MINIMAL ZERO-SUM SEQUENCES OF MAXIMUM LENGTH IN THE GROUP $C_3 \oplus C_{3k}$

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Abstract

A sequence α in an additively written abelian group G is called a minimal zero-sum sequence if its sum is the zero element of G and none of its proper subsequences has sum zero. This note characterizes the minimal zero-sum sequences of maximum length in the group $C_3 \oplus C_{3k}$.

Let α be a sequence in an additively written finite abelian group G; its sum and length will be denoted by $\sigma(\alpha)$ and $|\alpha|$, respectively. We call α a *zero-sum sequence* or a *zero sum* if $\sigma(\alpha) = 0$, and a *minimal zero-sum sequence* if $\sigma(\alpha) = 0$ and $\sigma(\beta) \neq 0$ for each proper subsequence β of α.

The *Davenport constant* $D(G)$ of G is defined as the maximum length of a minimal zero-sum sequence in G. The value of the Davenport constant for groups of rank 2 was determined independently by Olson [3] and Kruyswijk (see [4]): *If* $G = C_m \oplus C_n$ *where* m *divides* n, *then* $D(G) = m + n - 1$. Here and further on C_n denotes the cyclic group of order n.

We describe the minimal zero-sum sequences of maximum length in the group $C_3 \oplus C_{3k}$, for $k \geq 2$. This maximum length is $D(C_3 \oplus C_{3k}) = 3k + 2$ by the result of Olson and Kruyswijk. Such an explicit description is known only for the groups of the form $C_2 \oplus C_{2k}$ (apart from obvious cases like cyclic groups and elementary 2-groups). It was obtained by Gao and Geroldinger in [1], Theorem 3.3.

The exposition employs the shape of the minimal zero-sum sequences of maximum length in the group $C_3 \oplus C_3$, i. e. $C_3 \oplus C_{3k}$ with $k = 1$. It is straightforward to derive that they are only of one type: a^2b^2c where a, b, c are different, nonzero and such that $c = a + b$. (We use

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multiplicative notation for sequences; exponents indicate term multiplicities.) In contrast, the respective sequences in groups $C_3 \oplus C_{3k}$ for larger k have several essentially different forms. So we leave $C_3 \oplus C_3$ out of consideration but the Davenport constant $D(C_3 \oplus C_3) = 5$ is used, and also the η-invariant of $C_3 \oplus C_3$. This is the minimum length $\eta(C_3 \oplus C_3)$ of a sequence α in $C_3 \oplus C_3$ which ensures that α has a *short* zero-sum subsequence, i. e. a nonempty zero-sum subsequence of length at most 3. It is well known and easy to verify that $\eta(C_3 \oplus C_3) = 7$. (In general, $\eta(C_m \oplus C_n) = 2m + n - 2$ for all positive integers m and n such that m divides n; see e. g. [2], Theorem 5.8.3.)

Let $G = C_3 \oplus C_{3k}$ where $k \geq 2$. The subgroup $H = 3G = \{3x \,|\, x \in G\}$ is cyclic of order k, and the factor group G/H is isomorphic to $C_3 \oplus C_3$. We denote by $\overline{x}$ the image of $x \in G$ under the canonical epimorphism of G onto G/H; and if $\alpha = x_1 \dots x_\ell$ is a sequence in G then $\overline{\alpha} = \overline{x_1} \dots \overline{x_\ell}$ denotes the image of α.

Let α be a minimal zero-sum sequence in G of maximum length $D(G) = 3k + 2$. Then $\overline{\alpha}$ is a zero sum of length $3k + 2$ in the group $G/H \cong C_3 \oplus C_3$. Now, because $\eta(C_3 \oplus C_3) = 7$ and $|\overline{\alpha}| = 3k + 2 = 3(k - 2) + 8$, one can separate $k - 1$ pairwise disjoint short zero-sum subsequences $\overline{\alpha_1}, \dots, \overline{\alpha_{k-1}}$ of $\overline{\alpha}$, of length $1, 2$ or 3. The remaining part $\overline{\alpha_k} = \overline{L}$ of $\overline{\alpha}$ also must be a zero sum in G/H, and its length is at least 5. Consequently, α is partitioned into k disjoint subsequences $\alpha_1, \dots, \alpha_{k-1}, \alpha_k$ with sums $\sigma(\alpha_1), \dots, \sigma(\alpha_{k-1}), \sigma(\alpha_k)$ in H. Since $\sigma(\alpha_1) + \dots + \sigma(\alpha_{k-1}) + \sigma(\alpha_k) = 0$ and α is a minimal zero-sum sequence in G, it follows that $\sigma(\alpha_1) \cdots \sigma(\alpha_{k-1})\sigma(\alpha_k)$ is a minimal zero-sum sequence of length k in H, which is a cyclic group of order k. So the sums $\sigma(\alpha_i)$ are all equal to a certain generator of H. Moreover, $\overline{\alpha_1}, \dots, \overline{\alpha_{k-1}}, \overline{\alpha_k} = \overline{L}$ are minimal zero-sum sequences in G/H. In particular $|L| = 5$ as $D(C_3 \oplus C_3) = 5$. This implies $|\overline{\alpha_i}| = 3$ for all $i = 1, \dots, k - 1$.

This separation procedure can start out with any short zero-sum subsequence of $\overline{\alpha}$. So we infer from the above that $\overline{\alpha}$ has no zero sums of length 1 or 2, i. e. no term of α is in H and neither is the sum of two terms of α. Next, for each term t of α there is a partition such that t is in the leftover 5-tuple L. Indeed, it is possible to choose $k - 2$ triples with sums in H without involving t. Apart from t, there remain 7 terms, so one more triple with sum in H can be chosen out of these seven. Hence t is in the leftover of the partition obtained.

Define a *numerous coset* to be a coset of H which contains at least 3 terms of α. A term of α will be called *numerous* if it belongs to a numerous coset. In the next lemma, by *partition* we mean a partition into $k - 1$ triples and one 5-tuple like above.

Lemma 1. Let $G = C_3 \oplus C_{3k}$ where $k \geq 2$, and let $H = 3G$. Each minimal zero-sum sequence α of maximum length $D(G) = 3k + 2$ in G has the following properties:

a) α represents at most 4 proper cosets of H.

b) Three terms of α with sum in H are either from the same coset of H or from three different cosets of H.

c) All terms of α in a numerous coset are equal.

d) Each numerous term of α has order $3k$.

e) Every partition of α has a leftover of the form $L = a_1a_2b_1b_2c$, with $a_i \in \overline{a}$, $b_i \in \overline{b}$, $i = 1, 2$, and $c \in \overline{c}$, where $\overline{a}, \overline{b}, \overline{c}$ are different proper cosets of H.

Proof. a) The nonzero elements of $G/H \cong C_3 \oplus C_3$ can be partitioned into 4 disjoint pairs of the form $x, 2x$. Since $x + 2x = 0$ in $C_3 \oplus C_3$, $\overline{\alpha}$ can contain at most one element from each pair (or else it would have a zero sum of length 2).

b) Let $a + b + c \in H$, with a, b in different cosets of H. If $c \in \overline{a}$ or $c \in \overline{b}$, say $c \in \overline{a}$, then $a + a + c \in H$. This implies $b - a \in H$, contrary to the assumption $\overline{a} \neq \overline{b}$.

c) Let $\overline{a}$ be a numerous coset. Consider a partition S in which the leftover L contains at least one term from $\overline{a}$. Because $3\overline{a} = 0$ in G/H and the coset $\overline{a}$ is numerous, not all terms from $\overline{a}$ are in L (otherwise a proper subsum of L would belong to H). So there exist terms $a, a' \in \overline{a}$ such that $a \in L$, $a' \notin L$, and it suffices to prove that $a = a'$ for every such pair a, a'. Now a' is in a triple of S, and all triples in S have sum equal to some generator g of H, as well as the leftover L. Because $a - a' \in H$, interchanging a and a' yields another partition S', but the respective generator of H for S' is the same g. This is clear for $k \geq 3$ where at least one triple of S remains intact after the swap. As for $k = 2$, in this case H is a cyclic group of order 2, hence it has a unique generator. We infer in particular that the sum of the leftover before and after the swap is the same which implies the desired $a = a'$.

d) Let a be a numerous term. As $3a \in H$, there is a partition containing a triple (a, a, a). Hence $3a = g$ for some generator g of H. Now H has order k, so $\text{ord}(g) = k$. On the other hand $a \notin H$ means that $\overline{a}$ is a nonzero element of $G/H \cong C_3 \oplus C_3$, therefore $\text{ord}(\overline{a}) = 3$. Because $\text{ord}(\overline{a})$ divides $\text{ord}(a)$, we obtain that $\text{ord}(a)$ is a multiple of 3. Let $\text{ord}(a) = 3\ell$, then $3\ell a = 0$ which can be written as $\ell g = 0$. So ℓ is divisible by k, implying the claim.

e) If L is the leftover 5-tuple of a partition, then the 5-term sequence $\overline{L}$ is a minimal zero-sum sequence of maximum length in $G/H \cong C_3 \oplus C_3$. By the introductory remark about $C_3 \oplus C_3$, we have $\overline{L} = \overline{a}^2\overline{b}^2\overline{c}$ where $\overline{a}, \overline{b}, \overline{c} \in G/H$ are different and nonzero. This implies the conclusion.

The characterization below uses a partition S of α defined as follows: All possible triples of equal terms in α are separated first, and then the partition is completed in an arbitrary fashion. We show that in fact *each* triple in S consists of three equal terms.

Suppose on the contrary that S has a triple (x, y, z) such that x, y, z are not all equal. By Lemma 1(c), x, y and z are not from the same coset of H, hence by Lemma 1(b) they come from three different cosets of H. By Lemma 1(e), the leftover L of S is $L = a_1a_2b_1b_2c$, with $a_i \in \overline{a}$, $b_i \in \overline{b}$, $i = 1, 2$, and $c \in \overline{c}$, where $\overline{a}, \overline{b}, \overline{c}$ are different proper cosets of H. Since by Lemma 1(a) α represents at most 4 different cosets of H, one of x, y and z belongs to one of the cosets $\overline{a}$ and $\overline{b}$, say $x \in \overline{a}$. Then $\overline{a}$ is a numerous coset, containing the terms a_1, a_2 and x, and so $a_1 = a_2 = x$ by Lemma 1(c). Hence a triple of equal terms of α was not separated while forming the partition S, contrary to its definition.

Therefore S has $k-1$ triples consisting of equal terms and one leftover 5-tuple L. Since $k \geq 2$, α has at least one numerous term. It also follows that S is uniquely determined, and we call it the *special partition* of α.

Lemma 2. Let α be a minimal zero-sum sequence of maximum length $D(G) = 3k+2$ in the group $G = C_3 \oplus C_{3k}$ where $k \geq 2$. Denote by S the special partition of α, with a leftover 5-tuple L, and let a be a numerous term of α. Then L has one of the following forms:

(1) $L = a^2 b_1 b_2 b_3$ where b_1, b_2, b_3 are in the same proper coset of $\langle a \rangle$ and $b_1 + b_2 + b_3 = a$: the leftover contains two a's;

(2) $L = ab^2(a-b)^2$ where $b \notin \langle a \rangle$: the leftover contains one a;

(3) $L = b^2(a+b)^2(a+2b)$ where $b \notin \langle a \rangle$ and $\mathrm{ord}(b) = 3$: the leftover contains no a's.

Proof. The sum of each triple in S and the sum of L are equal to a certain generator g of H. In particular $3a = g$ because S contains a triple $T = (a, a, a)$. Recall that $\mathrm{ord}(a) = 3k$ by Lemma 1(d) and consider the distinct elements $a, 2a, \ldots, (3k-3)a = -3a$ of $\langle a \rangle$. Each one of them can be expressed by using several a's from T (0, 1, 2 or 3) and several of the remaining $k-2$ complete triples from S. Hence all elements of $\langle a \rangle$ except $-a$ and $-2a$ are expressible as subsequence sums of α not intersecting the leftover L. Thus if a proper subsum of L belongs to $\langle a \rangle$, the value of the subsum must equal a or $2a$, otherwise a contradiction with the minimality of α is obtained.

Furthermore, $\mathrm{ord}(a) = 3k$ implies $G/\langle a \rangle \cong C_3$. Observe also that α does not contain elements of $\langle a \rangle$ except a. Indeed, $\langle a \rangle = H \cup (a+H) \cup (2a+H)$. Now α has no terms in H, and because a is a term in $a+H$, there are no terms in $2a+H$ either (or else $\overline{\alpha}$ would have a zero sum of length 2). Also, all terms in $a+H$ are equal to a as a is numerous.

By the previous remark, all terms different from a in the leftover L do not belong to $\langle a \rangle$. Since L contains at most two a's, three cases are possible.

Case 1: L contains two a's: $L = a^2 b_1 b_2 b_3$ where $b_i \notin \langle a \rangle$, $i = 1, 2, 3$. Then $\sigma(L) = g = 3a$ implies $b_1 + b_2 + b_3 = a$. Because $G/\langle a \rangle \cong C_3$, b_1, b_2, b_3 must be all in the same coset of $\langle a \rangle$. So L has the form (1).

Case 2: L contains one a; its remaining four terms are not in $\langle a \rangle$. If three of them are in the same coset of $\langle a \rangle$, their sum is in $\langle a \rangle$ as $G/\langle a \rangle \cong C_3$. Since $\sigma(L) = 3a$, the remaining term of L is also in $\langle a \rangle$ which is impossible. So $L = ab_1b_2c_1c_2$ where b_1, b_2 are in one of the proper cosets of $\langle a \rangle$ and c_1, c_2 are in the other. Then $b_i + c_j \in \langle a \rangle$ for $i, j = 1, 2$. Hence $b_i + c_j \in \{a, 2a\}$, $i, j = 1, 2$. On the other hand $b_1 + b_2 + c_1 + c_2 = \sigma(L) - a = 2a$, and now the minimality of α implies $b_i + c_j = a$ for $i, j = 1, 2$. Therefore $b_1 = b_2 = b$, $c_1 = c_2 = c$, with $b + c = a$. So L is of the form (2).

Case 3: L contains no a's. Suppose each of the two proper cosets of $\langle a \rangle$ contains at least two terms of L. Then the sum of these four terms belongs to $\langle a \rangle$, in view of $G/\langle a \rangle \cong C_3$ again.

It follows from $\sigma(L) = 3a$ that the fifth term of L is in $\langle a\rangle$ which is a contradiction. Also, L has a term in each proper coset of $\langle a\rangle$: otherwise $\sigma(L) \notin \langle a\rangle$ which contradicts $\sigma(L) = 3a$. Hence $L = b_1b_2b_3b_4c$ where b_1, b_2, b_3, b_4 are in one of the proper cosets of $\langle a\rangle$ and c is in the other. Now $b_i + c \in \langle a\rangle$ for $i = 1, 2, 3, 4$, implying $b_i + c \in \{a, 2a\}$. So the b_i's take on at most 2 distinct values, in fact exactly 2, or else 4 terms of L would be the same. Let these values be b_1 and b_2. If $b_1 + c = a$ then $b_2 + c = 2a$ and vice versa, so that $b_2 = b_1 + a$ or $b_1 = b_2 + a$. In conclusion, $L = b^2(b + a)^2c$ for some b, c in different proper cosets of $\langle a\rangle$.

Next, $b+b+(b+a) \in \langle a\rangle$, hence $b+b+(b+a) = a$ or $b+b+(b+a) = 2a$. The second equality leads to $a = 3b$ which is false as $b \notin \langle a\rangle$ and $\text{ord}(a) = 3k$. So $b + b + (b + a) = a$, implying $3b = 0$. The remaining two terms $b + a$ and c of L add up to $\sigma(L) - a = 2a$ which gives $b + c = a$. Hence L has the form (3).

Now we characterize the minimal zero-sum sequences of maximum length in $G = C_3 \oplus C_{3k}$.

Theorem. Let $G = C_3 \oplus C_{3k}$ where $k \geq 2$. A sequence α of length $D(G) = 3k + 2$ in G is a minimal zero-sum sequence if and only if it has one of the following forms:

(i) $\alpha = a^{3k-1}b_1b_2b_3$ where $\text{ord}(a) = 3k$, b_1, b_2, b_3 are in the same proper coset of $\langle a\rangle$ and $b_1 + b_2 + b_3 = a$;

(ii) $\alpha = a^{3k-2}b^2(a - b)^2$ where $\text{ord}(a) = 3k$ and $b \notin \langle a\rangle$;

(iii) $\alpha = a^u(b + a)^v(b - a)$ where $\text{ord}(a) = 3k$, $b \notin \langle a\rangle$, $\text{ord}(b) = 3$ and u, v are nonnegative integers satisfying $u + v = 3k + 1$, $u \equiv v \equiv -1 \pmod 3$;

(iv) $\alpha = b^2a^u(b + a)^v(2b + a)^w$ where $\text{ord}(a) = 3k$, $b \notin \langle a\rangle$, $\text{ord}(b) = 3$ and u, v, w are nonnegative integers satisfying $u + v + w = 3k$, $v + 2w \equiv 1 \pmod 3$.

Proof. It is straightforward to check that each of the sequences (i)–(iv) is a minimal zero-sum sequence in G, of length $D(G) = 3k+2$. Conversely, let α be a minimum zero-sum sequence in G of (maximum) length $D(G) = 3k+2$. Consider the special partition S of α with leftover 5-tuple L, and let g be the associated generator of H: the sum of L and the sum of each triple in S are equal to g. For each numerous term a there is a triple (a, a, a) in S, hence $3a = g = \sigma(L)$.

Case 1: Suppose that α contains a numerous term a and an order 3 term b with multiplicity 2 such that $b \notin \langle a\rangle$. Then (a, b) is a basis of G as $\text{ord}(a) = 3k$; so $G/\langle b\rangle \cong \langle a\rangle \cong C_{3k}$. Delete the two occurrences of the term b from α to obtain a sequence α' of length $3k$ in G. Let $\varphi\colon G \to G/\langle b\rangle$ be the canonical epimorphism. It is immediate that $\varphi(\alpha')$ is a minimal zero-sum sequence of length $3k$ in $G/\langle b\rangle$, which is a cyclic group of order $3k$. Hence all terms of $\varphi(\alpha')$ are equal to some generator of $G/\langle b\rangle$. Equivalently, all terms of α different from b are in the same coset of $\langle b\rangle$ which generates the factor group $G/\langle b\rangle$. This coset can be only $a + \langle b\rangle$. In addition, $\sigma(\alpha') = \sigma(\alpha) - 2b = 0 - 2b = b$. One infers that α has the form (iv).

From now on, assume that α is not as in case 1. Then, for any numerous term a, the leftover L of S is not of the form (3) in Lemma 2. So each numerous term of α occurs in L.

Case 2: There exists a numerous term $a \in \alpha$ that occurs in L exactly once. Then by Lemma 2 the leftover has the form (2): $L = ab^2(a-b)^2$ where $b \notin \langle a \rangle$. We prove that α has the form (ii). Observe that neither b nor $a-b$ is numerous. Indeed, if b is numerous then $3b = g = 3a$, so that $a-b$ has order 3. On the other hand, $a-b$ occurs at least twice in α, hence its multiplicity is 2; also $a-b \notin \langle a \rangle$ because $b \notin \langle a \rangle$. These conclusions contradict the assumption that α is not as in case 1. Thus b is not numerous and, by symmetry, $a-b$ is not numerous either. Hence a is the unique numerous term of α, and because every term not in L is numerous, the sequence has the form (ii).

Case 3: Each numerous term $a \in \alpha$ occurs in the leftover of S twice. Then, by Lemma 2, for every numerous term a the leftover has the form (1): $L = a^2b_1b_2b_3$ where b_1, b_2, b_3 are in the same proper coset of $\langle a \rangle$ and $b_1 + b_2 + b_3 = a$.

If there is a unique numerous term a in α then clearly α has the form (i).

Let α have at least two numerous terms. Since the leftover 5-tuple L contains two occurrences of each one of them, the numerous terms are exactly two, say a and c, $a \neq c$. Like before, we have $3a = g = 3c$, yielding $\mathrm{ord}(c-a) = 3$. Let $L = a^2c^2b_1$ where the term b_1 is not numerous. Then b_1 is the only term in α different from a and c as each term outside L is numerous. Furthermore, $\sigma(L) = 3a$ gives $2c + b_1 = a$, i. e. $b_1 = a - 2c$. Finally, the multiplicities u and v of a and c are both 2 modulo 3. Therefore $\alpha = a^uc^v(a-2c)$, with $\mathrm{ord}(c-a) = 3$ and $u \equiv v \equiv -1 \pmod 3$. Now set $b = c - a$. Then $b \notin \langle a \rangle$, $\mathrm{ord}(b) = 3$, $c = b + a$ and $a - 2c = b - a$. So α is of the form (iii) which completes the proof.

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