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RECOUNTING DETERMINANTS FOR A CLASS OF HESSENBERG MATRICES

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Abstract

We provide combinatorial interpretations for determinants which are Fibonacci numbers of several recently introduced Hessenberg matrices. Our arguments make use of the basic definition of the determinant as a signed sum over the symmetric group.

1. Introduction

In what follows, $\mathbb{W}$ and $\mathbb{N}$ will denote the nonnegative and positive integers, respectively. Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 3$, and L_n be the nth Lucas number, defined by $L_1 = 1, L_2 = 3$ with $L_n = L_{n-1} + L_{n-2}$ if $n \geqslant 3$. Given an $n \times n$ matrix A, the *determinant of A,* denoted $|A|$, is defined by

$$|A| = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}, \tag{1.1}$$

where S_n is the set of permutations of $\{1, 2, ..., n\}$.

A matrix is said to be (lower) Hessenberg [7] if all entries above the superdiagonal are zero. The Hessenberg matrix [7]

$$A_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ & & & & 1 \\ 1 & \cdots & & 1 & 2 \end{pmatrix}_{n \times n} \tag{1.2}$$

has as its determinant F_{n+2}. Several other Hessenberg matrices whose determinants are Fibonacci numbers were introduced in [5, 7], where cofactor expansion was used to obtain these determinants. In this note, we provide combinatorial interpretations and generalizations for several of these determinants using definition (1.1). Recently, similar combinatorial proofs were given for the determinants of Vandermonde's matrix [4] and of matrices whose entries are Fibonacci [2] and Catalan [3] numbers.

A *composition* of a positive integer n is a sequence of positive integers, called *parts*, summing to n. For instance, the sequences (1, 3, 2) and (2, 4) are compositions of 6. The Fibonacci number F_{n+1} [8, p. 46] counts the compositions of n with parts belonging to $\{1, 2\}$.

2. Combinatorial Proofs

We first consider two families of Hessenberg matrices [7] which alternate 1s and 0s below the main diagonal. If t is an indeterminate, let $A_{n,t}$ be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are 1, all main diagonal entries are 1 except the last one, which is $t+1$, and the entries of each column below the main diagonal alternate 0s and 1s, starting with 0. Let $B_{n,t}$ be the Hessenberg matrix obtained from $A_{n,t}$ by alternately replacing the 1s on the superdiagonal with $\boldsymbol{i}$s and $-\boldsymbol{i}$s, where $\boldsymbol{i}=\sqrt{-1}$. The matrices $A_{5,t}$ and $B_{5,t}$ are shown below.

$$A_{5,t} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & t+1 \end{pmatrix} \qquad B_{5,t} = \begin{pmatrix} 1 & \boldsymbol{i} & 0 & 0 & 0 \\ 0 & 1 & -\boldsymbol{i} & 0 & 0 \\ 1 & 0 & 1 & \boldsymbol{i} & 0 \\ 0 & 1 & 0 & 1 & -\boldsymbol{i} \\ 1 & 0 & 1 & 0 & t+1 \end{pmatrix}$$

Proposition 2.1. *If* $n \geqslant 2$, *then* $|A_{n,t}| = |B_{n,t}| = F_n + tF_{n-1}$.

Proof. Notice that the only nonzero terms of $|A_{n,t}|$ in the expansion given in (1.1) are those corresponding to permutations of $\{1, 2, ..., n\}$ each of whose cycles (when expressed disjointly) consists of an odd number of consecutive integers. These permutations are all even and can clearly be identified with *compositions* of n with odd parts, where the sequence of cycle lengths determines the sequence of parts. Note further that the terms in $|A_{n,t}|$ corresponding to these permutations are either $t+1$ or 1, depending on whether or not n goes in a cycle by itself.

If $t \in \mathbb{W}$, then $|A_{n,t}|$ gives the cardinality of the set of "marked" compositions of n with odd parts in which a terminal 1 may either be marked with one of t colors or be unmarked. From the well known description of F_m [8, p. 46] as the number of compositions of $m \in \mathbb{N}$ with odd parts, there are F_n such compositions of n that do not end in a marked 1 and tF_{n-1}

such compositions that do, implying $|A_{n,t}| = F_n + tF_{n-1}$, as desired. Replacing the 1s on the superdiagonal of $A_{n,t}$ with $\boldsymbol{i}$s as described doesn't change any of the terms in the expansion of $|A_{n,t}|$, whence $|B_{n,t}| = |A_{n,t}|$. □

Letting $t = 1$ and $t = 2$ in Proposition 2.1 gives

Corollary 2.1. *If* $n \geqslant 2$, *then*

$$|A_{n,1}| = |B_{n,1}| = F_{n+1},$$

and

$$|A_{n,2}| = |B_{n,2}| = F_{n+1} + F_{n-1} = L_n.$$

We now turn to the Hessenberg matrices $C_{n,t}$ [7] in which the superdiagonal entries are $-1s$, all main diagonal entries are 2s except the last one, which is $t+1$, and all entries below the diagonal are 1s. The matrix $C_{5,t}$ is shown below.

$$C_{5,t} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{pmatrix}$$

Proposition 2.2. *If* $n \geqslant 1$, *then* $|C_{n,t}| = F_{2n} + tF_{2n-1}$.

Proof. Notice that the only nonzero terms of $|C_{n,t}|$ are those corresponding to permutations of $\{1, 2, ..., n\}$ each of whose cycles consists of a set of consecutive integers. These permutations are clearly synonymous with compositions of n. A cycle of length at least two in such a permutation π contributes one towards the product in (1.1), as the factor (± 1) contributed by the cycle towards $\mathrm{sgn}(\pi)$ equals the factor contributed by the cycle towards the product of matrix entries. A 1-cycle clearly contributes $t + 1$ or 2 towards this product.

If $t \in \mathbb{W}$, then $|C_{n,t}|$ gives the cardinality of the set of "marked" compositions of n in which one may circle non-terminal 1s and may either mark a terminal 1 with one of t colors or leave it unmarked. Convert these compositions of n into ones of length $2n$ by replacing each part $m \geqslant 1$ (uncircled and uncolored if $m = 1$) with the parts $12^{m-1}1$, replacing any circled 1s with 2s, and replacing a terminal colored 1 with a 2 of the same color. This gives all $F_{2n} + tF_{2n-1}$ compositions of $2n$ in $\{1, 2\}$ that end either in a 1 or in a 2 marked with one of t colors, which implies $|C_{n,t}| = F_{2n} + tF_{2n-1}$. □

Letting $t = 1$ and $t = 2$ in Proposition 2.2 gives

Corollary 2.2. *If* $n \geqslant 1$, *then* $|C_{n,1}| = F_{2n+1}$ *and* $|C_{n,2}| = L_{2n}$.

Next, let D_n be the $n \times n$ matrix with 1s on the main diagonal, $\boldsymbol{i}$s on the sub- and superdiagonals, and 0s elsewhere and E_n the $n \times n$ matrix with 3s on the main diagonal, 1s on both the sub- and superdiagonals, and 0s elsewhere [5, 6].

$$D_4 = \begin{pmatrix} 1 & \boldsymbol{i} & 0 & 0 \\ \boldsymbol{i} & 1 & \boldsymbol{i} & 0 \\ 0 & \boldsymbol{i} & 1 & \boldsymbol{i} \\ 0 & 0 & \boldsymbol{i} & 1 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Proposition 2.3. *If $n \geqslant 1$, then $|D_n| = F_{n+1}$ and $|E_n| = F_{2n+2}$.*

Proof. Note that the only nonzero terms in the expansion of $|D_n|$ are those corresponding to permutations that can be expressed as products of disjoint transpositions consisting of two consecutive integers. These permutations are synonymous with compositions of n in $\{1, 2\}$ and each corresponding term is clearly one, which implies $|D_n| = F_{n+1}$.

The same permutations contribute nonzero terms to $|E_n|$, but now fixed points and transpositions each add factors of 3 and -1, respectively, to these terms. Let T_n denote the set of "marked" compositions of n with parts in $\{1, 2\}$ in which the 1s are marked black (B), white (W), or green (G). Then

$$|E_n| = \sum_{\lambda \in T_n} (-1)^{v(\lambda)},$$

where $v(\lambda)$ records the number of 2s in $\lambda \in T_n$.

Let $T_n^* \subseteq T_n$ comprise those compositions having no 2s as well as no occurrences of a white 1 directly preceding a green 1. That T_n^* has cardinality F_{2n+2} follows from a well known description [8, p. 46] of F_{2n+2} as the number of words of length n in the alphabet $\{B, W, G\}$ in which WG is disallowed. Define a sign-changing involution of $T_n - T_n^*$ by identifying the first occurrence of a 2 or a WG and switching to the other option. Since all members of T_n^* clearly have positive sign, it follows that $|E_n| = |T_n^*| = F_{2n+2}$. $\square$

Let $D_{n,t}$ be the matrix obtained from D_n by replacing the lowest superdiagonal $\boldsymbol{i}$ with $\boldsymbol{i}t$ and $E_{n,t}$ the matrix obtained from E_n by replacing the lowest superdiagonal 1 with t. By additionally allowing a terminal 2 in the compositions of n described above to be marked with one of t colors, the determinants in Proposition 2.3 can be generalized to

$$|D_{n,t}| = F_n + tF_{n-1}, \quad n \geqslant 1,$$

and

$$|E_{n,t}| = F_{2n+2} - (t-1)F_{2n-2}, \quad n \geqslant 1.$$

Let G_n be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are 1s, the main diagonal entries are 2s, and the entries of each column below the main diagonal alternate -1s and 1s, starting with -1. Let H_n be the matrix obtained by changing the superdiagonal entries of G_n to -1s.

$$G_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & 2 \end{pmatrix} \qquad H_5 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & 2 \end{pmatrix}$$

Proposition 2.4. *If $n \geqslant 1$, then $|G_n| = F_{2n+1}$ and $|H_n| = F_{n+2}$.*

Proof. Note that $|G_n|$ equals $|C_{n,t}|$ when $t = 1$ and thus counts the compositions of n where 1s may be circled (which we will denote by R_n). The determinant for H_n is the same as that of G_n except that the terms in (1.1) are signed according to the number of even cycles in the corresponding permutation of $\{1, 2, ..., n\}$. That is,

$$|H_n| = \sum_{\lambda \in R_n} (-1)^{v(\lambda)},$$

where $v(\lambda)$ records the number of even parts of $\lambda \in R_n$.

Let $R_n^* \subseteq R_n$ comprise those compositions possessing only odd parts and not containing a circled 1 directly following an uncircled part. Define a sign-changing involution of $R_n - R_n^*$ by identifying the first occurrence of an even part or of an uncircled odd part directly followed by a circled 1 and switching to the other option. Since each member of R_n^* clearly has positive sign, it follows that $|H_n| = |R_n^*|$.

That $|R_n^*| = F_{n+2}$ can be reasoned as follows. Suppose a member of R_n^* begins with k circled 1s, where $k \geqslant 0$. If k is odd, replace these k 1s with a single part k to get a composition of n with odd parts. If k is even, replace these k 1s with a single part $k+1$ to get a composition of $n+1$ with odd parts, which implies that there are $F_n + F_{n+1} = F_{n+2}$ members of R_n^* altogether. □

Let $G_{n,t}$ and $H_{n,t}$ be the matrices obtained by replacing the lowest diagonal 2 in G_n and H_n, respectively, with $t+1$. Allowing a terminal 1 to be marked with one of $t+1$ colors yields the generalizations

$$|G_{n,t}| = F_{2n} + tF_{2n-1}, \quad n \geqslant 1,$$

and

$$|H_{n,t}| = F_n + tF_{n+1}, \quad n \geqslant 1,$$

which were shown in [7] using cofactor expansion and recursion.

3. Generalizations

We close by highlighting generalizations of the results above which are readily obtained from the preceding combinatorial arguments. Consider the generalized Fibonacci numbers [1, p. 23] given by $G_1 = s, G_2 = t + s$, and $G_n = G_{n-1} + G_{n-2}$ if $n \geqslant 3$, where s and t are natural numbers. If $n \geqslant 1$, the number G_n counts the compositions of n in $\{1, 2\}$ in which a terminal 1 may be marked with one of s colors and a terminal 2 with one of t colors. It is easy to see that

$$G_n = sF_n + tF_{n-1}, \quad n \geqslant 2.$$

We now introduce matrices with two free parameters generalizing those in the prior section which will have generalized Fibonacci numbers as determinants. Let $D_{n,t,s}$ be the matrix obtained from $D_{n,t}$ by replacing the lowest diagonal 1 with s. Let $A_{n,t,s}$, $B_{n,t,s}$, $C_{n,t,s}$, $G_{n,t,s}$, and $H_{n,t,s}$ be the matrices obtained from $A_{n,t}$, $B_{n,t}$, $C_{n,t}$, $G_{n,t}$, and $H_{n,t}$, respectively, by replacing $t + 1$ with $t + s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. The matrices $A_{5,t,s}$ and $H_{5,t,s}$ are shown below.

$$A_{5,t,s} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & s \\ 1 & 0 & 1 & 0 & t+s \end{pmatrix} \qquad H_{5,t,s} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -s \\ 1 & -1 & 1 & -1 & t+s \end{pmatrix}$$

Extending the arguments of the prior section yields the following results:

Proposition 3.1. *If $n \geqslant 2$, then $|A_{n,t,s}| = |B_{n,t,s}| = G_n$.*

Proposition 3.2. *If $n \geqslant 1$, then $|C_{n,t,s}| = G_{2n}$.*

Proposition 3.3. *If $n \geqslant 1$, then $|D_{n,t,s}| = G_n$.*

Proposition 3.4. *If $n \geqslant 1$, then $|G_{n,t,s}| = G_{2n}$ and $|H_{n,t,s}| = G^*_{n+1}$, where G^*_i denotes the generalized Fibonacci sequence in which the parameters s and t are exchanged.*

We outline the proof of Proposition 3.1, and leave the others as exercises for the interested reader. If $s, t \in \mathbb{N}$, the proof of Proposition 2.1 shows that $|A_{n,t,s}|$ counts the "marked" compositions of n with odd parts in which a terminal 1 may either be marked above with one of s colors or marked below with one of t colors and a terminal part $m \geqslant 3$ may be marked above with one of s colors. There are sF_n such compositions whose last part is marked above and tF_{n-1} whose last part is a 1 marked below, implying $|A_{n,t,s}| = sF_n + tF_{n-1} = G_n$.

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