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A BIJECTIVE PROOF OF $f_{n+4} + f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} + 3$

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Abstract

In *Proofs that Really Count*, Benjamin and Quinn mentioned that there was no known bijective proof for the identity $f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$ for $n \geq 0$, where f_k is the k-th Fibonacci number. In this paper, we interpret f_k as the cardinality of the set F_k consisting of all ordered lists of 1's and 2's whose sum is k. We then demonstrate a bijection between the sets $F_{n+4} \cup \bigcup_{k=1}^{n} (\{1, 2, \ldots, k\} \times F_k)$ and $(\{1, 2, \ldots, n+1\} \times F_{n+2}) \cup \{1, 2, 3\}$, which gives a bijective proof of the identity.

1. Introduction

We will interpret the k-th Fibonacci number f_k as the cardinality of the set F_k of all ordered lists of 1's and 2's that have sum k. Thus, $(f_0, f_1, f_2, f_3, f_4, f_5, \ldots) = (1, 1, 2, 3, 5, 8, \ldots)$. For an integer m, the number mf_k will be interpreted as the cardinality of the Cartesian product $[m] \times F_k$, where $[m] := \{1, 2, 3, \ldots, m\}$.

On page 14 of *Proofs that Really Count* [1], Benjamin and Quinn mentioned that there was no known bijective proof for the identity $f_1 + 2f_2 + \ldots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$ for $n \geq 0$. In Section 2 we define a map

$$\phi : F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k) \longrightarrow \{1, 2, 3\} \cup ([n+1] \times F_{n+2}),$$

and in Section 3 we describe why ϕ is a bijection. This provides a bijective proof of the identity $f_{n+4} + f_1 + 2f_2 + \ldots + nf_n = (n+1)f_{n+2} + 3$ for $n \geq 0$. For completeness, we also define the inverse map

$$\psi : \{1, 2, 3\} \cup ([n+1] \times F_{n+2}) \longrightarrow F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k)$$

in Section 4, and the cases in the definition of ψ correspond to those for ϕ.

When examples are given below, ordered lists are denoted using angled brackets, e.g., $\langle a_1, a_2, \ldots, a_m\rangle$. Also, Doron Zeilberger [4] has written a Maple package that implements the bijection, which may be downloaded from

$$\texttt{http://www.math.rutgers.edu/}\sim\texttt{zeilberg/tokhniot/PHIL} \tag{1}$$

2. The bijection ϕ

In this section we define the bijection

$$\phi : F_{n+4} \cup \bigcup_{k=1}^{n}([k] \times F_k) \longrightarrow \{1, 2, 3\} \cup ([n + 1] \times F_{n+2}).$$

For $X \in F_{n+4}$ where the last n numbers in the list X are 1's, define ϕ according to the chart below.

$$\begin{array}{rrcc}
\phi : & \langle 1,1,1,1,\overbrace{1,1,\ldots,1}^{n}\rangle & \mapsto & (1, \langle\overbrace{1,1,\ldots,1}^{n+2}\rangle) \\
\phi : & \langle 2,1,1,\overbrace{1,1,\ldots,1}^{n}\rangle & \mapsto & 1 \\
\phi : & \langle 1,2,1,\overbrace{1,1,\ldots,1}^{n}\rangle & \mapsto & 2 \\
\phi : & \langle 1,1,2,\overbrace{1,1,\ldots,1}^{n}\rangle & \mapsto & 3 \\
\phi : & \langle 2,2,\overbrace{1,1,\ldots,1}^{n}\rangle & \mapsto & (1, \langle 2,\overbrace{1,1,\ldots,1}^{n}\rangle)
\end{array}$$

For all cases **not** covered by the chart above, we define ϕ by the two cases below.

Case 1: Consider $X \in F_{n+4}$, so X is a list of 1's and 2's that sums to $n + 4$. By the above special cases above, we know that X ends in a string of exactly ℓ 1's, where $0 \le \ell < n$ (so X has a 2 followed by ℓ 1's at the end). Take X and delete the last 2 in X to get $\widehat{X}$, which is an element of F_{n+2}, and define $\phi : X \mapsto (n - \ell + 1, \widehat{X})$.

Examples for $n = 3$:
$$\begin{array}{lrcl}
\phi : & \langle 1,1,2,1,2\rangle & \mapsto & (4, \langle 1,1,2,1\rangle) \\
\phi : & \langle 1,1,2,2,1\rangle & \mapsto & (3, \langle 1,1,2,1\rangle) \\
\phi : & \langle 1,1,1,2,1,1\rangle & \mapsto & (2, \langle 1,1,1,1,1\rangle)
\end{array}$$

Case 2: Consider (i, X) where $X \in F_k$ and $i \in [k]$ (and thus $i \le k$). Take X and append a 2 followed by $(n-k)$ 1's to get $\widetilde{X}$, which is an element of F_{n+2}, and define $\phi : (i, X) \mapsto (i, \widetilde{X})$.

Examples for $n = 3$:
$$\begin{array}{lrcl}
\phi : & (1, \langle 1\rangle) & \mapsto & (1, \langle 1,2,1,1\rangle) \\
\phi : & (1, \langle 1,1\rangle) & \mapsto & (1, \langle 1,1,2,1\rangle) \\
\phi : & (2, \langle 2\rangle) & \mapsto & (2, \langle 2,2,1\rangle) \\
\phi : & (2, \langle 2,1\rangle) & \mapsto & (2, \langle 2,1,2\rangle)
\end{array}$$

3. Showing ϕ is bijective

The following three facts (which may be easily verified) help show that ϕ is injective:

1. The image of ϕ from the five special cases consists of $\{1, 2, 3\}$ and all elements (i, Y) of $[n + 1] \times F_{n+2}$ where $i = 1$ and Y ends in at least $n = (n + 1 - i)$ 1's.
2. The image of ϕ from Case 1 consists of all elements (i, Y) of $[n + 1] \times F_{n+2}$ where $2 \leq i \leq n + 1$ and Y ends in at least $(n + 1 - i)$ 1's.
3. The image of ϕ from Case 2 consists of all elements (i, Y) of $[n + 1] \times F_{n+2}$ where $1 \leq i \leq n$ and one of the last $(n + 1 - i)$ entries in Y is a 2.

It is easily seen from the definition that ϕ restricted to Case 1 is injective; and similarly, ϕ is injective when restricted to Case 2 or to the five special cases. Thus, since the three images described above are distinct, ϕ as a whole is injective. Furthermore, the union of the three images above consists of all of $\{1, 2, 3\} \cup ([n + 1] \times F_{n+2})$ (note that there is no element $(i, Y) \in [n + 1] \times F_{n+2}$ with $i = n + 1$ and Y containing a 2 in the last $(n + 1 - i)$ entries). Thus ϕ is a bijection.

4. The inverse bijection ψ

In this section we define the inverse bijection

$$\psi : \{1, 2, 3\} \cup ([n + 1] \times F_{n+2}) \longrightarrow F_{n+4} \cup \bigcup_{k=1}^{n} ([k] \times F_k).$$

For elements of $\{1, 2, 3\}$ and for the elements $(1, \langle\overbrace{1, 1, \ldots, 1}^{n+2}\rangle)$ and $(1, \langle 2, \overbrace{1, 1, \ldots, 1}^{n}\rangle)$ of $[n + 1] \times F_{n+2}$, define ψ according to the chart below.

$$\begin{array}{lccc}
\psi : & (1, \langle\overbrace{1, 1, \ldots, 1}^{n+2}\rangle) & \mapsto & \langle 1, 1, 1, 1, \overbrace{1, 1, \ldots, 1}^{n}\rangle \\
\psi : & 1 & \mapsto & \langle 2, 1, 1, \overbrace{1, 1, \ldots, 1}^{n}\rangle \\
\psi : & 2 & \mapsto & \langle 1, 2, 1, \overbrace{1, 1, \ldots, 1}^{n}\rangle \\
\psi : & 3 & \mapsto & \langle 1, 1, 2, \overbrace{1, 1, \ldots, 1}^{n}\rangle \\
\psi : & (1, \langle 2, \overbrace{1, 1, \ldots, 1}^{n}\rangle) & \mapsto & \langle 2, 2, \overbrace{1, 1, \ldots, 1}^{n}\rangle
\end{array}$$

For all cases **not** covered by the chart above, we define ψ as follows. Consider (i, Y), where $Y \in F_{n+2}$ and $i \in [n+1]$.

Case 1: If Y ends with at least $(n+1-i)$ 1's, then insert a 2 before the last $(n+1-i)$ 1's to get $\widetilde{Y}$ and define $\psi : (i, Y) \mapsto \widetilde{Y}$.

Examples for $n = 3$:
$$\begin{aligned} \psi :& \quad (4, \langle 1,1,1,2 \rangle) & \mapsto & \quad \langle 1,1,1,2,2 \rangle \\ \psi :& \quad (3, \langle 2,1,1,1 \rangle) & \mapsto & \quad \langle 2,1,1,2,1 \rangle \\ \psi :& \quad (2, \langle 1,1,1,1,1 \rangle) & \mapsto & \quad \langle 1,1,1,2,1,1 \rangle \end{aligned}$$

Case 2: If one of the last $n+1-i$ entries in Y is a 2, then delete the last 2 in Y and all 1's following that 2 to get $\widehat{Y}$. Define $\psi : (i, Y) \mapsto (i, \widehat{Y})$.

Examples for $n = 3$:
$$\begin{aligned} \psi :& \quad (1, \langle 1,2,1,1 \rangle) & \mapsto & \quad (1, \langle 1 \rangle) \\ \psi :& \quad (1, \langle 1,1,2,1 \rangle) & \mapsto & \quad (1, \langle 1,1 \rangle) \\ \psi :& \quad (2, \langle 2,2,1 \rangle) & \mapsto & \quad (2, \langle 2 \rangle) \\ \psi :& \quad (2, \langle 2,1,2 \rangle) & \mapsto & \quad (2, \langle 2,1 \rangle) \end{aligned}$$

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