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ON THE GROWTH OF A VAN DER WAERDEN-LIKE FUNCTION

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Abstract

Let $\overline{W}(3,k)$ denote the largest integer w such that there is a red/blue coloring of $\{1,2,\ldots,w\}$ which has no red 3-term arithmetic progression and no block of k consecutive blue integers. We show that for some absolute constant c, $\overline{W}(3,k) \geq k^{c\log k}$ for all k.

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1. Introduction

A classic theorem of van der Waerden [13], [8] asserts that for all k and r, there is a least integer $W_r(k)$ such that any r-coloring of $[W_r(k)] := \{1,2,\ldots,W_r(k)\}$ contains a monochromatic k-term arithmetic progression (k-AP). The true order of growth of $W_r(k)$ (and especially $W(k) := W_2(k)$) has attracted the interest of many researchers since van der Waerden's theorem first appeared in 1927 ([1], [3], [4], [6], [7], [11], [12]). The best current upper bound on $W(k)$ is the striking result of Gowers [7]:

$$W(k) < 2^{2^{2^{2^{2^{k+9}}}}}.$$

On the other hand, the best lower bound available is due to Berlekamp in 1968 ([3]), and asserts that

$$W(p+1) \geq p\,2^p$$

for p prime.

In order to obtain a better understanding of $W(k)$, it is natural to study the so-called "off-diagonal" van der Waerden number $W(k,l)$, which is defined to be the least integer w such that any red/blue coloring of $[w]$ either has a red k-AP or a blue l-AP.

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A complete list of the known values of $W(k,l)$ appears in the recent paper of Landman, Robertson and Culver [10]. In particular, they have computed the following values of $W(3,k)$:

k	3	4	5	6	7	8	9	10	11	12	13
$W(3,k)$	6	18	22	32	46	58	77	97	114	135	160

In [10], it is suggested that $W(3,k)$ might be bounded by some polynomial in k (perhaps even a quadratic!). We don't resolve this question here. Instead we study the related function $\overline{W}(3,k)$, defined to be the least integer w such that any red/blue coloring of $[w]$ either has a red 3-AP or a block of k *consecutive* blue integers. Since a block of k consecutive integers is a k-AP, then we have $\overline{W}(3,k) \geq W(3,k)$.

What we show in this note is that $\overline{W}(3,k)$ grows faster than any polynomial in k.

We note that the function $\overline{W}(3,k)$ is closely related to the function $\Gamma_k(3)$ discussed in Nathanson [11] as well as Landman and Robertson [9]. This is defined to be the least integer t such that any sequence $x_1 < x_2 < \cdots < x_t$ with $x_{i+1} - x_i \leq k$ for $1 \leq i \leq t-1$ must contain a 3-AP. Since it is easy to show that $\overline{W}(3,k) \leq k\,\Gamma_k(3)$, then our result also gives non-polynomial growth bounds to this function as well.

2. The Main Result

Theorem. For all $m > 0$,

$$\overline{W}(3,3m) \geq 2m\,(W_{r_3(m)}(3) - 1).$$

where $r_3(m)$ is defined by

$$r_3(m) = \max_{S\subseteq[m]}\{|S| : S \text{ has no 3-AP}\}.$$

Proof. By definition, there is a set $S(m) = \{s_1, s_2, \ldots, s_r\} \subseteq [m]$ with no 3-AP, where $r = r_3(m)$. Also, by definition, with $w := W_r(3) - 1$, there is an r-coloring $\chi : [w] \to [r]$ with no monochromatic 3-AP. Let I_k denote the interval $\{2(k-1)m+1, \ldots, (2k-1)m\}$ for $1 \leq k \leq w$.

For $1 \leq k \leq w$, select the element

$$x_k = 2(k-1)m + s_{\chi(k)}.$$

In other words, thinking of each I_k as a copy of $[m]$, x_k corresponds to

$$s_{\chi(k)} \in S(m) = \{s_1, \ldots, s_r\} \subseteq [m].$$

We claim that the set $X = \{x_1, x_2, \ldots, x_w\}$ contains no 3-AP. Suppose to the contrary that x_i, x_j and x_k, $i < j < k$, form a 3-AP. Thus,

$$x_i \in I_i = [2(i-1)m+1, (2i-1)m],$$
$$x_j \in I_j = [2(j-1)m+1, (2j-1)m],$$
$$x_k \in I_k = [2(k-1)m+1, (2k-1)m].$$

Therefore,

$$2(j-1)m+1-(2i-1)m \le x_j - x_i \le (2j-1)m - 2(i-1)m - 1,$$
$$2(k-1)m+1-(2j-1)m \le x_k - x_j \le (2k-1)m - 2(j-1)m - 1,$$

i.e.,

$$2(j-i)m - m + 1 \le x_j - x_i \le 2(j-i)m + m - 1,$$
$$2(k-j)m - m + 1 \le x_k - x_j \le 2(k-j)m + m - 1.$$

However, since x_i, x_j and x_k form a 3-AP then $x_j - x_i = x_k - x_j$. This implies that $j-i = k-j$, i.e., i, j and k form a 3 -AP. Furthermore, since

$$x_i = 2(i-1)m + s_{\chi(i)},$$
$$x_j = 2(j-1)m + s_{\chi(j)},$$
$$x_k = 2(k-1)m + s_{\chi(k)},$$

then we can conclude that $s_{\chi(i)}, s_{\chi(j)}$ and $s_{\chi(k)}$ form a 3-AP. However, by definition, S has no *non-trivial* 3-AP. Hence, the only possibility is that $s_{\chi(i)} = s_{\chi(j)} = s_{\chi(k)}$, which implies $\chi(i) = \chi(j) = \chi(k)$. Thus, i, j and k form a monochromatic 3-AP, which is a contradiction.

Note that since every interval I_k contains a point of X, then the difference between consecutive terms of X is less than $3m$.

Finally, define the red/blue coloring $\chi^* : [2mw] \to \{red, blue\}$ by:

$$\chi^*(i) = \begin{cases} \textit{red} & : \quad \text{if} \quad i = x_k \quad \text{for some } k, \\ \textit{blue} & : \quad \text{otherwise.} \end{cases}$$

Thus, χ^* has no red 3-AP and no blue $3m$-block. Therefore,

$$\overline{W}(3, 3m) > 2mw = 2m(W_r(3) - 1) = 2m(W_{r_3(m)}(3) - 1)$$

and the theorem is proved. □

Corollary. For some absolute constant c,

$$\overline{W}(3, k) > k^{c \log k}.$$

Proof. It is known [8] that

$$W_k(3) > k^{c_1 \log k}$$

for a suitable constant $c_1 > 0$. Also, it is known [2] that

$$r_3(k) > k \exp(-c_2\sqrt{\log k})$$

for a suitable constant $c_2 > 0$. Thus,

$$\begin{aligned} W_{r_3(k)}(3) &> r_3(k)^{c_1 \log r_3(k)} \\ &= \exp(c_1 \log^2(r_3(k))) \\ &> \exp(c_1(\log k - c_2\sqrt{\log k})^2) \\ &> \exp((c_1/2)\log^2 k) \\ &= k^{(c_1/2)\log k} \end{aligned}$$

for $k > k_0(c_2)$ sufficiently large. Now setting $m = k/3$ in the preceding theorem (together with a little algebra) gives the desired inequality. This completes the proof. □

3. Concluding Remarks.

The best available upper bound on $\overline{W}(3,k)$ comes from the upper bound estimate on $r_3(k)$ due to Bourgain [5]:

$$r_3(k) = O\left(k\sqrt{\frac{\log\log k}{\log k}}\right).$$

Using this estimate, we can obtain an upper bound for $\overline{W}(3,k)$ as follows. First, suppose $[N]$ is *red/blue*-colored, and let $x_1 < x_2 < \cdots < x_t$ denote the red integers in $[N]$. Hence, by Bourgain's estimate, if $t > cN\sqrt{\frac{\log\log N}{\log N}}$ for a sufficiently large c, then we have a red 3-AP. If not, then we must have

$$x_{i+1} - x_i > c'\sqrt{\frac{\log N}{\log\log N}}$$

for some i and suitable constant c'. Hence, if $N > k^{ck^2}$ for a suitable constant c, then the RHS is greater than k, i.e., we have a block of k consecutive blue integers. This shows that $\overline{W}(3,k) < k^{ck^2}$ for a suitable constant $c > 0$.

Whether this is close to the true behavior of $\overline{W}(3,k)$, and whether our result suggests that the function $W(3,k)$ is also non-polynomial, we leave for the reader to decide.

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