INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A08

CONJUGATE SEQUENCES IN A FIBONACCI-LUCAS SENSE AND SOME IDENTITIES FOR SUMS OF POWERS OF THEIR ELEMENTS

Roman Wituła
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

Damian Słota
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland
d.slota@polsl.pl

Received: 5/30/06, Revised: 11/16/06, Accepted: 2/3/07, Published: 2/9/07

Abstract

The aim of this paper is to introduce the notion of a pair of conjugate sequences in a Fibonacci-Lucas sense. New identities are generated by means of such sequences (including separate derivations for Fibonacci and Lucas numbers) and by means of Lucas and Vieta-Lucas polynomials.

1. Introduction

The paper, which is a follow-up of [17], is still focused on analyzing the notion of a pair of sequences $\{x_n\}$ and $\{y_n\}$ that are conjugate in a Fibonacci-Lucas sense (Sections 2 and 4). A number of identities are derived for such pairs of conjugate sequences (see Lemma 7).

In Section 3, by means of Lucas and Vieta-Lucas polynomials, some identities (Lemma 9) are presented for sequences $\{z_n\}$ defined by recurrence formula ($z_0, z_1 \in \mathbb{C}$):

$$z_{n+2} = a\, z_{n+1} + z_n, \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

and for auxiliary sequences ($\xi_{0,n} = z_n$):

$$\xi_{k,n} = z_{n+k} + (-1)^k\, z_{n-k}, \quad k = 1, \dots, n, \quad n \in \mathbb{N}.$$

In Section 5 the decompositions of some special symmetric polynomials of three, four, and five variables are given on the grounds of Vieta-Lucas polynomials. Finally, in Section 6 the decompositions of the polynomials discussed in Section 5 are applied to generate the corresponding identities for the pairs of conjugate sequences in a Fibonacci-Lucas sense. Special cases of such identities for Fibonacci and Lucas numbers are discussed separately ((6.42)–(6.45)).

Several publications were concerned with identities for the sums of the powers of Fibonacci and Lucas numbers (see [7, 8, 9, 10, 11, 17] and references therein). However, it should be emphasized that the research tasks undertaken in this paper are substantively different from those already discussed elsewhere.

2. Conjugate Sequences

Let us assume that the elements of sequences $\{x_n\}$ and $\{y_n\}$ are determined by means of the same recurrence relation:

$$x_{n+1} = a\,x_n + b\,x_{n-1}, \qquad y_{n+1} = a\,y_n + b\,y_{n-1}, \quad n \in \mathbb{N}, \tag{2.1}$$

where $b = \pm 1$, $a \neq 0$, $a^2 + 4b > 0$. First our objective is to find some adjoint-conditions – connecting elements of sequences $\{x_n\}$ and $\{y_n\}$. These conditions are derived from the following identities for Fibonacci and Lucas numbers:

$$\begin{aligned} F_{n-1} + F_{n+1} &= L_n, \\ L_{n-1} + L_{n+1} &= 5\,F_n. \end{aligned}$$

Lemma 1 *If there exists $A \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $x_{n_0} \neq 0$,*

$$y_n = x_{n-1} + x_{n+1}, \quad \text{for} \quad n = n_0 - 1, n_0 + 1,$$

and

$$Ax_n = y_{n-1} + y_{n+1}, \quad \text{for} \quad n = n_0,$$

then, $A = 2 + 2b + a^2$.

Proof. We have

$$\begin{aligned} Ax_{n_0} &= y_{n_0-1} + y_{n_0+1} = x_{n_0-2} + 2x_{n_0} + x_{n_0+2} \\ &= x_{n_0-2} + abx_{n_0-1} + (2 + b + a^2)x_{n_0} = (2 + 2b + a^2)x_{n_0} \end{aligned}$$

so $A = 2 + 2b + a^2$. □

Lemma 2 *Let $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, $x_n = u\alpha^n + v\beta^n$, $y_n = \zeta\alpha^n + \xi\beta^n$, $n \in \mathbb{N}_0$. If $|\alpha| \neq |\beta|$ and identities:*

$$\left\{ \begin{array}{l} y_n = x_{n-1} + x_{n+1} \\ (2 + 2b + a^2)x_n = y_{n-1} + y_{n+1} \end{array} \right. \tag{2.2}$$

hold for every $n \in \mathbb{N}$, then the following identities are satisfied:

$$\begin{aligned} y_n &= (\alpha^{-1}+\alpha)\, u\, \alpha^n + (\beta^{-1}+\beta)\, v\, \beta^n \\ x_n &= \frac{\alpha+\alpha^{-1}}{2+2b+a^2}\, \zeta\, \alpha^n + \frac{\beta+\beta^{-1}}{2+2b+a^2}\, \xi\, \beta^n, \end{aligned} \tag{2.3}$$

for every $n \in \mathbb{N}$.

Proof. We have

$$y_n = x_{n-1} + x_{n+1} = u\alpha^{n-1} + u\alpha^{n+1} + v\beta^{n-1} + v\beta^{n+1} = u(\alpha^{-1}+\alpha)\alpha^n + v(\beta^{-1}+\beta)\beta^n.$$

□

Corollary 3 *If $b = -1$ then by (2.2) we obtain: $ax_n = y_n$, for all $n \in \mathbb{N}$.*

Definition 4 *Sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$, defined by the same relation*

$$x_{n+1} = ax_n + x_{n-1} \quad \text{and} \quad y_{n+1} = ay_n + y_{n-1},$$

are called conjugate sequences in a Fibonacci-Lucas sense (with parameter a) if the following conditions are satisfied:

$$\begin{aligned} y_n &= x_{n-1} + x_{n+1}, \\ (4+a^2)x_n &= y_{n-1} + y_{n+1}, \end{aligned}$$

for every $n \in \mathbb{N}$.

Remark 5 *If $a = 1$, then $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. Additionally, if $x_n = u\alpha^n + v\beta^n$, $y_n = \zeta\alpha^n + \xi\beta^n$, $n \in \mathbb{N}_0$ and identities (2.2) hold, then*

$$y_n = \sqrt{5}u\alpha^n - \sqrt{5}v\beta^n \quad \text{and} \quad x_n = \frac{\sqrt{5}}{5}u\alpha^n - \frac{\sqrt{5}}{5}v\beta^n, \quad n \in \mathbb{N}_0.$$

In the language of conjugate sequences, we may recapitulate the discussion to this point as follows:

Lemma 6 *If elements of two sequences $\{x_n\}$ and $\{y_n\}$ are conjugates in a Fibonacci-Lucas sense with parameter a and $x_n = u\alpha^n + v\beta^n$, $n \in \mathbb{N}_0$, where α, β are roots of the characteristic equation $\mathbb{X}^2 - a\mathbb{X} - 1 = 0$, then*

$$\begin{aligned} \alpha\beta &= -1, \\ \alpha + \alpha^{-1} &= -(\beta+\beta^{-1}) = \alpha - \beta = \pm\sqrt{a^2+4} \end{aligned}$$

and, consequently, we obtain

$$y_n = \pm\sqrt{a^2+4}(u\alpha^n - v\beta^n), \quad n \in \mathbb{N},$$

or

$$y_n = \pm\frac{1}{\sqrt{a^2+4}}(u\alpha^n - v\beta^n), \quad n \in \mathbb{N}.$$

Proof. We note that

$$(\alpha-\beta)^2+4\alpha\beta=(\alpha+\beta)^2$$

i.e.,

$$\alpha-\beta=\pm\sqrt{a^2+4}$$

and

$$\alpha+\alpha^{-1}+\beta+\beta^{-1}=(\alpha+\beta)(\alpha\beta+1)(\alpha\beta)^{-1}=0.$$

□

Lemma 7 *Let us assume that the elements of sequences $\{x_n\}$ and $\{y_n\}$ are conjugates in a Fibonacci-Lucas sense with parameter a, or, more precisely, let us assume that the following conditions are satisfied:*

$$x_{n+1}=ax_n+x_{n-1}, \qquad y_{n+1}=ay_n+y_{n-1}, \tag{2.4}$$

$$y_n=x_{n-1}+x_{n+1} \tag{2.5}$$

and

$$(a^2+4)x_n=y_{n-1}+y_{n+1}. \tag{2.6}$$

Then, the following identities hold:

$$z_n+az_{n+3}=(a^2+1)z_{n+2}, \tag{2.7}$$

$$\begin{aligned} z_{n+4}+cz_{n+2}+z_n &= az_{n+3}+(1+c)z_{n+2}+z_n \\ &= (a^2+1+c)z_{n+2}+az_{n+1}+z_n \\ &= (a^2+2+c)z_{n+2} \end{aligned} \tag{2.8}$$

for every $c\in\mathbb{C}$ and $z\in\{x,y\}$,

$$(a^2+4)x_{n+2}+y_{n-1}=y_{n+3}+y_{n+1}+y_{n-1}=(a^2+3)y_{n+1}, \tag{2.9}$$

$$y_{n+2}+x_{n-1}=x_{n+3}+x_{n+1}+x_{n-1}=(a^2+3)x_{n+1}, \tag{2.10}$$

$$\begin{aligned} x_n+x_{n+4}+ax_{n+5} &= (a^2+1)x_{n+4}+ax_{n+3}+x_n \\ &= (a^2+1)x_{n+4}+(a^2+1)x_{n+2} \\ &= (a^2+1)y_{n+3}, \end{aligned} \tag{2.11}$$

$$x_{n+4}-x_n=x_{n+4}+x_{n+2}-(x_{n+2}+x_n)=y_{n+3}-y_{n+1}=ay_{n+2}, \tag{2.12}$$

$$x_n + ax_{n+5} = a^2 y_{n+3} + x_{n+2}, \tag{2.13}$$

$$\begin{aligned} y_n + y_{n+4} + ay_{n+5} &= (a^2+1)y_{n+4} + ay_{n+3} + y_n \\ &= (a^2+1)y_{n+4} + (a^2+1)y_{n+2} \\ &= (a^2+1)(a^2+4)x_{n+3}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} y_{n+4} - y_n = y_{n+4} + y_{n+2} - (y_{n+2} + y_n) = \\ = (a^2+4)(x_{n+3} - x_{n+1}) = a(a^2+4)x_{n+2} \end{aligned} \tag{2.15}$$

$$y_n + ay_{n+5} = a^2(a^2+4)x_{n+3} + y_{n+2}. \tag{2.16}$$

However, we have

$$z_n + z_{n+2} + z_{n+4} + z_{n+6} + a(z_{n+5} + z_{n+7}) = (a^2+1)(a^2+4)z_{n+4} \tag{2.17}$$

and

$$z_{n+8} - 2z_{n+4} + z_n = a^2(a^2+4)z_{n+4} \tag{2.18}$$

for every $z \in \{x, y\}$.

In the next section some generalizations of some of the identities from Lemma 7 will be presented.

3. Lucas and Vieta-Lucas Polynomials

Let us set

$$\Omega_n(x) := 2T_n(x/2)$$

and

$$W_n(x) := (-i)^n \Omega_n(ix)$$

for every $n = 0, 1, 2, \dots$, where $T_n(x)$ means the n-th Chebyshev polynomial of the first kind. We note that the polynomials $\Omega_n(x)$, called Vieta-Lucas polynomials (see [6, 15]), satisfy the following recurrence relations:

$$\Omega_0(x) = 2, \qquad \Omega_1(x) = x,$$

$$\Omega_{k+2}(x) = x\,\Omega_{k+1}(x) - \Omega_k(x), \qquad k \in \mathbb{N}.$$

Table 1: Triangle of the coefficients of polynomials $W_n(x)$

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
$W_0(x)$	2									
$W_1(x)$		1								
$W_2(x)$	2		1							
$W_3(x)$		3		1						
$W_4(x)$	2		4		1					
$W_5(x)$		5		5		1				
$W_6(x)$	2		9		6		1			
$W_7(x)$		7		14		7		1		
$W_8(x)$	2		16		20		8		1	
$W_9(x)$		9		30		27		9		1

Legende:
b
a
a+b

Hence, we obtain

$$\Omega_2(x) = x^2 - 2, \qquad \Omega_3(x) = x^3 - 3x, \qquad \Omega_4(x) = x^4 - 4x^2 + 2,$$
$$\Omega_5(x) = x^5 - 5x^3 + 5x, \qquad \Omega_6(x) = x^6 - 6x^4 + 9x^2 - 2,$$
$$\Omega_7(x) = x^7 - 7x^5 + 14x^3 - 7x, \qquad \text{etc.}$$

Similarly, we have

$$W_0(x) = 2, \qquad W_1(x) = x,$$
$$W_{n+2}(x) = x\, W_{n+1}(x) + W_n(x),$$

which generate the so-called Lucas polynomials [1, 2, 3, 4, 6]):

$$W_2(x) = x^2 + 2, \qquad W_3(x) = x^3 + 3x, \qquad W_4(x) = x^4 + 4x^2 + 2,$$
$$W_5(x) = x^5 + 5x^3 + 5x, \qquad W_6(x) = x^6 + 6x^4 + 9x^2 + 2,$$
$$W_7(x) = x^7 + 7x^5 + 14x^3 + 7x, \qquad \text{etc.}$$

Lemma 8 *We have*

$$\operatorname{coeff}\big[x^2; W_{2k}(x)\big] = k^2, \quad \operatorname{coeff}\big[x^3; W_{2k+1}(x)\big] = \sum_{l=1}^{k} l^2 = \frac{1}{6}\, k\,(k+1)\,(2k+1),$$

$$\operatorname{coeff}\big[x^4; W_{2k}(x)\big] = \sum_{l=1}^{k} (k-l) l^2 = \frac{1}{12}\, k^2\,(k^2-1),$$

$$W_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \tag{3.19}$$

$$x \sum_{r=1}^{s} W_{2r-1}(x) = W_{2s}(x) - 2, \tag{3.20}$$

and

$$x \sum_{r=0}^{s} W_{2r}(x) = W_{2s+1}(x). \tag{3.21}$$

The polynomials $W_n(x)$, $n \in \mathbb{N}$, enable the description of many fundamental identities, involving the elements of the sequence $\{z_n\}_{n=1}^{\infty}$ determined by the following one-parameter recurrence relation:

$$z_0, z_1 \in \mathbb{C} \qquad \text{and} \qquad z_{n+2} = a\, z_{n+1} + z_n, \qquad n \in \mathbb{N}_0. \tag{3.22}$$

Additionally, let us assume that $\xi_{0,n} := z_n$ and

$$\xi_{k,n} := z_{n+k} + (-1)^k\, z_{n-k}$$

for every $k = 1, 2, \ldots, n-1$.

Some of the identities are generalized forms of the identities from Lemma 7 (especially identities (2.4)–(2.8) and (2.17)–(2.18)). All the identities presented below seem to be original.

Lemma 9 *The following identities hold true*

$$\begin{aligned}
\xi_{k+1,n} &= a\xi_{k,n} + \xi_{k-1,n}, && (3.23)\\
\xi_{k+1,n} &= W_{k-s}(a)\xi_{s+1,n} + W_{k-s-1}(a)\xi_{s,n}, \quad s = 1 \le k, && (3.24)\\
\xi_{k,n} &= W_k(a) z_n, && (3.25)\\
\sum_{r=-s}^{s} z_{n-2r} &= \left(\sum_{r=0}^{s} W_{2r}(a)\right) z_n = \frac{1}{a} W_{2s+1}(a) z_n, && (3.26)\\
\sum_{r=1}^{s} \big(z_{n+2r-1} - z_{n-2r+1}\big) &= \left(\sum_{r=1}^{s} z_{n+2r-1}\right) - \left(\sum_{t=1}^{s} z_{n-2r+1}\right) = \\
&= \left(\sum_{r=1}^{s} W_{2r-1}(a)\right) z_n = \frac{1}{a}\big(W_{2s}(a) - 2\big). && (3.27)
\end{aligned}$$

Proof. First, we note that

$$\begin{aligned}
z_{n+2k+1} - z_{n-2k-1} &= a z_{n+2k} + z_{n+2k-1} - \big(z_{n-2k+1} - a z_{n-2k}\big)\\
&= a\big(z_{n+2k} + z_{n-2k}\big) + z_{n+2k-1} - z_{n-2k+1},\\
z_{n+2k} + z_{n-2k} &= a\big(z_{n+2k-1} - z_{n-2k+1}\big) + z_{n+2k-2} + z_{n-2k+2},
\end{aligned}$$

i.e., that

$$\xi_{k+1,n} = a\xi_{k,n} + \xi_{k-1,n} = W_1(a)\xi_{k,n} + W_0(a)\xi_{k-1,n}.$$

Now, suppose that for some $s \in \mathbb{N}_0$, $s \leqslant k$ we have

$$\xi_{k+1,n} = W_{k-s}(a)\xi_{s+1,n} + W_{k-s-1}(a)\xi_{s,n}.$$

Hence, we get

$$\begin{aligned}\xi_{k+1,n} &= W_{k-s}(a)(a\xi_{s,n} + \xi_{s-1,n}) + W_{k-s-1}(a)\xi_{s,n} \\ &= W_{k-s-1}(a)\xi_{s,n} + W_{k-s}(a)\xi_{s-1,n}.\end{aligned}$$

This leads to

$$\begin{aligned}\xi_{k+1,n} &= W_k(a)\xi_{1,n} + W_{k-1}(a)\xi_{0,n} \\ &= (aW_k(a) + W_{k-1}(a))z_n = W_{k+1}(a)z_n.\end{aligned}$$

□

Corollary 10 *We have*

$$z_{n+8} - 2z_{n+4} + 2z_n - 2z_{n-4} + z_{n-8} = a^2(a^2+4)W_4(a)z_n,$$

$$z_{2k} + (-1)^k z_0 = W_k(a)z_k, \tag{3.28}$$

$$az_{2k+1} + 2(-1)^{k+1}z_0 = W_{k+1}(a)z_{k+1} - W_k(a)z_k, \tag{3.29}$$

$$a\sum_{k=0}^{2s} z_{2k} = W_{2s+1}(a)z_{2s}, \tag{3.30}$$

$$a\sum_{k=0}^{s} z_{2k+1} = W_{s+1}(a)z_{s+1} + ((-1)^s - 1)z_0, \tag{3.31}$$

$$a\left(\sum_{k=s}^{2s-1} z_{2k} - \sum_{l=0}^{s-1} z_{2l}\right) = (W_{2s}(a) - 2)z_{2s-1}, \tag{3.32}$$

$$a\sum_{k=0}^{4s} z_k = W_{2s+1}(a)z_{2s} + W_{2s-1}(a)z_{2s-1}, \tag{3.33}$$

$$\begin{aligned}a\sum_{k=1}^{s} W_k(a)z_k &= a\sum_{k=1}^{s} z_{2k} + \frac{a}{2}((-1)^s - 1)z_0 \\ &= \begin{cases} W_{s+1}(a)z_s - az_0 & \text{for } s \in 2\mathbb{N}, \\ W_s(a)z_{s-1} - az_s - 2az_0 & \text{for } s \in 2\mathbb{N} - 1. \end{cases}\end{aligned} \tag{3.34}$$

4. Some Generalizations of Conjugate Sequences Concept

Let us assume that the sequences $\{x_n\}$ and $\{y_n\}$ are determined by the same recurrence equations (2.1). The following result is a response to an attempt to substitute conditions (2.2) by some other conditions connecting the elements of the sequences $\{x_n\}$ and $\{y_n\}$.

Lemma 11 *Let $A, B, C, D, \alpha, \beta \in \mathbb{C}$, $|\alpha| \neq |\beta|$. Let us assume that there exist $u, v \in \mathbb{C}$ so that for every $n \in \mathbb{N}$ the following identities are satisfied:*

$$x_n = A\alpha^n + B\beta^n \tag{4.35}$$

$$y_n = C\alpha^n + D\beta^n \tag{4.36}$$

$$x_n + x_{n+2} + x_{n+4} + x_{n+6} = u \cdot y_{n+3} \tag{4.37}$$

and

$$y_n + y_{n+2} + y_{n+4} + y_{n+6} = v \cdot x_{n+3}. \tag{4.38}$$

If $ABCD \neq 0$ then $-b = \alpha\beta = \pm 1$. Moreover, if $AC \neq 0$ (resp. $BD \neq 0$) then

$$u = \frac{A}{C}(\alpha + \alpha^{-1})[(\alpha + \alpha^{-1})^2 - 2] \qquad \text{and} \qquad v = \frac{C}{A}(\alpha + \alpha^{-1})[(\alpha + \alpha^{-1})^2 - 2]$$

(resp., $u = \frac{B}{D}(\beta + \beta^{-1})[(\beta + \beta^{-1})^2 - 2]$ and $v = \frac{D}{B}(\beta + \beta^{-1})[(\beta + \beta^{-1})^2 - 2]$).

Proof. The following four identities from (4.37) and (4.38) can be generated:

$$\begin{cases} A\frac{\alpha^8-1}{\alpha^3(\alpha^2-1)} = A\frac{\alpha^4-\alpha^{-4}}{\alpha-\alpha^{-1}} = uC, \\ B\frac{\beta^8-1}{\beta^3(\beta^2-1)} = B\frac{\beta^4-\beta^{-4}}{\beta-\beta^{-1}} = uD, \\ C\frac{\alpha^8-1}{\alpha^3(\alpha^2-1)} = C\frac{\alpha^4-\alpha^{-4}}{\alpha-\alpha^{-1}} = vA, \\ D\frac{\beta^8-1}{\beta^3(\beta^2-1)} = D\frac{\beta^4-\beta^{-4}}{\beta-\beta^{-1}} = uB. \end{cases} \tag{4.39}$$

Hence

$$A\left(\frac{\alpha^4 - \alpha^{-4}}{\alpha - \alpha^{-1}}\right)^2 = uvA \qquad \text{and} \qquad A\left(\frac{\beta^4 - \beta^{-4}}{\beta - \beta^{-1}}\right)^2 = uvB.$$

If $ABCD \neq 0$ then

$$\left(\frac{\alpha^4 - \alpha^{-4}}{\alpha - \alpha^{-1}}\right)^2 = \left(\frac{\beta^4 - \beta^{-4}}{\beta - \beta^{-1}}\right)^2,$$

i.e.,

$$\alpha^3 + \alpha + \alpha^{-1} + \alpha^{-3} = \pm(\beta^3 + \beta + \beta^{-1} + \beta^{-3})$$

or

$$\begin{aligned}(\alpha+\alpha^{-1})^3-2(\alpha+\alpha^{-1}) \pm \big((\beta+\beta^{-1})^3-2(\beta+\beta^{-1})\big) = \\ = \big((\alpha+\alpha^{-1}) \pm (\beta+\beta^{-1})\big)\big((\alpha+\alpha^{-1})^2 \mp (\beta+\beta^{-1})(\alpha+\alpha^{-1})+(\beta+\beta^{-1})^2-2\big),\end{aligned}$$

while

$$\triangle_{\alpha+\alpha^{-1}} = (\beta+\beta^{-1})^2-4(\beta+\beta^{-1})^2+8 = -3\beta^{-2}\left[\beta^4-\frac{2}{3}\beta^2+1\right] < 0.$$

Therefore

$$\alpha+\alpha^{-1} = \pm(\beta+\beta^{-1}),$$

i.e.,

$$\alpha^2 \pm (\beta+\beta^{-1})\alpha+1 = 0,$$

which implies for plus sign

$$(\alpha+\beta)(\alpha+\beta^{-1}) = 0, \quad \text{i.e.,} \quad \alpha\beta = -1$$

and for minus sign

$$(\alpha-\beta)(\alpha-\beta^{-1}) = 0, \quad \text{i.e.,} \quad \alpha\beta = 1$$

(since $|\alpha| \neq |\beta|$). □

Corollary 12 *If $ABCD \neq 0$ and $\alpha\beta = 1$ then*

$$\alpha+\alpha^{-1} = \alpha+\beta = \beta^{-1}+\beta$$

and

$$AD = BC.$$

So, sequences $\{x_n\}$ and $\{y_n\}$ are linearly dependent; or, more precisely, we have $x_n = \frac{A}{D}y_n$, $n \in \mathbb{N}$.

Corollary 13 *If $ABCD \neq 0$ and $\alpha\beta = -1$ then*

$$\alpha+\alpha^{-1} = \alpha-\beta = -(\beta^{-1}+\beta)$$

and

$$AD+BC = 0.$$

Moreover, we have

$$x_n+x_{n+2} = \frac{B}{D}(\beta-\alpha)y_{n+1} \qquad \text{and} \qquad y_n+y_{n+2} = \frac{D}{B}(\beta-\alpha)x_{n+1};$$

or

$$x_n^*+x_{n+2}^* = (\beta-\alpha)^2 y_{n+1} \qquad \text{and} \qquad y_n+y_{n+2} = x_{n+1}^*$$

where $x_n^* := \frac{D}{B}(\beta-\alpha)x_n$, $n \in \mathbb{N}$.

In the sequel if $B(\beta-\alpha) = D$ or $D(\beta-\alpha) = B$ then sequences $\{x_n\}$ and $\{y_n\}$ are conjugates in a Fibonacci-Lucas sense with parameter $a = \alpha+\beta$.

Proof. We have

$$\begin{aligned} x_n + x_{n+2} &= A(1+\alpha^2)\alpha^n + B(1+\beta^2)\beta^n \\ &= -\frac{BC}{D}\alpha(\alpha+\alpha^{-1})\alpha^n + B\beta(\beta+\beta^{-1})\beta^n \\ &= \frac{B}{D}(\beta-\alpha)[C\alpha^{n+1} + D\beta^{n+1}] = \frac{B}{D}(\beta-\alpha)y_{n+1}. \end{aligned}$$

□

5. Decompositions of Some Special Polynomials of Many Variables

In this section an attempt is made at decomposing some symmetric polynomials of 3, 4, and 5 variables into factors (such decompositions may easily be generalized to polynomials of 6 or even more variables). The decompositions are derived by means of modified Chebyshev polynomials $\Omega_n(x)$ (Vieta-Lucas polynomials).

Lemma 14 *The following identities hold*

$$\begin{aligned} &p_n^{(3)}(x,y,z) \coloneqq (x+y+z)^n - (x+y)^n - (x+z)^n - (y+z)^n + x^n + y^n + z^n \\ &= nxyz\Big[\frac{(x+y+z)^{n-1}}{xy+xz+yz} - (xy+xz+yz)^{(n-3)/2}\Omega_{n-1}\Big(\frac{x+y+z}{(xy+xz+yz)^{1/2}}\Big) \\ &\quad + F_{3,n}\big(xyz, x+y+z, xy, x+y, xz, x+z, yz, y+z\big)\Big] \end{aligned} \tag{5.40}$$

(in the following formulas, the compact form of the respective identities is given)

$$= \begin{cases} 0, & \text{for } n \leqslant 2, \\ 6xyz, & \text{for } n = 3, \\ 12xyz(x+y+z), & \text{for } n = 4, \\ 10xyz\big[2(x+y+z)^2 - xy - xz - yz\big], & \text{for } n = 5, \\ 30xyz(x+y+z)\big[(x+y+z)^2 - xy - xz - yz\big], & \text{for } n = 6, \\ 7xyz\big[6(x+y+z)^4 - 9(xy+xz+yz)(x+y+z)^2 & \\ \quad +2(xy+xz+yz)^2 - xyz(x+y+z)\big], & \text{for } n = 7, \\ 56xyz(x+y+z)\Big[\big((x+y+z)^2 - xy - xz - yz\big)^2 & \\ \quad -\frac{1}{2}xyz(x+y+z)\Big], & \text{for } n = 8, \\ 18xyz\big[4(x+y+z)^6 - 10(xy+xz+yz)(x+y+z)^4 & \\ \quad +8(xy+xz+yz)^2(x+y+z)^2 - (xy+xz+yz)^3 & \\ \quad -4xyz(x+y+z)^3 + xyz\big(xy(x+y) + xz(x+z) & \\ \quad +yz(y+z)\big) + \frac{10}{3}(xyz)^2\big], & \text{for } n = 9. \end{cases}$$

Moreover, we have

$$F_{3,n}\big(xyz, x+y+z, xy, x+y, xz, x+z, yz, y+z\big) = \\ = \begin{cases} 0, & \text{for} \quad n \leqslant 6, \\ -xyz(x+y+z), & \text{for} \quad n=7, \\ -\frac{7}{2}xyz(x+y+z)^2, & \text{for} \quad n=8, \\ -8xyz(x+y+z)^3+2xyz\big(xy(x+y) \\ \quad +xz(x+z)+yz(y+z)\big)+\frac{20}{3}(xyz)^2, & \text{for} \quad n=9, \\ -15xyz(x+y+z)^4+9xyz\big(xy(x+y)^2 \\ \quad +xz(x+z)^2+yz(y+z)^2\big)+48x^2y^2z^2(x+y+z), & \text{for} \quad n=10, \\ -25xyz(x+y+z)^5+25xyz\big(xy(x+y)^3 \\ \quad +xz(x+z)^3+yz(y+z)^3\big)-3xyz\big(x^3(y+z)^2 \\ \quad +y^3(x+z)^2+z^3(x+y)^2\big)+92(xyz)^2\times \\ \quad \times\big((x+y)^2+(x+z)^2+(y+z)^2\big) \\ \quad +117(xyz)^3(x+y+z), & \text{for} \quad n=11, \end{cases}$$

and

$$F_{3,12} = -\frac{77}{2}x^2y^2z^2(x+y+z)^6 \\ +55x^2y^2z^2\big(xy(x+y)^4+xz(x+z)^4+yz(y+z)^4\big) \\ -\frac{33}{2}x^2y^2z^2\big(x^2y^2(x+y)^2+x^2z^2(x+z)^2+y^2z^2(y+z)^2\big) \\ +484x^3y^3z^3(x+y+z)^3-319x^3y^3z^3(xy+xz+yz)(x+y+z)+\frac{6699}{2}x^4y^4z^4.$$

Remark 15 *We note that xyz divides $p_n^{(3)}(x,y,z)$ for every $n \in \mathbb{N}$.*

Lemma 16 *We have*

$$p_n^{(4)}(x,y,z,u) := (x+y+z+u)^n-(x+y+z)^n-(x+y+u)^n- \\ -(x+z+u)^n-(y+z+u)^n+(x+y)^n+(x+z)^n+(x+u)^n+ \\ (y+z)^n++(y+u)^n+(z+u)^n-x^n-y^n-z^n-u^n= \\ =n(n-1)xyzu\Bigg[\frac{(x+y+z+u)^{n-2}}{xy+xz+xu+yz+yu+zu}- \\ -(xy+xz+xu+yz+yu+zu)^{(n-4)/2}\Omega_{n-2}\left(\frac{x+y+z+u}{(xy+xz+xu+yz+yu+zu)^{1/2}}\right)+ \\ +F_{4,n}(xyzu, x+y+z+u, xyz, x+y+z, xyu, x+y+u, xzu, \\ x+z+u, yzu, y+z+u)\Bigg]= \tag{5.41}$$

$$= \begin{cases} 0, & \text{for} \quad n \leqslant 3, \\ 24xyzu, & \text{for} \quad n = 4, \\ 60xyzu(x+y+z+u), & \text{for} \quad n = 5, \\ 60xyzu\big[2(x+y+z+u)^2 - \\ \quad -(xy+xz+xu+yz+yu+zu)\big], & \text{for} \quad n = 6, \\ 210xyzu(x+y+z+u)\big[(x+y+z+u)^2 - \\ \quad -(xy+xz+xu+yz+yu+zu)\big], & \text{for} \quad n = 7, \\ 56xyzu\big[6(x+y+z+u)^4 - 9(xy+xz+xu+ \\ \quad +yz+yu+zu)(x+y+z+u)^2 + 2(xy+xz+ \\ \quad +xu+yz+yu+zu)^2 - xyz(x+y+z)- \\ \quad -xyu(x+y+u) - xuz(x+u+z)- \\ \quad -yuz(y+u+z) - 3xyzu\big] = \\ \quad = 28xyzu\big[5(x+y+z+u)^4 + 5(x^2+y^2+z^2+u^2)\times \\ \quad \times(x+y+z+u)^2 + 2(x^2+y^2+z^2+u^2)^2 - \\ \quad -30xyzu - 10xyz(x+y+z) - 10xyu(x+y+u)- \\ \quad -10xuz(x+u+z) - 10uyz(u+y+z)- \\ \quad -4(x^2y^2+x^2u^2+x^2z^2+y^2u^2+y^2z^2+u^2z^2)\big], & \text{for} \quad n = 8, \end{cases}$$

and for $n = 9$:

$$= 252xyzu(x+y+z+u)\big[2(x+y+z+u)^4 - 4(xy+xz+xu+yz+yu+zu)(x+y+z+u)^2 + \\ + 2(xy+xz+xu+yz+yu+zu)^2 - xyz(x+y+z)^2 - xuz(x+u+z)^2 - xuy(x+u+y)^2 - \\ - uyz(u+y+z)^2 - 42xyuz(x+y+u+z)\big],$$

where

$$F_{4,n}(xyzu, x+y+z+u, xyz, x+y+z, xyu, x+y+u, xzu, x+z+u, yzu, y+z+u) =$$

$$= \begin{cases} 0 & \text{for} \quad n \leqslant 7, \\ -xyz(x+y+z) - xyu(x+y+u)- \\ \quad -xuz(x+u+z) - yuz(y+u+z) - 3xyzu, & \text{for} \quad n = 8, \\ -\frac{7}{2}xyz(x+y+z)^2 - \frac{7}{2}xyu(x+y+u)^2 \\ \quad -\frac{7}{2}xuz(x+u+z)^2 - \frac{7}{2}yuz(y+u+z)^2 - \\ \quad -147xyuz(x+y+u+z) & \text{for} \quad n = 9. \end{cases}$$

Remark 17 *We note that $xyzu$ divides $p_n^{(4)}(x,y,z,u)$ for every $n \in \mathbb{N}$.*

Lemma 18 *We have the following decompositions:*

$$\begin{aligned}
p_n^{(5)}(x,y,z,u,v) &:= (x+y+z+u+v)^n-(x+y+z+u)^n-(x+y+z+v)^n-\\
&-(x+y+u+v)^n-(x+z+u+v)^n-(z+y+u+v)^n+(x+y+z)^n+(x+y+u)^n+\\
&+(x+y+v)^n+(x+z+u)^n+(x+z+v)^n+(x+u+v)^n+(y+z+u)^n+(y+z+v)^n+\\
&+(y+u+v)^n+(z+u+v)^n-(x+y)^n-(x+z)^n-(x+u)^n-(x+v)^n-(y+z)^n-\\
&-(y+u)^n-(y+v)^n-(z+u)^n-(z+v)^n-(u+v)^n+x^n+y^n+z^n+u^n+v^n=
\end{aligned}$$

$$=\begin{cases} 0 & \text{for } n\leqslant 4,\\ 5!xyzuv & \text{for } n=5,\\ \frac{6!}{2!}xyzuv(x+y+z+u+v) & \text{for } n=6,\\ \frac{7!}{2!3!}xyzuv\big[3(x+y+z+u+v)^2-\\ \quad -x^2-y^2-z^2-u^2-v^2\big]=\\ \quad =\frac{7!}{3!}xyzuv[(x+y+z+u+v)^2+xy+xz+\\ \qquad +xu+xv+yz+yu+yv+zu+zv+uv] & \text{for } n=7.\end{cases}$$

6. Applications

6.1 Identities for the Powers of Elements of Conjugate Sequences

We now present some identities obtained by applying identity (5.40) to the elements of sequences $\{x_n\}$ and $\{y_n\}$ satisfying (2.4)–(2.6).

a) For $x = x_{n+4}$, $y = -x_n$, $z = -ay_{n+2}$ and odd powers only:

$$x_{n+4}^k - x_n^k - a^k y_{n+2}^k = \begin{cases} 0 & \text{for } k=1,\\ 3ax_ny_{n+2}x_{n+4} & \text{for } k=3,\\ 5ax_ny_{n+2}x_{n+4}(x_nx_{n+4}+a^2y_{n+2}^2) & \text{for } k=5,\\ 7ax_ny_{n+2}x_{n+4}(x_nx_{n+4}+a^2y_{n+2}^2)^2 & \text{for } k=7,\\ 9ax_ny_{n+2}x_{n+4}\big[\frac{1}{3}(ax_ny_{n+2}x_{n+4})^2-\\ \quad -(x_nx_{n+4}+a^2y_{n+2}^2)^3\big] & \text{for } k=9;\end{cases}$$

b) for $x = x_{n+4}$, $y = -x_n$, $z = ay_{n+2}$:

$$(2a)^k y_{n+2}^k + x_{n+4}^k + (-1)^k x_n^k - (ay_{n+2} - x_n)^k - (2ay_{n+2} + x_n)^k =$$
$$= \begin{cases} 0 & \text{for } k \leqslant 2, \\ -6ax_n y_{n+2} x_{n+4} & \text{for } k = 3, \\ -24a^2 x_n y_{n+2}^2 x_{n+4} & \text{for } k = 4, \\ -10ax_n y_{n+2} x_{n+4}[7a^2 y_{n+2}^2 + x_n x_{n+4}] & \text{for } k = 5, \\ -60a^2 x_n y_{n+2}^2 x_{n+4}[3a^2 y_{n+2}^2 + x_n x_{n+4}] & \text{for } k = 6, \\ -112a^2 x_n y_{n+2}^2 x_{n+4}[(3a^2 y_{n+2}^2 + x_n x_{n+4})^2 + \\ \quad + a^2 x_n y_{n+2}^2 x_{n+4}] & \text{for } k = 8; \end{cases}$$

c) for $x = x_{n+4}$, $y = x_n$, $z = -ay_{n+2}$:

$$x_{n+4}^k - (a^2 + 2)^k x_{n+2}^k + 2^k x_n^k + (-a)^k y_{n+2}^k - (x_n - ay_{n+2})^k =$$
$$= \begin{cases} 0 & \text{for } k \leqslant 2, \\ -6ax_n y_{n+2} x_{n+4} & \text{for } k = 3, \\ -24ax_n^2 y_{n+2} x_{n+4} & \text{for } k = 4, \\ -10ax_n y_{n+2} x_{n+4}[9x_n^2 - ay_{n+2} x_{n+4}] & \text{for } k = 5, \\ -60ax_n^2 y_{n+2} x_{n+4}[5x_n^2 - ay_{n+2} x_{n+4}] & \text{for } k = 6, \\ -112ax_n^2 y_{n+2} x_{n+4}[(5x_n^2 - ay_{n+2} x_{n+4})^2 + \\ \quad + ax_n^2 y_{n+2} x_{n+4}] & \text{for } k = 8; \end{cases}$$

d) for $x = x_{n+4}$, $y = x_n$, $z = ay_{n+2}$:

$$(2^k - 1)(ay_{n+2} + x_n)^k - (2x_n + ay_{n+2})^k - (x_n + 2ay_{n+2})^k +$$
$$+ x_{n+4}^k + x_n^k + a^k y_{n+2}^k =$$
$$= \begin{cases} 0 & \text{for } k \leqslant 2, \\ 6ax_n y_{n+2} x_{n+4} & \text{for } k = 3, \\ 24ax_n y_{n+2} x_{n+4}^2 & \text{for } k = 4, \\ 10ax_n y_{n+2} x_{n+4}[7x_{n+4}^2 - ax_n y_{n+2}] & \text{for } k = 5, \\ 60ax_n y_{n+2} x_{n+4}^2[3x_{n+4}^2 - ax_n y_{n+2}] & \text{for } k = 6, \\ 112ax_n y_{n+2} x_{n+4}^2[(3x_{n+4}^2 - ax_n y_{n+2})^2 - \\ \quad - ax_n y_{n+2} x_{n+4}^2] & \text{for } k = 8; \end{cases}$$

e) for $x = y_{n+4}$, $y = -y_n$, $z = -a(a^2+4)x_{n+2}$ and odd powers only:

$$y_{n+4}^k - y_n^k - (a(a^2+4))^k x_{n+2}^k = \\ = \begin{cases} 0 & \text{for } k=1, \\ 3a(a^2+4)y_n x_{n+2} y_{n+4} & \text{for } k=3, \\ 5a(a^2+4)y_n x_{n+2} y_{n+4}(a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4}) & \text{for } k=5, \\ 7a(a^2+4)y_n x_{n+2} y_{n+4}(a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4})^2 & \text{for } k=7, \\ 9a(a^2+4)y_n x_{n+2} y_{n+4}[(a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4})^3 + \\ \quad +\frac{1}{3}(a(a^2+4)y_n x_{n+2} y_{n+4})^2] & \text{for } k=9; \end{cases}$$

f) for $x = y_{n+4}$, $y = -y_n$, $z = a(a^2+4)x_{n+2}$:

$$(2a(a^2+4))^k x_{n+2}^k + y_{n+4}^k + (-1)^k y_n^k - (a(a^2+4)x_{n+2} - y_n)^k - \\ - (2a(a^2+4)x_{n+2} + y_n)^k = \\ = \begin{cases} 0 & \text{for } k \leqslant 2, \\ -6a(a^2+4)y_n x_{n+2} y_{n+4} & \text{for } k=3, \\ -24a^2(a^2+4)^2 y_n x_{n+2}^2 y_{n+4} & \text{for } k=4, \\ -10a(a^2+4)y_n x_{n+2} y_{n+4} \times \\ \quad \times[7a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4}] & \text{for } k=5, \\ -60a^2(a^2+4)^2 y_n x_{n+2}^2 y_{n+4} \times \\ \quad \times[3a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4}] & \text{for } k=6, \\ -112a^2(a^2+4)^2 y_n x_{n+2}^2 y_{n+4} \times \\ \quad \times[(3a^2(a^2+4)^2 x_{n+2}^2 + y_n y_{n+4})^2 + \\ \quad +a^2(a^2+4)^2 y_n x_{n+2}^2 y_{n+4}] & \text{for } k=8; \end{cases}$$

g) for $x = y_{n+4}$, $y = y_n$, $z = -a(a^2+4)x_{n+2}$:

$$y_{n+4}^k - (a^2+2)^k y_{n+2}^k + 2^k y_n^k + \\ + (-a(a^2+4))^k x_{n+2}^k - (y_n - a(a^2+4)x_{n+2})^k = \\ = \begin{cases} 0 & \text{for } k \leqslant 2, \\ -6a(a^2+4)y_n x_{n+2} y_{n+4} & \text{for } k=3, \\ -24a(a^2+4)y_n^2 x_{n+2} y_{n+4} & \text{for } k=4, \\ -10a(a^2+4)y_n x_{n+2} y_{n+4} \times \\ \quad \times[7y_n^2 + a(a^2+4)x_{n+2} y_{n+4}] & \text{for } k=5, \\ -60a(a^2+4)y_n^2 x_{n+2} y_{n+4} \times \\ \quad \times[3y_n^2 + a(a^2+4)x_{n+2} y_{n+4}] & \text{for } k=6, \\ -112a(a^2+4)y_n^2 x_{n+2} y_{n+4} \times \\ \quad \times[(3y_n^2 + a(a^2+4)x_{n+2} y_{n+4})^2 + \\ \quad\quad +a(a^2+4)y_n^2 x_{n+2} y_{n+4}] & \text{for } k=7; \end{cases}$$

h) for $x = x_n$, $y = ax_{n+1}$, $z = ax_{n+3}$:

$$x_{n+4}^k + a^k x_{n+3}^k - [1 + (a^2+1)^k] x_{n+2}^k - a^k y_{n+2}^k + a^k x_{n+1}^k + x_n^k = \\ = \begin{cases} 0 & \text{for } k \leqslant 2, \\ 6a^2 x_n x_{n+1} x_{n+3} & \text{for } k = 3, \\ 12a^2 x_n x_{n+1} x_{n+3} x_{n+4} & \text{for } k = 4, \\ 10a^2 x_n x_{n+1} x_{n+3} \times \\ \quad \times [2x_{n+4}^2 - a x_n x_{n+1} - a x_{n+2} x_{n+3}] & \text{for } k = 5, \\ 30a^2 x_n x_{n+1} x_{n+3} x_{n+4} \times \\ \quad \times [a^2 x_{n+3}^2 + x_{n+2} x_{n+4} - a x_n x_{n+1}] & \text{for } k = 6; \end{cases}$$

i) for $x = y_n$, $y = ay_{n+1}$, $z = ay_{n+3}$:

$$y_{n+4}^k + a^k y_{n+3}^k - [(a^2+1)^k + 1] y_{n+2}^k + \\ + a^k y_{n+1}^k + y_n^k - a^k (a^2+4)^k x_{n+2}^k = \\ = \begin{cases} 0 & \text{for } k \leqslant 2, \\ 6a^2 y_n y_{n+1} y_{n+3} & \text{for } k = 3, \\ 12a^2 y_n y_{n+1} y_{n+3} y_{n+4}, & \text{for } k = 4, \\ 10a^2 y_n y_{n+1} y_{n+3} \times \\ \quad \times [2y_{n+4}^2 - a(y_n y_{n+1} + y_{n+2} y_{n+3})], & \text{for } k = 5, \\ 30a^2 y_n y_{n+1} y_{n+3} y_{n+4} \times \\ \quad \times [y_{n+4}^2 - a(y_n y_{n+1} + y_{n+2} y_{n+3})], & \text{for } k = 6, \\ 56a^2 y_n y_{n+1} y_{n+3} y_{n+4} \times \\ \quad \times [(y_{n+4}^2 - a(y_n y_{n+1} + y_{n+2} y_{n+3}))^2 - \\ \quad\quad - \frac{1}{2} a^2 y_n y_{n+1} y_{n+3} y_{n+4}] & \text{for } k = 8; \end{cases}$$

j) for $x = x_{n-1}$, $y = x_{n+1}$, $z = y_{n+2}$:

$$y_{n+2}^k + [(4+a^2)^k - (3+a^2)^k + 1] x_{n+1}^k + x_{n-1}^k - y_n^k - (x_{n+1} + y_{n+2})^k = \\ = \begin{cases} 0 & \text{for } k \leqslant 2, \\ 6x_{n-1} x_{n+1} y_{n+2} & \text{for } k = 3, \\ 12(2 + 2b + a^2) x_{n-1} x_{n+1}^2 y_{n+2} & \text{for } k = 4, \\ 10 x_{n-1} x_{n+1} y_{n+2} \times \\ \quad \times [(2(4+a^2)^2 - (3+a^2)^2) x_{n+1}^2 - x_{n-1} y_{n+2}] & \text{for } k = 5, \\ 30(4+a^2) x_{n-1} x_{n+1}^2 y_{n+2} \times \\ \quad \times [(2a^2+7) x_{n+1}^2 - x_{n-1} y_{n+2}] & \text{for } k = 6, \\ 56(a^2+4) x_{n-1} x_{n+1}^2 y_{n+2} \times \\ \quad \times [((2a^2+7) x_{n+1}^2 - x_{n-1} y_{n+2})^2 - \\ \quad\quad - \frac{1}{2}(4+a^2) x_{n-1} x_{n+1}^2 y_{n+2}] & \text{for } k = 8; \end{cases}$$

k) for $x = bx_n$, $y = y_{n-1}$, $z = y_{n+1}$, $b \in \mathbb{C}$:

(then $bx_n + y_{n+1} = (b+1)x_n + x_{n+2}$ and $bx_n + y_{n-1} = (b+1)x_n + x_{n-2}$) we obtain

$$\begin{aligned}
&[(4+b+a^2)^k - (4+a^2)^k + b^k]x_n^k + y_{n-1}^k + y_{n+1}^k - \\
&\quad - (bx_n + y_{n+1})^k - (bx_n + y_{n-1})^k = \\
&\quad = \begin{cases}
0 & \text{for } k \leqslant 2,\\
6bx_ny_{n-1}y_{n+1} & \text{for } k = 3,\\
12b(4+b+a^2)x_n^2y_{n-1}y_{n+1} & \text{for } k = 4,\\
10bx_ny_{n-1}y_{n+1}[(2(4+b+a^2)^2 - \\
\quad -b(4+a^2))x_n^2 - y_{n-1}y_{n+1}] & \text{for } k = 5,\\
30b(4+b+a^2)x_n^2y_{n-1}y_{n+1}\times \\
\quad \times[((4+b+a^2)^2 - b(4+a^2))x_n^2 - y_{n-1}y_{n+1}] & \text{for } k = 6,\\
56b(4+b+a^2)x_n^2y_{n-1}y_{n+1}\times \\
\quad \times[(\{(4+b+a^2)^2 - b(4+a^2)\}x_n^2 - \\
\quad -y_{n-1}y_{n+1})^2 - \frac{1}{2}b(4+b+a^2)x_n^2y_{n-1}y_{n+1}] & \text{for } k = 8;
\end{cases}
\end{aligned}$$

l) for $x = w_n$, $y = bw_{n+2}$, $z = w_{n+4}$, where $b \in \mathbb{C}$ and $w \in \{x, y\}$:

$$\begin{aligned}
&w_{n+4}^k + [(a^2+2+b)^k - (a^2+2)^k + b^k]w_{n+2}^k + w_n^k - \\
&\quad - (bw_{n+2} + w_n)^k - ((a^2+2+b)w_{n+2} - w_n)^k = \\
&= \begin{cases}
0, & \text{for } k \leqslant 2,\\
6bw_nw_{n+2}w_{n+4}, & \text{for } k = 3,\\
12(a^2+2+b)bw_nw_{n+2}^2w_{n+4}, & \text{for } k = 4,\\
10bw_nw_{n+2}w_{n+4}[(2(a^2+2+b)^2 - \\
\quad -b(a^2+2))w_{n+2}^2 - w_nw_{n+4}], & \text{for } k = 5,\\
30b(a^2+2+b)w_nw_{n+2}^2w_{n+4}\times \\
\quad \times[((a^2+2+b)^2 - b(a^2+2))w_{n+2}^2 - w_nw_{n+4}], & \text{for } k = 6,\\
56b(a^2+2+b)w_nw_{n+2}^2w_{n+4}\times \\
\quad \times[(((a^2+2+b)^2 - b(a^2+2))w_{n+2}^2 - w_nw_{n+4})^2 - \\
\quad -\frac{1}{2}b(a^2+2+b)w_nw_{n+2}^2w_{n+4}] & \text{for } k = 8;
\end{cases}
\end{aligned}$$

m) for $x=(a^2+3)x_{n+5}$, $y=-y_{n+2}$, $z=-x_{n-1}$:

$$\begin{aligned}&(a^2+3)^k[x_{n+5}^k+a^ky_{n+3}^k-(-1)^kx_{n+1}^k]+(-y_{n+2})^k+(-x_{n-1})^k-\\&-((a^2+3)x_{n+5}-y_{n+2})^k-((a^2+3)x_{n+5}-x_{n-1})^k=\\&=\begin{cases}0, & \text{for } k\leqslant 2,\\ 6(a^2+3)x_{n-1}y_{n+2}x_{n+5}, & \text{for } k=3,\\ 12a(a^2+3)^2x_{n-1}y_{n+2}y_{n+3}x_{n+5}, & \text{for } k=4,\\ 10(a^2+3)x_{n-1}y_{n+2}x_{n+5}\times\\ \quad\times[2a^2(a^2+3)^2y_{n+3}^2-x_{n-1}y_{n+2}+\\ \quad\quad+(a^2+3)^2x_{n+1}x_{n+5}], & \text{for } k=5,\\ 30a(a^2+3)^2x_{n-1}y_{n+2}y_{n+3}x_{n+5}\times\\ \quad\times[a^2(a^2+3)^2y_{n+3}^2-x_{n-1}y_{n+2}+\\ \quad\quad+(a^2+3)^2x_{n+1}x_{n+5}], & \text{for } k=6,\\ 56a(a^2+3)^2x_{n-1}y_{n+2}y_{n+3}x_{n+5}\times\\ \quad\times[(a^2(a^2+3)^2y_{n+3}^2-x_{n-1}y_{n+2}+\\ \quad\quad+(a^2+3)^2x_{n+1}x_{n+5})^2-\\ \quad\quad-\frac{1}{2}a(a^2+3)^2x_{n-1}y_{n+2}y_{n+3}x_{n+5}] & \text{for } k=8.\end{cases}\end{aligned}$$

Below, the identities generated by (5.41) and the elements of sequences $\{x_n\}$ and $\{y_n\}$ satisfying relations (2.4)–(2.6) will be given:

a) for $x=x_n$, $\quad y=x_{n+2}$, $\quad z=x_{n+2}$, $\quad u=x_{n+4}$

$$\begin{aligned}&\big[(a^2+2b+2)^k-2(a^2+2b+1)^k+(a^2+2b)^k+2^k\big]x_{n+2}^k+\\&+2y_{n+1}^k+2y_{n+3}^k-(y_{n+1}+x_{n+2})^k-(y_{n+3}+x_{n+2})^k=\\&=\begin{cases}0 & \text{for } k\leqslant 3,\\ 24x_nx_{n+2}^2x_{n+4} & \text{for } k=4,\\ 60(a^2+2b+2)x_nx_{n+2}^3x_{n+4} & \text{for } k=5,\\ 60x_nx_{n+2}^2x_{n+4}\big[2\big((a^2+2b+2)^2-\\ \quad-(a^2+2b+\frac{1}{2})\big)x_{n+2}^2-x_nx_{n+4}\big] & \text{for } k=6,\\ 210(a^2+2b+2)x_nx_{n+2}^3x_{n+4}\big[\big((a^2+2b+2)^2-\\ \quad-2(a^2+2b+\frac{1}{2})\big)x_{n+2}^2-x_nx_{n+4}\big] & \text{for } k=7;\end{cases}\end{aligned}$$

b) for $x = w_n, \quad y = w_{n+2}, \quad z = w_{n+4}, \quad u = w_{n+6}, w \in \{x, y\}$ respectively:

$$\begin{aligned}
&(a^2+4)^k x_{n+5}^k + (a^2+4)^k((a^2+2)^k+1)x_{n+3}^k + (a^2+4)^k x_{n+1}^k - y_{n+6}^k - \\
&- y_n^k + ((a^2+2)^k - (a^2+3)^k - 1)(y_{n+4}^k + y_{n+2}^k) + (y_n + y_{n+6})^k - \\
&- [(a^2+2)(a^2+4)x_{n+3} - y_{n+2}]^k - [(a^2+2)(a^2+4)x_{n+3} - y_{n+4}]^k = \\
&= \begin{cases} 0 & \text{for } k \leqslant 3, \\ 24y_n y_{n+2} y_{n+4} y_{n+6} & \text{for } k = 4, \\ 60(a^2+2)(a^2+4)y_n y_{n+2} x_{n+3} y_{n+4} y_{n+6} & \text{for } k = 5, \\ 60y_n y_{n+2} y_{n+4} y_{n+6}[2(a^2+2)^2(a^2+4)^2 x_{n+3}^2 - \\ \quad -(a^2+3)y_{n+2}y_{n+6} - (a^2+4)x_{n+1}y_{n+4} - y_n y_{n+2}] & \text{for } k = 6, \\ 210(a^2+2)(a^2+4)y_n y_{n+2} x_{n+3} y_{n+4} y_{n+6} \times \\ \quad \times[(a^2+2)^2(a^2+4)^2 x_{n+3}^3 - (a^2+3)y_{n+2}y_{n+6} \\ \quad -(a^2+4)x_{n+1}y_{n+4} - y_n y_{n+2}] & \text{for } k = 7, \end{cases}
\end{aligned}$$

$$\begin{aligned}
&y_{n+5}^k + ((a^2+2)^k+1)y_{n+3}^k + y_{n+1}^k - x_{n+6}^k - x_n^k + \\
&+ ((a^2+2)^k - (a^2+3)^k - 1)(x_{n+4}^k + x_{n+2}^k) + (x_n + x_{n+6})^k - \\
&- ((a^2+2)y_{n+3} - x_{n+4})^k - ((a^2+2)y_{n+3} - x_{n+2})^k = \\
&= \begin{cases} 0 & \text{for } k \leqslant 3, \\ 24x_n x_{n+2} x_{n+4} x_{n+6} & \text{for } k = 4, \\ 60(a^2+2)x_n x_{n+2} x_{n+4} x_{n+6} y_{n+3} & \text{for } k = 5, \\ 60x_n x_{n+2} x_{n+4} x_{n+6}[2(a^2+2)^2 y_{n+3}^2 \\ \quad -(a^2+3)x_{n+2}x_{n+6} - y_{n+1}x_{n+4} - x_n x_{n+2}] & \text{for } k = 6, \\ 210(a^2+2)x_n x_{n+2} x_{n+4} x_{n+6} y_{n+3} \times \\ \quad \times[(a^2+2)^2 y_{n+3}^2 - (a^2+3)x_{n+2}x_{n+6} \\ \quad -y_{n+1}x_{n+4} - x_n x_{n+2}] & \text{for } k = 7. \end{cases}
\end{aligned}$$

6.2 Identities for Powers of Fibonacci and Lucas Numbers

Applications of identity (5.40) to generate some identities for Fibonacci and Lucas numbers will now be presented.

a) For $x = F_n$, $y = F_{n+1}$, $z = F_{n+2}$:

$$2^k F_{n+2}^k - L_{n+1}^k - F_{n+3}^k + F_n^k + F_{n+1}^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 6F_nF_{n+1}F_{n+2}, & \text{for } k = 3, \\ 24F_nF_{n+1}F_{n+2}^2, & \text{for } k = 4, \\ 10F_nF_{n+1}F_{n+2}[7F_{n+2}^2 - F_nF_{n+1}], & \text{for } k = 5, \\ 60F_nF_{n+1}F_{n+2}^2[3F_{n+2}^2 - F_nF_{n+1}], & \text{for } k = 6, \\ 14F_nF_{n+1}F_{n+2}[(6F_{n+2}^2 - F_nF_{n+1})^2 - \\ \quad -5F_{n+2}^2(F_{n+2}^2 + F_nF_{n+1})], & \text{for } k = 7, \\ 112F_nF_{n+1}F_{n+2}^2[(3F_{n+2}^2 - F_nF_{n+1})^2 - \\ \quad -F_nF_{n+1}F_{n+2}^2], & \text{for } k = 8; \end{cases}$$

b) for $x = F_n$, $y = F_{n+1}$, $z = F_{n+3}$:

$$F_{n+4}^k - (2^k + 1)F_{n+2}^k - L_{n+2}^k + F_n^k + F_{n+1}^k + F_{n+3}^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 6F_nF_{n+1}F_{n+3}, & \text{for } k = 3, \\ 12F_nF_{n+1}F_{n+3}F_{n+4}, & \text{for } k = 4, \\ 10F_nF_{n+1}F_{n+3}\times \\ \quad \times[2F_{n+4}^2 - F_{n+2}F_{n+3} - F_nF_{n+1}], & \text{for } k = 5, \\ 30F_nF_{n+1}F_{n+3}F_{n+4}\times \\ \quad \times[F_{n+4}^2 - F_{n+2}F_{n+3} - F_nF_{n+1}], & \text{for } k = 6, \\ 56F_nF_{n+1}F_{n+3}F_{n+4}\times \\ \quad \times[(F_{n+4}^2 - F_nF_{n+1} - F_{n+2}F_{n+3})^2 - \\ \quad -\frac{1}{2}F_nF_{n+1}F_{n+3}F_{n+4}], & \text{for } k = 8; \end{cases}$$

c) for $x = F_n$, $y = F_{n+1}$, $z = F_{n+4}$:

$$L_{n+3}^k + F_{n+4}^k - 2^k F_{n+3}^k - (1 + 3^k)F_{n+2}^k + F_{n+1}^k + F_n^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 6F_nF_{n+1}F_{n+4}, & \text{for } k = 3, \\ 12F_nF_{n+1}L_{n+3}F_{n+4}, & \text{for } k = 4, \\ 10F_nF_{n+1}F_{n+4}[2L_{n+3}^2 - F_nF_{n+1} - F_{n+2}F_{n+4}] \\ \quad = 2F_nF_{n+1}F_{n+4}[9L_{2n+6} - L_{2n+1} - 16(-1)^n], & \text{for } k = 5, \\ 30F_nF_{n+1}L_{n+3}F_{n+4}[L_{n+3}^2 - F_nF_{n+1} - F_{n+2}F_{n+4}] \\ \quad = 6F_nF_{n+1}L_{n+3}F_{n+4}[4L_{2n+6} - L_{2n+1} - 6(-1)^n], & \text{for } k = 6, \\ 56F_nF_{n+1}L_{n+3}F_{n+4}[(L_{n+3}^2 - F_nF_{n+1} - F_{n+2}F_{n+4})^2 \\ \quad -\frac{1}{2}F_nF_{n+1}L_{n+3}F_{n+4}] & \text{for } k = 8; \end{cases}$$

d) for $x = F_n$, $y = F_{n+2}$, $z = F_{n+4}$:

$$F_{n+4}^k - L_{n+3}^k + (4^k - 3^k + 1)F_{n+2}^k - L_{n+1}^k + F_n^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 6F_nF_{n+2}F_{n+4}, & \text{for } k = 3, \\ 48F_nF_{n+2}^2F_{n+4}, & \text{for } k = 4, \\ 10F_nF_{n+2}F_{n+4}[29F_{n+2}^2 - F_nF_{n+4}] & \\ \quad = 2F_nF_{n+2}F_{n+4}[28L_{2n+4} - 51(-1)^n] & \\ \quad = 10F_nF_{n+2}F_{n+4}[28F_{n+2}^2 + (-1)^n], & \text{for } k = 5, \\ 120F_nF_{n+2}^2F_{n+4}[13F_{n+2}^2 - F_nF_{n+4}] & \\ \quad = 24F_nF_{n+2}^2F_{n+4}[12L_{2n+4} - 19(-1)^n] & \\ \quad = 120F_nF_{n+2}^2F_{n+4}[12F_{n+2}^2 + (-1)^n], & \text{for } k = 6, \\ 224F_nF_{n+2}^2F_{n+4}[(13F_{n+2}^2 - F_nF_{n+4})^2 - & \\ \quad -2F_nF_{n+2}^2F_{n+4}] & \text{for } k = 8; \end{cases}$$

e) for $x = 2F_n$, $y = F_{n+2}$, $z = F_{n+5}$:

$$F_{n+5}^k - 2^kF_{n+4}^k + (6^k - 5^k + 1)F_{n+2}^k - (F_{n+1} + 3F_n)^k + 2^kF_n^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 12F_nF_{n+2}F_{n+5}, & \text{for } k = 3, \\ 144F_nF_{n+2}^2F_{n+5}, & \text{for } k = 4, \\ 20F_nF_{n+2}F_{n+5}[72F_{n+2}^2 - 4F_nF_{n+4} - F_{n+2}F_{n+5}] & \\ \quad = 4F_nF_{n+2}F_{n+5}[71L_{2n+4} - 10F_{2n+5} - 112(-1)^n], & \text{for } k = 5, \\ 360F_nF_{n+2}^2F_{n+5}[36F_{n+2}^2 - 4F_nF_{n+4} - F_{n+2}F_{n+5}] & \\ \quad = 360F_nF_{n+2}^2F_{n+5}[7L_{2n+4} - 2F_{2n+5} - 8(-1)^n], & \text{for } k = 6; \end{cases}$$

f) for $x = F_n$, $y = F_{n+3}$, $z = F_{n+4}$:

$$(F_n + F_{n+5})^k - F_{n+5}^k + F_{n+4}^k + F_{n+3}^k - (2^k + 3^k)F_{n+2}^k + F_n^k =$$
$$= \begin{cases} 0 & \text{for } k = 1, 2, \\ 6F_nF_{n+3}F_{n+4}, & \text{for } k = 3, \\ 12F_nF_{n+3}F_{n+4}(F_n + F_{n+5}), & \text{for } k = 4, \\ 10F_nF_{n+3}F_{n+4}[2(F_n + F_{n+5})^2 - F_{2n+6} + 2(-1)^n], & \text{for } k = 5, \\ 30F_nF_{n+3}F_{n+4}(F_n + F_{n+5})\times & \\ \quad \times[(F_n + F_{n+5})^2 - F_{2n+6} + 2(-1)^n], & \text{for } k = 6; \end{cases}$$

g) for $x = 2F_n$, $y = F_{n+3}$, $z = F_{n+4}$:

$$\begin{aligned} &F_{n+4}^k + F_{n+3}^k + (5F_{n+2})^k + (2F_n)^k - F_{n+5}^k - \\ &\quad - (F_n + 2F_{n+2})^k - (F_n + 3F_{n+2})^k = \\ &\quad = \begin{cases} 0 & \text{for } k = 1, 2, \\ 12F_nF_{n+3}F_{n+4}, & \text{for } k = 3, \\ 120F_nF_{n+2}F_{n+3}F_{n+4}, & \text{for } k = 4, \\ 20F_nF_{n+3}F_{n+4}\times \\ \quad \times[F_{n+2}^2 - 2F_nF_{n+5} - F_{n+3}F_{n+4}] \\ \ = 4F_nF_{n+3}F_{n+4}\times \\ \quad \times[45L_{2n+4} - 4L_{2n+3} - 79(-1)^n], & \text{for } k = 5, \\ 300F_nF_{n+2}F_{n+3}F_{n+4}\times \\ \quad \times[25F_{n+2}^2 - 2F_nF_{n+5} - F_{n+3}F_{n+4}] \\ \ = 60F_nF_{n+2}F_{n+3}F_{n+4}\times \\ \quad \times[20L_{2n+4} - 4L_{2n+3} - 29(-1)^n], & \text{for } k = 6; \end{cases} \end{aligned}$$

h) for $x = L_n$, $y = L_{n+1}$, $z = L_{n+2}$:

$$\begin{aligned} &2^kL_{n+2}^k - 5^k \cdot F_{n+1}^k - L_{n+3}^k + L_n^k + L_{n+1}^k = \\ &\quad = \begin{cases} 0 & \text{for } k = 1, 2, \\ 6L_nL_{n+1}L_{n+2}, & \text{for } k = 3, \\ 24L_nL_{n+1}L_{n+2}^2, & \text{for } k = 4, \\ 10L_nL_{n+1}L_{n+2}[7L_{n+2}^2 - L_nL_{n+1}], & \text{for } k = 5, \\ 60L_nL_{n+1}L_{n+2}^2[3L_{n+2}^2 - L_nL_{n+1}], & \text{for } k = 6; \end{cases} \end{aligned}$$

i) for $x = L_n$, $y = L_{n+1}$, $z = L_{n+3}$:

$$\begin{aligned} &L_{n+4}^k - 5^k \cdot F_{n+2}^k - (2^k + 1)L_{n+2}^k + L_n^k + L_{n+1}^k + L_{n+3}^k = \\ &\quad = \begin{cases} 0 & \text{for } k = 1, 2, \\ 6L_nL_{n+1}L_{n+3}, & \text{for } k = 3, \\ 12L_nL_{n+1}L_{n+3}L_{n+4}, & \text{for } k = 4, \\ 10L_nL_{n+1}L_{n+3}\times \\ \quad \times[2L_{n+4}^2 - L_{n+2}L_{n+3} - L_nL_{n+1}], & \text{for } k = 5, \\ 30L_nL_{n+1}L_{n+3}L_{n+4}\times \\ \quad \times[L_{n+4}^2 - L_{n+2}L_{n+3} - L_nL_{n+1}], & \text{for } k = 6. \end{cases} \end{aligned}$$

In the next step, identity (5.41) shall be applied to generate some selected identities for Fibonacci and Lucas numbers.

a) For $x = F_n$, $y = F_{n+1}$, $z = F_{n+3}$, $u = F_{n+5}$ we obtain the following identities:

$$\begin{aligned} &F_{n+6}^k - F_{n+5}^k - (2^k+1)F_{n+4}^k - (4^k - 3^k + 1)F_{n+3}^k + (2^k+1)F_{n+2}^k - \\ &\quad - F_{n+1}^k - F_n^k + L_{n+2}^k + L_{n+4}^k + (F_n + F_{n+5})^k - (F_n + L_{n+4})^k = \\ &= \begin{cases} 0 & \text{for } k = 1,2,3, \\ 24F_nF_{n+1}F_{n+3}F_{n+5}, & \text{for } k = 4, \\ 60F_nF_{n+1}F_{n+3}F_{n+5}F_{n+6}, & \text{for } k = 5, \\ 60F_nF_{n+1}F_{n+3}F_{n+5}(2F_{n+6}^2 - F_{n+5}F_{n+4} - \\ \quad -F_{n+3}F_{n+2} - F_{n+1}F_n), & \text{for } k = 6, \\ 210F_nF_{n+1}F_{n+3}F_{n+5}F_{n+6}(F_{n+6}^2 - \\ \quad -F_{n+5}F_{n+4} - F_{n+3}F_{n+2} - F_{n+1}F_n), & \text{for } k = 7, \end{cases} \end{aligned} \tag{6.42}$$

where the following identity was utilized: $F_{n+1} + F_{n+5} = 3F_{n+3}$ and $L_{n+1} + L_{n+5} = 3L_{n+3}$;

b) for $x = L_n$, $y = L_{n+1}$, $z = L_{n+3}$, $u = L_{n+5}$:

$$\begin{aligned} &L_{n+6}^k - L_{n+5}^k - (2^k+1)L_{n+4}^k - (4^k - 3^k + 1)L_{n+3}^k + (2^k+1)L_{n+2}^k - \\ &\quad - L_{n+1}^k - L_n^k + 5^kF_{n+2}^k + 5^kF_{n+4}^k + (L_n + L_{n+5})^k - (L_n + 5F_{n+4})^k = \\ &= \begin{cases} 0 & \text{for } k = 1,2,3, \\ 24L_nL_{n+1}L_{n+3}L_{n+5} & \text{for } k = 4, \\ 60L_nL_{n+1}L_{n+3}L_{n+5}L_{n+6} & \text{for } k = 5, \\ 60L_nL_{n+1}L_{n+3}L_{n+5}[2L_{n+6}^2 - L_{n+5}L_{n+4} - \\ \quad -L_{n+3}L_{n+2} - L_{n+1}L_n], & \text{for } k = 6; \end{cases} \end{aligned} \tag{6.43}$$

c) for $x = F_n$, $y = F_{n+2}$, $z = F_{n+4}$, $u = F_{n+6}$:

$$\begin{aligned} &L_{n+5}^k + (3^k+1)L_{n+3}^k + L_{n+1}^k - F_{n+6}^k - F_n^k + (3^k - 4^k - 1)(F_{n+4}^k + F_{n+2}^k) + \\ &\quad + (F_n + F_{n+6})^k - (2F_{n+4} + 3F_{n+2})^k - (3F_{n+4} + 2F_{n+2})^k = \\ &= \begin{cases} 0 & \text{for } k = 1,2,3, \\ 24F_nF_{n+2}F_{n+4}F_{n+6} & \text{for } k = 4, \\ 180F_nF_{n+2}F_{n+4}F_{n+6}L_{n+3} & \text{for } k = 5, \\ 60F_nF_{n+2}F_{n+4}F_{n+6}[18L_{n+3}^2 - \\ \quad -4F_{n+2}F_{n+6} - L_{n+1}F_{n+4} - F_nF_{n+2}] & \text{for } k = 6, \\ 630F_nF_{n+2}F_{n+4}F_{n+6}L_{n+3}[9L_{n+3}^2 - \\ \quad -4F_{n+2}F_{n+6} - L_{n+1}F_{n+4} - F_nF_{n+2}] & \text{for } k = 7; \end{cases} \end{aligned} \tag{6.44}$$

d) for $x = L_n$, $y = L_{n+2}$, $z = L_{n+4}$, $u = L_{n+6}$:

$$\begin{aligned}(5F_{n+5})^k + (15^k + 5^k)F_{n+3}^k + (5F_{n+1})^k - L_{n+6}^k - (4^k - 3^k + 1)L_{n+4}^k - \\ - (4^k - 3^k + 1)L_{n+2}^k - L_n^k + (6L_{n+2} + 2L_{n+1})^k - \\ - (4L_{n+2} + L_{n+5})^k - (4L_{n+4} - L_{n+1})^k = \\ = \begin{cases} 0 & \text{for } k = 1, 2, 3, \\ 24L_nL_{n+2}L_{n+4}L_{n+6} & \text{for } k = 4, \\ 900L_nL_{n+2}L_{n+4}L_{n+6}F_{n+3} & \text{for } k = 5, \\ 60F_nF_{n+2}F_{n+4}F_{n+6}[18L_{n+3}^2 - \\ \quad -4F_{n+2}F_{n+6} - L_{n+1}F_{n+4} - F_nF_{n+2}] & \text{for } k = 6, \\ 630F_nF_{n+2}F_{n+4}F_{n+6}L_{n+3}[9L_{n+3}^2 - \\ \quad -4F_{n+2}F_{n+6} - L_{n+1}F_{n+4} - F_nF_{n+2}] & \text{for } k = 7. \end{cases}\end{aligned} \tag{6.45}$$

References

[1] M. Benoumhani, *A Sequence of Binomial Coefficients Related To Lucas and Fibonacci Numbers*, J. Int. Sequences **6** (2003), 1–10.

[2] M. Catalani, *Identities for Fibonacci and Lucas Polynomials Derived from a Book of Gould*, arXiv:math.CO/0407105 (2004), 1–7.

[3] A. Constandache, A. Das, F. Toppan, *Lucas Polynomials and a Standard Lax Representation for he Polytropic Gas Dynamics*, Letters in Math. Phys. **60** (2002), 197–209.

[4] P. Filipponi, A. F. Horadam, *First Derivative Sequences of Extended Fibonacci and Lucas Polynomials*, in: G. E. Bergum et al. (eds.), Applications of Fibonacci Numbers **7** (1998), 115–128.

[5] R. Grzymkowski, R. Wituła, *Calculus Methods in Algebra, Part One*, WPKJS, Gliwice 2000 (in Polish).

[6] A. F. Horadam, *Vieta Polynomials*, Fibonacci Quarterly **40** (2002), 223–232.

[7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York 2001.

[8] R. S. Melham, *Sums of Certain Products of Fibonacci and Lucas Numbers – Part I*, Fibonacci Quaeterly **37** (1999), 248–251.

[9] R. S. Melham, *Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers*, Fibonacci Quarterly **37** (1999), 315–319.

[10] R. S. Melham, *Sums of Certain Products of Fibonacci and Lucas Numbers – Part II*, Fibonacci Quarterly **38** (2000), 3–7.

[11] R. S. Melham, *Alternating Sums of Fourth Powers of Fibonacci and Lucas Numbers*, Fibonacci Quarterly **38** (2000), 254–259.

[12] P. S. Modenov, *Special Problems of Elementary Mathematics*, Sovietskaja Nauka, Moscow 1957 (in Russian).

[13] S. Paszkowski, *Numerical Applications of Chebyshev Polynomials and Series*, PWN, Warsaw 1975 (in Polish).

[14] T. Rivlin, *Chebyshef Polynomials from Approximation Theory to Algebra and Number Theory*, Wiley, New York 1990.

[15] N. Robbins, *Vieta's Triangular Array and a Related Family of Polynomials*, Int. J. Math. & Math. Sci. **14** (1991), 239–244.

[16] R. Wituła, D. Słota, *On Modified Chebyshev Polynomials*, J. Math. Anal. Appl. **324** (2006), 321–343.

[17] R. Wituła, D. Słota, *Cauchy, Ferrers-Jackson and Chebyshev Polynomials and identities for the powers of elements of some conjugate recurrence sequences*, Central Eur. J. Math. **4** (2006), 531–546.