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MORE ON THE FIBONACCI SEQUENCE AND HESSENBERG MATRICES

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Abstract

Five new classes of Fibonacci-Hessenberg matrices are introduced. Further, we introduce the notion of two-dimensional Fibonacci arrays and show that three classes of previously known Fibonacci-Hessenberg matrices and their generalizations satisfy this property. Simple systems of linear equations are given whose solutions are Fibonacci fractions.

1. Introduction

The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$. An $n \times n$ matrix $\mathcal{A}$ is called a (lower) Hessenberg matrix if all entries above the superdiagonal are zero. As an example set $\mathcal{A}_1 = (1)$ and define $\mathcal{A}_n$ by:

$$\mathcal{A}_n := \begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & 1 \end{pmatrix}_{n \times n.}$$

These matrices are Hessenberg and the determinant of $\mathcal{A}_n$ is the nth Fibonacci number f_n. Several Hessenberg matrices whose determinants are Fibonacci numbers have been introduced in [1], [2], [4], and [5]. Strang [5] has introduced real tridiagonal matrices such that their determinants are Fibonacci numbers, while in [2] we see complex Hessenberg matrices with this property. It has been shown in [4] that the maximum determinant achieved by $n \times n$ Hessenberg $(0, 1)$-matrices is the nth Fibonacci number f_n and a class of matrices (denoted in this paper by $\mathcal{E}_{n,0}$) achieving this bound has been introduced.

In this paper, we consider sequences of Hessenberg matrices whose determinants are in the form $tf_{n-1}+f_{n-2}$ or $f_{n-1}+tf_{n-2}$ for some real or complex number t. Such matrices will be referred to as *Fibonacci-Hessenberg matrices*.

In Section 2 we introduce five new classes of Fibonacci-Hessenberg matrices. As a new concept, the two-dimensional Fibonacci array is introduced in Section 3. Three classes of Fibonacci-Hessenberg matrices satisfying this property are given.

2. More Fibonacci-Hessenberg Matrices

As mentioned above, several connections between the Fibonacci sequence and Hessenberg matrices have been given in [1], [2], [4], and [5]. In this section we develop some of these connections and provide more examples.

The Fibonacci recurrence relation $a_n = a_{n-1} + a_{n-2}$ beginning with $a_1 = 1$ and $a_2 = t$ produces the sequence $1, t, t+1, 2t+1, 3t+2, 5t+3, \cdots$. Thus $a_n = tf_{n-1} + f_{n-2}$ for $n \geq 1$ and $a_n = f_n$ if and only if $t = 1$. On the other hand, the sequence $a_n = a_{n-1} + a_{n-2}$ starting at $a_1 = t$ and $a_2 = 1$ satisfies $a_n = tf_{n-2} + f_{n-1}$.

Definition 1 Given a real or complex number t and an integer n, we refer to numbers $tf_{n-1} + f_{n-2}$ and $f_{n-1} + tf_{n-2}$ as type 1 and type 2, respectively, (t, n)-Fibonacci, briefly t-Fibonacci, numbers. A sequence of Hessenberg matrices $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots$, where $\mathcal{A}_n$ is an $n \times n$ matrix, is defined to be a *Fibonacci-Hessenberg* matrix if there exists an integer $m > 0$ and a number t such that, for each $n \geq m$, the determinant of $\mathcal{A}_n$ is a t-Fibonacci number and such that the determinants are of the same type.

Example 1 Given a number t, let $\mathcal{R}_{n,t}$ denote the $n \times n$ matrix given below.

$$\mathcal{R}_{n,t} := \begin{pmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 2 & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & t+1 \end{pmatrix}_{n \times n.}$$

The determinant of $\mathcal{R}_{n,t}$ is denoted by $r_{n,t}$. It can be shown that $r_{n,t} = tf_{n+1} + f_n,\ n \geq 1$, (see relation (10) in the proof of Theorem 2) and hence $\mathcal{R}_{n,t}$ is a Fibonacci-Hessenberg matrix. Thus, for instance, $r_{n,-1} = -f_{n+1} + f_n = -f_{n-1}$; that is $r_{n,-1}$ generates the additive inverse of the Fibonacci sequence. The Lucas numbers are defined by $l_1 = 1$, $l_2 = 3$ and $l_n = l_{n-1} + l_{n-2}$ for $n > 2$. One can easily verify by induction that $l_n = f_{n-1} + f_{n+1}$. Hence $r_{n,3} = 3f_{n+1} + f_n = f_{n+1} + f_{n+3} = l_{n+2}$. For $t = 0$ we have $r_{n,0} = f_n$,

Table 1: Determinant $r_{n,t}$ for $1 \leq n \leq 5$ and $t = -1, 0, 1, 2, 3$.

$t \backslash n$	1	2	3	4	5
3	4	7	11	18	29
2	3	5	8	13	21
1	2	3	5	8	13
0	1	1	2	3	5
-1	0	-1	-1	-2	-3

while $r_{n,1} = f_{n+1} + f_n = f_{n+2}$ and $r_{n,2} = 2f_{n+1} + f_n = f_{n+3}$. These are illustrated by Table 1.

Given a positive integer n, let $\mathcal{C}_{n,t}$ be the $n \times n$ matrix in which the entries below the diagonal are 1, the lowest entry of the nth column is $t+1$ and the other diagonal entries are 2, the entries on the superdiagonal are -1 and the entries above the superdiagonal are zero. Changing the first element of the first column in $\mathcal{C}_{n,t}$ to 2, we get another Hessenberg matrix denoted by $\mathcal{B}_{n,t}$. Matrices $\mathcal{C}_{5,t}$ and $\mathcal{B}_{5,t}$ are given below.

$$\mathcal{C}_{5,t} := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{pmatrix} \quad \mathcal{B}_{5,t} := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{pmatrix}$$

Proposition 1 The determinant of $\mathcal{C}_{n,t}$, denoted $c_{n,t}$, is $c_{n,t} = f_{2n} + tf_{2n-1},\ n \geq 1$, and $\mathcal{B}_{n,t}$ has determinant $b_{n,t} = f_{2n-1} + tf_{2n-2},\quad n \geq 1$.

Proof. We prove the statements by induction on n. It is obvious that these statements hold for $n = 1, 2$. Suppose $n \geq 3$. By the cofactor expansion along the first row we get $c_{n,t} = 2c_{n-1,t} + b_{n-1,t}$ and $b_{n,t} = c_{n-1,t} + b_{n-1,t}$. It follows from these two relations that

$$b_{n,t} = c_{n,t} - c_{n-1,t}. \tag{1}$$

Relations (1) and $c_{n,t} = 2c_{n-1,t} + b_{n-1,t}$ imply that $c_{n,t} = 3c_{n-1,t} - c_{n-2,t}$. Using the induction hypothesis we get

$$\begin{aligned} c_{n,t} &= 3c_{n-1,t} - c_{n-2,t} \\ &= 3(f_{2n-2} + tf_{2n-3}) - (f_{2n-4} + tf_{2n-5}) \\ &= (3f_{2n-2} - f_{2n-4}) + t(3f_{2n-3} - f_{2n-5}) \\ &= f_{2n} + tf_{2n-1}. \end{aligned}$$

Finally, it follows from $b_{n,t} = c_{n,t} - c_{n-1,t}$ and $c_{n,t} = f_{2n} + tf_{2n-1}$ that
$b_{n,t} = f_{2n-1} + tf_{2n-2}$. □

The three classes of Fibonacci-Hessenberg matrices given above are generalizations of matrices D_n, C_n, and B_n introduced in [1]. In fact the matrices D_n, C_n, and B_n given in [1] are $\mathcal{R}_{n,1}$, $\mathcal{C}_{n,1}$, and $\mathcal{B}_{n,1}$, respectively.

Now we introduce five new classes of Fibonacci-Hessenberg matrices. Given a number t, let $\mathcal{K}_{n,t}$ be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are -1, the entry located on the nth row and nth column is $t+1$ and the other diagonal entries are 2, and the entries on each column and below the diagonal are alternately -1 and 1 starting with -1. The matrix $\mathcal{K}_{5,t}$ is given by (2).

Replacing the top-left entry (the entry located in the first row and first column) in $\mathcal{K}_{n,t}$ with 1, we obtain another Hessenberg matrix denoted by $\mathcal{L}_{n,t}$. Replacing the superdiagonal entries in both $\mathcal{K}_{n,t}$ and $\mathcal{L}_{n,t}$ with 1, two more classes of Hessenberg matrices, denoted $\underline{\mathcal{K}}_{n,t}$ and $\underline{\mathcal{L}}_{n,t}$, respectively, are obtained.

$$\begin{aligned}
&\mathcal{K}_{5,t} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & t+1 \end{pmatrix} \mathcal{L}_{5,t} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & t+1 \end{pmatrix} \\
&\underline{\mathcal{K}}_{5,t} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & t+1 \end{pmatrix} \underline{\mathcal{L}}_{5,t} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & t+1 \end{pmatrix}
\end{aligned} \tag{2}$$

Theorem 1 Let $k_{n,t}$, $l_{n,t}$, $\underline{k}_{n,t}$ and $\underline{l}_{n,t}$ denote the determinants of $\mathcal{K}_{n,t}$, $\mathcal{L}_{n,t}$, $\underline{\mathcal{K}}_{n,t}$ and $\underline{\mathcal{L}}_{n,t}$, respectively. Then

$$\begin{cases} k_{n,t} = f_n + t f_{n+1}, \quad n \geq 1; \\ \\ \begin{cases} l_{1,t} = t+1, \\ l_{n,t} = k_{n-2,t} = f_{n-2} + t f_{n-1}, \quad n \geq 2; \end{cases} \\ \\ \underline{k}_{n,t} = f_{2n} + t f_{2n-1}, \quad n \geq 1; \\ \\ \begin{cases} \underline{l}_{1,t} = 1+t, \\ \underline{l}_{n,t} = \underline{k}_{n-1,t} + \underline{l}_{n-1,t} = f_{2n-1} + t f_{2n-2}, \quad n \geq 2. \end{cases} \end{cases}$$

Therefore, the four introduced classes of Hessenberg matrices are indeed Fibonacci-Hessenberg matrices.

Proof. The proof is by induction on n. Due to the similarity between matrices $\mathcal{K}_{n,t}$ ($\mathcal{L}_{n,t}$) and $\underline{\mathcal{K}}_{n,t}$ (resp. $\underline{\mathcal{L}}_{n,t}$), we just prove the first two statements. It is easily verified that the statements hold for $1 \leq n \leq 3$. Assume that $n \geq 4$. Using cofactor expansion along the first row we obtain:

$$\begin{cases} l_{n,t} = k_{n-1,t} - l_{n-1,t}, & n \geq 4; \\ k_{n,t} = 2k_{n-1,t} - l_{n-1,t}, & n \geq 4. \end{cases} \tag{3}$$

Therefore,

$$\begin{aligned} l_{n,t} &= k_{n-1,t} - l_{n-1,t} \\ &= (2k_{n-2,t} - l_{n-2,t}) - (k_{n-2,t} - l_{n-2,t}) \\ &= k_{n-2,t}. \end{aligned} \tag{4}$$

Relations (3) and (4) imply

$$k_{n,t} = 2k_{n-1,t} - l_{n-1,t} = 2k_{n-1,t} - k_{n-3,t}, \quad n \geq 4. \tag{5}$$

This, together with the induction hypothesis, result in:

$$\begin{aligned} k_{n,t} &= 2k_{n-1,t} - k_{n-3,t} = 2\,(f_{n-1} + tf_n) - (f_{n-3} + tf_{n-2}) \\ &= (f_{n-1} + f_{n-1} - f_{n-3}) + (tf_n + tf_n - tf_{n-2}) \\ &= (f_{n-1} + f_{n-2} + f_{n-3} - f_{n-3}) + (tf_n + tf_{n-1} + tf_{n-2} - tf_{n-2}) \\ &= f_n + tf_{n+1}. \end{aligned}$$

□

Define $\mathcal{E}_{1,t} = (t+1)$ and $\mathcal{E}_{2,t} = \begin{pmatrix} 1 & 1 \\ 0 & t+1 \end{pmatrix}$. Given the $n \times n$ matrix $\mathcal{E}_{n,t}$, a new matrix $\mathcal{E}_{n+1,t}$ is formed by adding one row of weight one and starting with 1 to the top of $\mathcal{E}_{n,t}$ and then adding a new column with alternating 1's and 0's, starting with a 1, to the left of the obtained matrix. The matrix $\mathcal{E}_{5,t}$ is given below. The matrix $\mathcal{E}_{n,0}$ was introduced in [4] and it was shown in [4] that the determinant of $\mathcal{E}_{n,0}$, $n \geq 1$, is f_n. Let $\mathbf{i}$ denote the usual complex unit with $\mathbf{i}^2 = -1$. Replacing the entry of $\mathcal{E}_{n,t}$ located in the ith row and $(i+1)$th column, $1 \leq i < n$, with $(-1)^{i+n}\mathbf{i}$, we obtain another Hessenberg matrix denoted by $\mathcal{H}_{n,t}$. The matrix $\mathcal{H}_{5,t}$ is shown below.

$$\mathcal{E}_{5,t} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & t+1 \end{pmatrix} \quad \mathcal{H}_{5,t} = \begin{pmatrix} 1 & \mathbf{i} & 0 & 0 & 0 \\ 0 & 1 & -\mathbf{i} & 0 & 0 \\ 1 & 0 & 1 & \mathbf{i} & 0 \\ 0 & 1 & 0 & 1 & -\mathbf{i} \\ 1 & 0 & 1 & 0 & t+1 \end{pmatrix}$$

Proposition 2 The determinants of $\mathcal{E}_{n,t}$ and $\mathcal{H}_{n,t}$, denoted by $e_{n,t}$ and $h_{n,t}$, respectively, are $e_{1,t} = h_{1,t} = t+1$ and $e_{n,t} = h_{n,t} = tf_{n-1} + f_n$ for $n \geq 2$. Thus $\mathcal{E}_{n,t}$ and $\mathcal{H}_{n,t}$ are Fibonacci-Hessenberg matrices.

Proof. Consider $\mathcal{H}_{n,t}$. Obviously we have $h_{1,t} = h_{2,t} = t+1$ and $h_{3,t} = 2t+3$. Suppose $n \geq 4$ and that the statement holds for each integer $0 < m < n$. Let $\mathcal{H}'_{n,t}$ be the matrix obtained from $\mathcal{H}_{n,t}$ by means of deleting the first row and second column. It is easy to see that using cofactor expansion along the first row on both $\mathcal{H}_{n,t}$ and $\mathcal{H}'_{n,t}$ results in $h_{n,t} = h_{n-1,t} + h_{n-2,t}$. Therefore, it follows from the induction hypothesis that $h_{n,t} = h_{n-1,t} + h_{n-2,t} = (tf_{n-2} + f_{n-1}) + (tf_{n-3} + f_{n-2}) = tf_{n-1} + f_n$. The same argument applies to $\mathcal{E}_{n,t}$. □

3. Two-dimensional Fibonacci Arrays

From among the introduced Fibonacci matrices, the matrices $\mathcal{R}_{n,t}$, $\mathcal{C}_{n,t}$ and $\mathcal{E}_{n,t}$ have a further interesting property. Given an $n \times n$ matrix $\mathcal{M}$, let $\mathcal{M}^{(i)}$ be the matrix obtained from $\mathcal{M}$ by replacing its ith column with the all one column vector $\mathbf{1}$. The mentioned matrices have the property that the determinants of their associated matrices $\mathcal{R}^{(i)}_{n,t}$, $\mathcal{C}^{(i)}_{n,t}$

and $\mathcal{E}_{n,t}^{(i)}$, $1 \leq i \leq n$, are t-Fibonacci numbers. This leads to a connection between Fibonacci fractions and the all one vector $\mathbf{1}$.

Theorem 2 Let $r_{n,t}^{(i)}$, $c_{n,t}^{(i)}$ and $e_{n,t}^{(i)}$ be determinants of $\mathcal{R}_{n,t}^{(i)}$, $\mathcal{C}_{n,t}^{(i)}$ and $\mathcal{E}_{n,t}^{(i)}$, respectively. Then we have

$$\begin{cases} r_{n,t}^{(i)} = tf_{n-i} + f_{n-i-1}, \quad n \geq i \geq 1; \\ r_{n,t} = t + \sum_{i=1}^{n} r_{n,t}^{(i)}; \\ \\ c_{n,t}^{(i)} = f_{2(n-i)+1} + tf_{2(n-i)}, \quad n \geq i \geq 1; \\ c_{n,t} = t + \sum_{i=1}^{n} c_{n,t}^{(i)}; \\ \\ e_{1,t}^{(1)} = 1; \; e_{n,t}^{(1)} = tf_{n-3} + f_{n-2}, \quad n \geq 2; \\ e_{n,t}^{(i)} = tf_{n-i} + f_{n-i+1}, \quad n \geq i \geq 2; \\ 2e_{n,t} = (t+1) + \sum_{i=1}^{n} e_{n,t}^{(i)}, \quad n \geq 2. \end{cases} \tag{6}$$

Proof. We use induction on n to prove the statements related to the Fibonacci-Hessenberg matrices $\mathcal{R}_{n,t}$; similar arguments apply to the other two classes of matrices. Setting $f_0 = 0$ and $f_{-1} = 1$, it is easy to verify that the statements hold for $1 \leq n \leq 3$. Evaluating the determinants of matrices $\mathcal{R}_{n,t}$ and $\mathcal{R}_{n,t}^{(1)}$ by cofactor expansion along the first row, we have

$$\begin{cases} r_{n,t}^{(1)} = r_{n-1,t} - r_{n-1,t}^{(1)}, & n \geq 3; \\ r_{n-1,t} = 2r_{n-2,t} - r_{n-2,t}^{(1)}, & n \geq 3. \end{cases} \tag{7}$$

Therefore,

$$\begin{aligned} r_{n,t}^{(1)} &= r_{n-1,t} - r_{n-1,t}^{(1)} \\ &= \left(2r_{n-2,t} - r_{n-2,t}^{(1)}\right) - \left(r_{n-2,t} - r_{n-2,t}^{(1)}\right) \\ &= r_{n-2,t}. \end{aligned} \tag{8}$$

It follows from (7) and (8) that

$$r_{n,t} = 2r_{n-1,t} - r_{n-1,t}^{(1)} = 2r_{n-1,t} - r_{n-3,t}, \quad n \geq 3. \tag{9}$$

This together with the induction hypothesis result in

$$\begin{aligned} r_{n,t} &= 2r_{n-1,t} - r_{n-3,t} \\ &= 2\,(tf_n + f_{n-1}) - (tf_{n-2} + f_{n-3}) \\ &= t(2f_n - f_{n-2}) + (2f_{n-1} - f_{n-3}) \\ &= tf_{n+1} + f_n, \end{aligned} \tag{10}$$

and hence

$$r_{n,t}^{(1)} = r_{n-2,t} = tf_{n-1} + f_{n-2}. \tag{11}$$

By cofactor expansion along the first row, one can easily verify that $r_{n,t}^{(i)} = r_{n-1,t}^{(i-1)}$ if $2 \leq i \leq n-1$, and thus

$$r_{n,t}^{(i)} = r_{n-i-1,t} = tf_{n-i} + f_{n-i-1}, \quad 2 \leq i \leq n-1. \tag{12}$$

The augmented matrices obtained from a given matrix $\mathcal{M}$ by adding the all one column vector $\mathbf{1}$ to the left and the right of $\mathcal{M}$ are denoted by $\mathbf{1}\mathcal{M}$ and $\mathcal{M}\mathbf{1}$, respectively. Evaluating the determinants by using cofactors along the first row, we obtain $r_{n,t}^{(n)} = 2r_{n-1,t}^{(n-1)} + (-1)^{n+1}\det(\mathbf{1}\mathcal{M})$ where $\mathcal{M}\mathbf{1}$ is the matrix $\mathcal{R}_{n-1,t}^{(n-1)}$. Therefore,

$$r_{n,t}^{(n)} = 2r_{n-1,t}^{(n-1)} + (-1)^{2n-1} r_{n-1,t}^{(n-1)} = r_{n-1,t}^{(n-1)} = 1 = tf_0 + f_{-1}. \tag{13}$$

It is easily checked by induction on n that $\sum_{j=-1}^{n} f_j = f_{n+2}$. This together with the equations $r_{n,t} = tf_{n+1} + f_n$ and $r_{n,t}^{(i)} = tf_{n-i} + f_{n-i-1}$ imply that $r_{n,t} = t + \sum_{i=1}^{n} r_{n,t}^{(i)}$. □

Corollary 1 (*Fibonacci Fractions and Hessenberg Matrices*) The system of equations $\mathcal{R}_{n,t}\mathbf{x} = \mathbf{1}$ has the unique solution

$$x_i = \frac{f_{n-i-1} + tf_{n-i}}{f_n + tf_{n+1}}, \quad 1 \le i \le n. \tag{14}$$

Similarly, the system of equations $\mathcal{C}_{n,t}\mathbf{x} = \mathbf{1}$ has the solution

$$x_i = \frac{f_{2(n-i)+1} + tf_{2(n-i)}}{f_{2n} + tf_{2n-1}}, \quad 1 \le i \le n. \tag{15}$$

We also have $x_1 = \frac{tf_{n-3}+f_{n-2}}{tf_{n-1}+f_n}$ and $x_i = \frac{tf_{n-i}+f_{n-i+1}}{tf_{n-1}+f_n}$, $2 \le i \le n$, as the unique solution of $\mathcal{E}_{n,t}\mathbf{x} = \mathbf{1}$.

Proof. It follows from (6) and Cramer's rule that the system $\mathcal{R}_{n,t}\mathbf{x} = \mathbf{1}$ has unique solution $x_i = \frac{r_{n,t}^{(i)}}{r_{n,t}} = \frac{f_{n-i-1}+tf_{n-i}}{f_n+tf_{n+1}}$, $1 \le i \le n$. The same argument applies to the systems $\mathcal{C}_{n,t}\mathbf{x} = \mathbf{1}$ and $\mathcal{E}_{n,t}\mathbf{x} = \mathbf{1}$. □

In particular, according to (14), for $t = 0, 1, 2$ the system $\mathcal{R}_{n,t}\mathbf{x} = \mathbf{1}$ has solutions:

$$\begin{cases} x_i = \frac{f_{n-i-1}}{f_n}, & t = 0; \\ x_i = \frac{f_{n-i}+f_{n-i-1}}{f_{n+1}+f_n} = \frac{f_{n-i+1}}{f_{n+2}}, & t = 1; \\ x_i = \frac{2f_{n-i}+f_{n-i-1}}{2f_{n+1}+f_n} = \frac{f_{n-i+2}}{f_{n+3}}, & t = 2. \end{cases}$$

It also follows from (15) that for $t = 0, 1, 2$ the system $\mathcal{C}_{n,t}\mathbf{x} = \mathbf{1}$ has solutions

$$\begin{cases} x_i = \frac{f_{2(n-i)+1}}{f_{2n}}, & t = 0; \\ x_i = \frac{f_{2(n-i+1)}}{f_{2n+1}}, & t = 1; \\ x_i = \frac{f_{2(n-i+1)}+f_{2(n-i)}}{f_{2n+1}+f_{2n-1}}, & t = 2. \end{cases}$$

Consider the $n \times n$ Fibonacci-Hessenberg matrix $\mathcal{R}_{n,t}$. For $1 \le i \le n$ we have $r_{n,t}^{(i)} = tf_{n-i} + f_{n-i+1}$. This t-Fibonacci number depends on both n and i, and hence we have a *two-dimensional array* $r(n, i; t)$ of t-Fibonacci numbers.

Table 2: The Values of $r_{n,t}^{(i)}$ for $1 \leq n, i \leq 6$.

$n \backslash i$	1	2	3	4	5	6
1	$tf_0 + f_{-1}$					
2	$tf_1 + f_0$	$tf_0 + f_{-1}$				
3	$tf_2 + f_1$	$tf_1 + f_0$	$tf_0 + f_{-1}$			
4	$tf_3 + f_2$	$tf_2 + f_1$	$tf_1 + f_0$	$tf_0 + f_{-1}$		
5	$tf_4 + f_3$	$tf_3 + f_2$	$tf_2 + f_1$	$tf_1 + f_0$	$tf_0 + f_{-1}$	
6	$tf_5 + f_4$	$tf_4 + f_3$	$tf_3 + f_2$	$tf_2 + f_1$	$tf_1 + f_0$	$tf_0 + f_{-1}$

Table 3: The Values of $r_{n,1}^{(i)}$ for $1 \leq n, i \leq 6$.

$n \backslash i$	1	2	3	4	5	6
1	1					
2	1	1				
3	2	1	1			
4	3	2	1	1		
5	5	3	2	1	1	
6	8	5	3	2	1	1

Table 2 represents $r(n, i; t)$ for $1 \leq n, i \leq 6$. Asymptotically, all rows and columns of the array $r(n, i; t)$ are the same. Table 3 represents $r(n, i; 1)$ for $1 \leq n, i \leq 6$. For a fixed n the nth row of the array consists of the first n Fibonacci numbers and, for each i, the ith column, starting at the ith entry, is also the Fibonacci sequence.

In the context of systems theory [3], we may consider the determinant function as an operator and interpret $r(n, i; t) = \mathrm{Det}(\mathcal{R}_{n,t}^{(i)})$ as a two-dimensional system. The results show that this system is invariant in the sense that its output is always a t-Fibonacci number. We can also say that the system is invariant with respect to fixing any of the two variables n and i; that is, its output with a fixed n is identical to the output when i is fixed and n varies.

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