > \frac{n}{k+1}$, which is impossible. This proves our claim.

Let P_k denote the product of the prime numbers not exceeding k. The easy estimate $P_k < 3^k$, together with an application of the Chinese Remainder Theorem, yields

$$|X| = \frac{n}{2(k+1)} \prod_{p \le k} \left(1 - \frac{1}{p}\right) + O(3^k).$$

By Mertens's theorem,

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) \ge \frac{1}{e^{\gamma+\delta} \ln x} \text{ where } |\delta| < \frac{4}{\ln(x+1)} + \frac{1}{2x} + \frac{2}{x \ln x}.$$

Computations for a bounded initial segment (suffices to consider $x < 12000$) establish that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) \ge \frac{\ln 3}{3 \ln x} \text{ for } x \ge 3.$$

Therefore, we obtain, for $k \ge 3$,

$$|X| > \frac{n}{8k \ln k}$$

and, for sufficiently large k,

$$|X| > \frac{n}{(2e^{\gamma} + \epsilon) k \ln k}.$$

References

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