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# REPRESENTATIONS OF SPLIT GRAPHS, THEIR COMPLEMENTS, STARS, AND HYPERCUBES

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## Abstract

A graph $G$ has a representation modulo $n$ if there exists an injective map $f : V(G) \to \{0, 1, \ldots, n\}$ such that vertices $u$ and $v$ are adjacent if and only if $|f(u) - f(v)|$ is relatively prime to $n$. The representation number $rep(G)$ is the smallest $n$ such that $G$ has a representation modulo $n$. We present new results involving representation numbers for stars, split graphs, complements of split graphs, and hypercubes.

## 1. Introduction

Let $G$ be a finite graph with vertices $\{v_1, \ldots, v_r\}$. A *representation of $G$ modulo $n$* is an assignment of distinct labels to the vertices such that the label $a_i$ assigned to vertex $v_i$ is in $\{0, 1, \ldots, n-1\}$ and such that $|a_i - a_j|$ and $n$ are relatively prime if and only if $(v_i, v_j) \in E(G)$. Erdős and Evans showed that every finite graph can be represented modulo some positive integer [1]. The representation number of a graph $G$, rep $(G)$, is the smallest $n$ such that $G$ has a representation modulo $n$. An example of a representation modulo 15 is given in Figure 1. In fact for this graph, rep $(G) = 15$. We note as part of this representation, the graph is completely described by the set of vertex labels $\{0, 1, 3, 5\}$ and the number 15.

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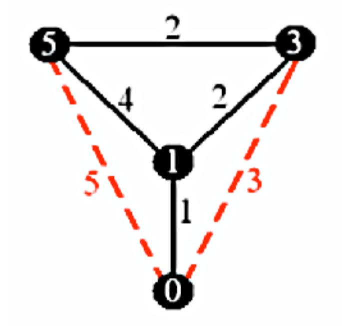


**Figure 1. Representation of a graph modulo 15.**

Modular representations have been studied extensively [1]-[12]. As part of an existence proof, Erdős and Evans established a general upper bound for the representation number of a graph [1]. Narayan later refined this bound by proving that a graph $G$ can be represented modulo a positive integer less than or equal to the product of the first $|V(G)| - 1$ primes greater than or equal to $|V(G)| - 1$ [10]. This new bound was also shown to be best possible [10].

The determination of $\text{rep}\,(G)$ for an arbitrary graph $G$ is a very difficult problem indeed. It seems to be as difficult, if not more so, than determining $p\dim(G)$ which has been shown to be NP-Complete [8]. In addition Evans, Isaak, and Narayan showed the representation numbers for the family consisting of the disjoint union of complete graphs is dependent upon the existence of families of mutually orthogonal Latin squares [4]. However calculation of $\text{rep}\,(G)$ for particular families of graphs is feasible. Representation numbers for several families of graphs including complete graphs, independent sets, matchings, and graphs of the form $K_m - P_l$, $K_m - C_l$, $K_m - K_{1,l}$ (each along with a set of isolated vertices) were determined in [3] and [4]. Recently, Evans used linked matrices and distance covering matrices to obtain new results involving representation numbers for the disjoint union of complete graphs [2].

In this paper, we investigate representation numbers for new families of graphs and present new results for each case. In Section 3, we examine representation numbers for stars. In Section 4, we determine new representation numbers for split graphs (graphs that are the disjoint union of a complete graph and an independent set). Later in Section 5, we investigate representation numbers for complements of split graphs. Finally in Section 6, we determine new results involving representation numbers for hypercubes.

## 2. Dimensions and Representations

The representation number of a graph is related to its product dimension as defined by Nešetřil and Pultr [12]. A *product representation* of *length* $t$ assigns distinct vectors of length $t$ to each vertex so that vertices $u$ and $v$ are adjacent if and only if their vectors differ in every position. The *product dimension* of a graph, denoted $\text{pdim}\, G$, is the minimum length of such a representation of $G$. The theory of product dimension has applications to constructions for perfect hashing and for qualitatively independent partitions [7]. Much of the seminal work on product dimension was done by Lovász, Nešetřil, and Pultr [9] and we shall build upon their results.

As developed in [3] and [4], there is a close correspondence between product representation and modular representation. From a representation of a graph $G$ modulo a product of primes $q_1, \dots, q_t$, we obtain a product representation of length $t$ as follows. The vector for vertex $v$ is $(v_1, \dots, v_t)$, where $v_i \equiv a \bmod q_i$ and $v_i \in \{0, \dots, q_{i-1}\}$ for $1 \le i \le t$. If $u$ has vector $(u_1, \dots, u_t)$ and $v$ has vector $(v_1, \dots, v_t)$, then the modular representation implies that $u$ and $v$ are adjacent if and only if $u_i \ne v_i$ for all $i$, making this assignment a product representation.

Conversely, given a product representation, a modular representation can be obtained by choosing distinct primes for the coordinates, provided that the prime for each coordinate is larger than the number of values used in that coordinate. The numbers assigned to the vertices can then be realized using the Chinese Remainder Theorem. The resulting modular representation generated from the product representation is called the *coordinate representation.*

## 3. Representation Numbers for Stars

We consider the representation number of the star $K_{1,m}$. It was shown in [4] that $\text{rep}(K_{1,m}) \le \min\{2^{\lceil \log_2 m \rceil + 1}, 2p\}$ where $p$ is the smallest prime number greater than or equal to $m+1$. However it is possible to have values of $m$ that do not fit one of the two possibilities. For example if $m = 24$, then $2^{\lceil \log_2 m \rceil + 1} = 64$ and $2p = 2 \cdot 29 = 58$. However assigning 0 to the root of the star and labeling the remaining vertices $\{1, 3, \dots, 11, 15, \dots, 37, 43, \dots, 51\}$ forms a representation of $G$ modulo $52 = 2^2 \cdot 13$.

We first show that without loss of generality it is possible to label the root vertex of the star with a 0. Assume that $K_{1,r}$ has a representation modulo $r$ with labels $\{a_0, a_1, \dots, a_r\}$ where $\{0, a_0, a_1 - a_0, \dots, a_r - a_0\}$ is also a representation of $K_{1,r}$, as the differences between the vertex labels are all preserved.

In a representation modulo $n$ with the root labeled 0 the remaining vertices must be labeled so that the following two conditions are satisfied: (i) any two labels are congruent modulo some prime dividing $n$, and (ii) each label is relatively prime to $n$. We give an example of such a representation in Figure 2.

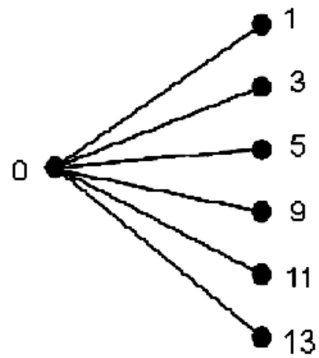


**Figure 2. A representation of $K_{1,6}$ modulo $2 \cdot 7 = 14$.**

The above idea is generalized in Theorem 1.

**Theorem 1.** *Let $n$ be a positive integer and let $p$ be the smallest prime dividing $n$. Then $K_{1,r}$ has a representation modulo $p^k n$ if $r \leq p^{k-1}\phi(n)$, where $\phi(n)$ denotes the Euler phi function.*

*Proof.* For a given $n$ and $p^k$, we construct a representation of $K_{1,p^{k-1}\phi(n)}$. We first label the root 0. The remaining vertices are then labeled with integers equivalent to 1 mod $p$ but not equivalent to 0 mod $q$ where $q$ is any other prime dividing $n$. This gives $p^{k-1}\phi(n)$ possible labels for the non-root vertices. □

## 4. Representations of Split Graphs

While it is not difficult to show that the product dimension of $K_m + tK_1$ is $m+1$ for all $t > (m-1)!$ the determination of the representation number for such graphs is not straightforward. We begin by reviewing some known results. Let $p_i$ denote the $i$th prime, and for any prime $p_i$ let $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ denote the next $k$ primes larger than $p_i$. We restate a lemma from [4] that gives the representation number of a split graph when the number of isolated vertices is not too large.

**Lemma 1.** *If $t < (m-1)!$ then Then* $\operatorname{rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$*where $p_s$ is the smallest prime greater than or equal to $m$.*

We will extend the above lemma to investigate cases where $t$ is arbitrarily large. We first consider the case where $G$ consists of one edge and an arbitrary number of isolated vertices.

### 4.1. Graphs with One Edge

Surprisingly the representation of a graph comprised of a single edge along with a set of isolated vertices is non-trivial. This problem is suggested in [3] where it is mentioned that rep$(K_2 \cup mK_1) \leq 6m$. They also noted that this bound is not optimal as rep$(K_2 \cup 5K_1) =$ rep $(K_2 \cup 6K_1) = 30$.

Let $G$ be a graph consisting of a single edge along with a set of isolated vertices. In the next lemma we show that we may label the two vertices of degree one with 0 and 1.

**Lemma 2.** *Let $\{a_1, a_2, \ldots, a_r\}$ be a representation of $K_2 \cup tK_1$ modulo $n$ where the endpoints $v_1$ and $v_2$ of $K_2$ are labeled $a_1$ and $a_2$ respectively. Then without loss of generality, $a_1 = 0$ and $a_2 = 1$.*

*Proof.* Let $\{a_1, a_2, \ldots, a_{t+2}\}$ be a representation of $G$ modulo $n$. Then it follows that $\{0,\ a_2 - a_1 \ (\bmod n),\ \ldots,\ a_{t+2} - a_1 (\bmod n)\}$ is a representation modulo $n$. Since $v_1$ and $v_2$ are adjacent it follows that gcd$(\ a_2 - a_1 - 0, n) = 1$. Then by Euler's Theorem, $(a_2 - a_1)^{\phi(n)} \equiv 1 \bmod n$. Hence $\{0,\ 1,\ a_3 a_2^{\phi(n)-1} (\bmod n),\ a_4 a_2^{\phi(n)-1} (\bmod n), \ldots, a_{t+2} a_2^{\phi(n)-1}\}$ is a representation modulo $n$. □

In Table 1 we give a representation of $K_2 + 11K_1$ modulo 42.

**Example 1.** Let $n = 2 \cdot 3 \cdot 7 = 42$. We will construct a coordinate representation of $K_2 + 11K_1$ modulo 42 where $t$ is as large as possible. Without loss of generality the two vertices with degree 1 can be labeled 0 and 1, or equivalently with coordinate vectors $(0, 0, 0)$ and $(1, 1, 1)$. We now seek a maximum sized collection of coordinate vectors that contain a 0 and a 1, and also intersect when taken pairwise. A set of nine such vectors can be constructed by having the vectors agree on a single coordinate. A coordinate representation of $K_2 + 9K_1$ modulo $2 \cdot 3 \cdot 7 = 42$ is given below:

| | mod 2 | mod 3 | mod 7 | # |
|---|---|---|---|---|
| $v_1$ | 0 | 0 | 0 | 0 |
| $v_2$ | 1 | 1 | 1 | 1 |
| $v_3$ | 0 | 1 | 0 | 28 |
| $v_4$ | 0 | 1 | 1 | 22 |
| $v_5$ | 0 | 1 | 2 | 16 |
| $v_6$ | 0 | 1 | 3 | 10 |
| $v_7$ | 0 | 1 | 4 | 4 |
| $v_8$ | 0 | 1 | 5 | 40 |
| $v_9$ | 0 | 1 | 6 | 34 |
| $v_{10}$ | 0 | 0 | 1 | 36 |
| $v_{11}$ | 0 | 2 | 1 | 8 |

**Table 1: A Representation of $K_2 + 11K_1$ modulo 42**

Next we will start with a given value of $n$ (and its prime factorization) and seek the maximum value of $t$ such that $K_2 + tK_1$ has a representation represented modulo $n$.

**Theorem 2.** *For odd $n > 1$, $K_2 + tK_1$ has a representation modulo $2^k n$ if $t < 2^{k-1}(n - \phi(n))$.*

*Proof.* By Lemma 2 we may label the vertices of degree 1, $v_1$ and $v_2$ with 0 and 1. For each of the remaining labels we choose numbers that are both odd and not relatively prime to $n$. Since there are $n - \phi(n)$ numbers that are not relatively prime to $n$ there must be $2^{k-1}(n - \phi(n))$ odd numbers less than $2^k n$ that are relatively prime to $n$. □

We then have the following corollary.

**Corollary 1.** *We have* $\text{rep}(K_2 + tK_1) \leq \min\{2^k n | n > 1$ *is odd and* $t < 2^{k-1}(n - \phi(n))\}$.

### 4.2. Split Graphs with More than One edge

Next we generalize the above results for graphs with one edge to the family of split graphs. We give an example that illustrates many of the key ideas.

**Example 2.** We have $\text{rep}(K_3 \cup 40K_1) \leq 3 \cdot 5 \cdot 7 \cdot 11 = 1155$.

| | mod 3 | mod 5 | mod 7 | mod 11 |
|---|---|---|---|---|
| $a_1$ | 0 | 0 | 0 | 0 |
| $a_2$ | 1 | 1 | 1 | 1 |
| $a_3$ | 2 | 2 | 2 | 2 |
| $a_4 - a_{12}$ | 2 | 0 | 1 | >1 |
| $a_{13} - a_{21}$ | 2 | 1 | 0 | >1 |
| $a_{22} - a_{26}$ | 2 | 0 | >1 | 1 |
| $a_{27} - a_{31}$ | 2 | 1 | >1 | 0 |
| $a_{32} - a_{34}$ | 2 | >1 | 0 | 1 |
| $a_{35} - a_{37}$ | 2 | >1 | 1 | 0 |
| $a_{38}$ | 2 | 0 | 1 | 1 |
| $a_{39}$ | 2 | 1 | 0 | 1 |
| $a_{40}$ | 2 | 1 | 1 | 0 |
| $a_{41}$ | 2 | 1 | 0 | 0 |
| $a_{42}$ | 2 | 0 | 1 | 0 |
| $a_{43}$ | 2 | 0 | 0 | 1 |

**Table 2: Calculating an upper bound for** rep $(K_3 \cup 40K_1)$

Observing the mod 3, mod 5 and mod 7 columns, the number of possible independent vertices is: $5 \cdot 7 \cdot 11 - \binom{2}{1}\phi_1(5 \cdot 7 \cdot 11) + \binom{2}{2}\phi_2(5 \cdot 7 \cdot 11) = 40$.

**Theorem 3.** *Let $p$ be the smallest prime dividing $n$. Assume that $p \geq m > 1$ where there are at least $m - 1$ distinct primes dividing $n$. Then $K_m + tK_1$ has a representation modulo $p^k n$ for all $t$ satisfying* $t \leq p^{k-1} \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \phi_i(n)$.

*Proof.* We begin by labeling the vertices of $K_m$ with $0, 1, \ldots, m-1$. Since the remaining vertices must be congruent to each of $0, 1, .., m-1$ modulo some prime dividing $p^k n$ there must be $m - 1$ distinct primes dividing $n$. Since each of the independent vertices must agree pairwise on some prime dividing $p^k n$, we can make their labels congruent to $m - 1$

on the smallest $p$. Thus for the remaining coordinates, we must include each of the integers $0, 1, \ldots, m-2$ at least once.

This becomes an inclusion-exclusion problem. First we must get rid of the vertices that are not congruent to $i$ where $0 \leq i \leq m-1$. There are $\binom{m-1}{i}$ such $i$ and for each there are $\phi_1(n)$ of these. However we must add back in the $\binom{m-1}{2}\phi_2(n)$ numbers that are not congruent to some $i, j$ where $0 \leq i < j < m-1$ that are double counted. Continuing in this manner, we keep adding in more $(-1)^i\binom{m-1}{i}\phi_i(n)$ until $i = m-1$. Thus we may include at least $p^{k-1}\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\phi_i(n)$ independent vertices along with $K_m$ in a representation modulo $p^k n$. □

**Corollary 2.** *We have* $\text{rep}(K_m + tK_1) \leq \min\{p^k n \mid$ *no prime less than or equal to* $p \geq m > 1$ *is one of the at least* $m-1$ *primes dividing* $n$ *and* $t \leq p^{k-1}\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\phi_i(n)\}$

## 5. Representation Numbers for Complements of Split Graphs

We next consider the representation for graphs which are the complements of split graphs. Let $G$ be a complement of a split graph. Then there exist disjoint sets $A$ and $B$ such that $V(G) = A \cup B$ and $E(G) = \{(u, v)|u \in A \text{ or } v \in A\}$. If $|A| = m$ and $|B| = n$. Note that when $m = 1$, $G = K_{1,n}$ and when $n = 1$, $G = K_{m+1}$. In order to find the representation number of such graphs, we define the following function on the integers in the definition below.

**Definition.** If $p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ is the prime factorization of $n$ where $p_j < p_{j+1}$, let $\phi_i(n)$ be the number of nonnegative integers less than $n$ that are not congruent to $0, 1, \ldots i-1$ modulo $p_j$ for $1 \leq j \leq r$. Equivalently, $\phi_i(n) = (p_1 - i)p_1^{k_1-1}(p_2 - i)p_2^{k_2-1}\cdots(p_r - i)p_r^{k_r-1}$ when $i \leq p_1$ and $\phi_i(n) = 0$ otherwise.

Note that $\phi_0(n) = n$ is the identity function and that $\phi_1(n) = \phi(n)$ is the Euler phi function.

**Theorem 4.** *Let* $r$ *be a positive integer and* $p$ *be a prime such that* $q$ *does not divide* $r$ *for* $1 < q \leq p$. *Then* $G$ *has a representation modulo* $p^k r$ *if* $m < p$ *and* $n \leq p^{k-1}\phi_n(r)$.

*Proof.* Since the subset $A$ of $G$ constitutes a complete graph on $m$ vertices we may label them $0, 1, \ldots, m-1$. Since the $n$ vertices on the outside must agree on some prime pairwise, label them so that they are all congruent to $m$ modulo $p$. Then, to make them disagree completely with the roots, make the components for primes dividing $r$ all at least $m$. Since each label corresponds to a number that is not congruent to $0, 1, \ldots, m-1$ for any prime dividing $r$, there are $\phi_m(r)$ of these. Finally, since there are $p^{k-1}$ copies of this, we may fit at least $p^{k-1}\phi_m(r)$ non-root vertices. □

**Corollary 3.** *Let $G$ be a complement of a split graph with disjoint sets $A$ and $B$ such that $V(G) = A \cup B$ and $E(G) = \{(u, v) | u \in A$ or $v \in A\}$ and $|A| = m$ and $|B| = n$. Then* $\text{rep}(FS_{m,n}) \leq \min\{p^k r \mid$ *no* $q \leq p$ *divides* $r$, *where* $p > m$ *is prime and* $n \leq p^{k-1}\phi_m(r)\}$.

## 6. Bounds for Representation Numbers of Hypercubes

Recall that for given graphs $G$ and $H$, their *(Cartesian) product*, $G \times H$, is the graph such that $V(G \times H) = V(G) \times V(H)$ and $((u_1, u_2), (v_1, v_2)) \in E(G \times H)$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(H)$, or $(u_1, v_1) \in E(G)$ and $u_2 = v_2$. Inductively, we denote $G^1 = G$ and for positive $n$, $G^{n+1} = G^n \times G$. The hypercube $K_2^d$ is the graph whose vertices are $d$-tuples with entries in $\{0, 1\}$ and whose entries are the pairs of $d$-tuples that differ in exactly one position.

The first two cases are clear, with $\text{rep}(K_2) = 2$ and $\text{rep}(K_2^2) = 4$. After a little more inspection one may note that $\text{rep}(K_2^3) = 10$. Next, we justify $\text{rep}(K_2^4) \leq 70$. To see this, convert the labels of $K_2^3$ to their coordinate representation. Since it is modulo 10, each label $a$ is put in the form $(a \bmod 2, a \bmod 5)$.

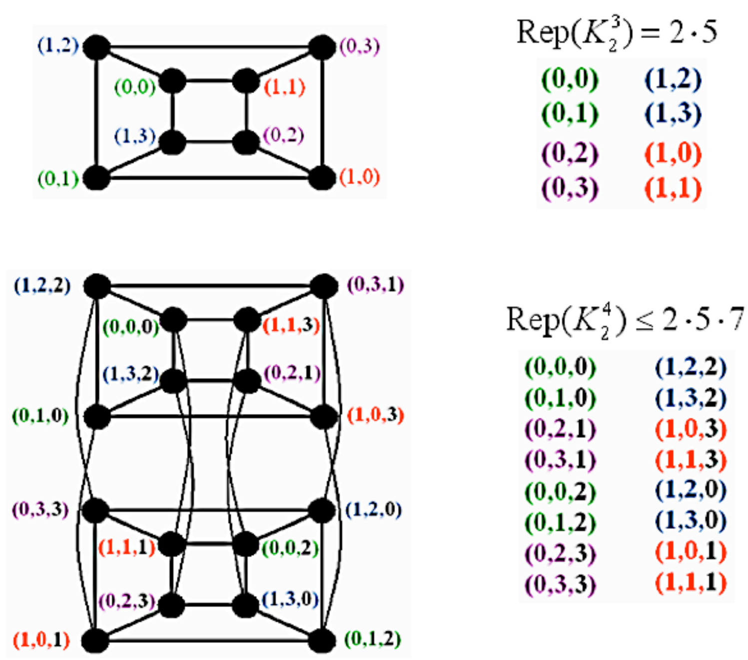


**Figure 3. Extending a representation of $K_2^3$ modulo 10 to a representation of $K_2^4$ modulo 70.**

If we divide the set of labels of $K_2^n$ into four sets $S_{ij}^n$ for $i, j \in \{0, 1\}$ where $S_{00}^n$ is the top left quadrant, $S_{01}^n$ is the bottom left, $S_{10}^n$ is the top right and $S_{11}^n$ is the bottom right, we may properly define the induction in the following mathematical form. Let $S_{00}^3 = \{0, 6\}$, $S_{01}^3 = \{2, 8\}$, $S_{10}^3 = \{3, 7\}$ and $S_{11}^3 = \{1, 5\}$ with inductive step

$$S_{ij}^{n+1} = \{x(\bmod \frac{1}{3}P_{n+1}) | x(\bmod \frac{1}{3}P_n) \in S_{ik}^n x \equiv [2(i+j) + k \bmod 4] \bmod p_{n+1}\}, \qquad (1)$$

where $P_n$ is the product of the first $n$ primes. That is, for each $x \in S_{ij}^n$, there exist $y \in S_{i0}^{n+1}$ and $z \in S_{i1}^{n+1}$ such that $x \equiv y \equiv z(\bmod \frac{1}{3}P_n)$, but that $y \equiv 2i + j \bmod p_{n+1}$ while $z \equiv [2(i+1) + j \bmod 4] \bmod p_{n+1}$. The corresponding $y$ and $z$ are found using the Chinese Remainder Theorem. It is also important to note that if $x \in S_{ij}^n$, then for every $a$ where $3 \leq a < n$, $x(\bmod \frac{1}{3}P_a) \in S_{ik}^a$ for some $k \in \{0, 1\}$. With this, we have a representation of $K_2^n$ by labeling $2^n$ vertices uniquely with the numbers in the set

$$\bigcup_{i,j \in \{0,1\}} S_{ij}^n.$$

We have proven the following theorem.

**Theorem 5.** *Let $P_n$ be the product of the first $n$ primes. Then* $\operatorname{rep}(K_2^n) \leq \frac{1}{3}P_n$ *for $n > 2$.*

*Proof.* First, we want to make sure that there are exactly $2^{n-2}$ numbers in each $S_{ij}^n$ and that none are in any other $S_{kl}^n$. Certainly this is true for $n = 3$, so we assume it holds for some $n \geq 3$. Then for every $x \in S_{ij}^n$, there exists a $y \in S_{i0}^{n+1}$ and a $z \in S_{i1}^{n+1}$ by the Chinese remainder theorem. Since $y \equiv [2(i+j) \bmod 4](\bmod p_{n+1})$, and $z \equiv [2(i+j) + 1 \bmod 4](\bmod p_{n+1})$, $y \neq z$. Furthermore there does not exist an $x' \in S_{kl}^n$ such that $x' \equiv x(\bmod \frac{1}{3}P_n)$ unless $x' = x$, $i = k$ and $j = l$. Hence there can not be a $y' \in S_{kl}^{n+1}$ such that $y' = y$ or $y' = z$. Thus, $|S_{ij}^{n+1}| = 2^{n-1}$ and $S_{ij}^{n+1} \cap S_{kl}^{n+1} = \emptyset$ when $i \neq k$ or $j \neq l$.

We next show that these four sets of labels form a representation of $K_2^n \bmod \frac{1}{3}P_n$. Let $A \subseteq \{0, 1, \ldots, m-1\}$ and denote $R(A, m)$ as the graph on $|A|$ vertices such that there exists a representation modulo $m$ using the entire set $A$ as labels. For example, $R(\{0, 2\}, 4) = 2K_1$ and $R(\{0, 1, 2, 3, 5, 6, 7, 8\}, 10) = K_2^3$. One may note that for the labeling given for $K_2^3$, the following properties hold, where $i, j, k, l \in \{0, 1\}$, $i \neq k$ and $j \neq l$.

$$R(S_{ij}^3 \cup S_{il}^3, 10) = 2^2 K_1$$

$$R(S_{ij}^3 \cup S_{kl}^3, 10) = 2^1 K_2$$

$$R(S_{ij}^3 \cup S_{kj}^3, 10) = K_2^2$$

$$R(S_{00}^3 \cup S_{01}^3 \cup S_{10}^3 \cup S_{11}^3, \frac{1}{3}(2 \cdot 3 \cdot 5)) = K_2^3.$$

□

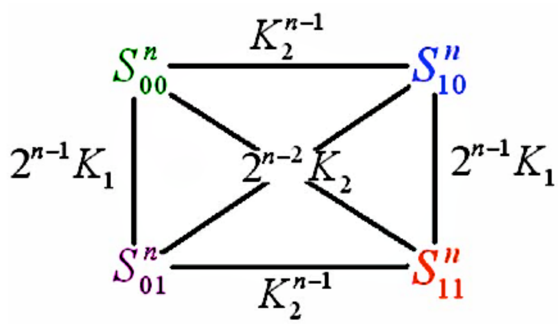


**Figure 4. The inductive step**

In general we show:

$$R(S_{ij}^n \cup S_{il}^n, \frac{1}{3}P_n) = 2^{n-1}K_1 \quad (2)$$

$$R(S_{ij}^n \cup S_{kl}^n, \frac{1}{3}P_n) = 2^{n-2}K_2 \quad (3)$$

$$R(S_{ij}^n \cup S_{kj}^n, \frac{1}{3}P_n) = K_2^{n-1} \quad (4)$$

$$R(S_{00}^n \cup S_{01}^n \cup S_{10}^n \cup S_{11}^n, \frac{1}{3}P_n) = K_2^n \quad (5)$$

Property (2) is trivial since the elements of the sets $S_{0j}^n$ are all even and those of $S_{1j}^n$ are odd. For the remaining properties we will proceed with induction. The inductive step is illustrated in Figure 4.

Suppose that $x \in S_{ij}^n$ and that $x' \in S_{kl}^n$ where $i \neq k$ and $j \neq l$ and there exists an edge between the vertices labelled $x$ and $x'$. This is the only edge between $x$ and any vertex with a label from $S_{kl}^n$ and the only one between the one labelled $y$ and any in $S_{ij}^n$. There exists a $y \in S_{ij}^{n+1}$ and $z \in S_{il}^{n+1}$ that correspond to $x$ and $y' \in S_{kj}^{n+1}$ and $z' \in S_{kl}^{n+1}$ that correspond to $x'$. Since all other vertices in $S_{ij}^n \cup S_{kl}^n$ are adjacent to neither $x$ nor $x'$, the only adjacencies to check are between $y$ and $z'$ and between $y'$ and $z$. It cannot be the case that $y \equiv z' (\bmod p_{n+1})$, because

$$y \equiv [2(i+j) + j \bmod 4] \equiv [2i + 3j \bmod 4] (\bmod p_{n+1})$$

and

$$z' \equiv [2(k+l) + l \bmod 4] \equiv [2k + 3l \bmod 4] (\bmod p_{n+1}).$$

To see this, note that if they were congruent, we would have

$$2(i-k) \equiv 3(l-j) (\bmod p_{n+1}),$$

which is impossible. The case for $z$ and $y'$ can be shown similarly.

Thus for every vertex labelled with an element of $S_{ij}^{n+1}$, there is exactly one other in $S_{kl}^{n+1}$ that is adjacent to it, and property (3) is shown.

To verify that (4) holds we first note that for every $x \in S_{ij}^n$ there is exactly one corresponding $y \in S_{im}^{n+1}$ for a given $m$ with $x \equiv y(\mod \frac{1}{3}P_n)$. Furthermore we note if $x' \in S_{kl}^n$ where $i \neq k$, there is a corresponding $y' \in S_{km}^{n+1}$. Also, note that

$$y \equiv [2(i+m) + j \bmod 4](\mod p_{n+1})$$

$$y' \equiv [2(k+m) + l \bmod 4](\mod p_{n+1}).$$

If $y \equiv y' \bmod p_{n+1}$, then it would be the case that

$$l - j \equiv 2(i-k) \equiv 2 \bmod 4.$$

With $l, j \in \{0, 1\}$, this is impossible. Because $R(S_{ij}^n \cup S_{il}^n, \frac{1}{3}P_n) = 2^{n-1}K_1$ and all adjacencies are preserved from $x \in S_{ij}$ and $x' \in S_{kl}^n$ to $y \in S_{im}^{n+1}$ and $y' \in S_{km}^{n+1}$ when $i \neq k$, $R(S_{ij}^{n+1} \cup S_{k,j}^{n+1}, \frac{1}{3}P_{n+1})$ still retains the shape of $K_2^n$, and (4) holds.

Finally we verify that (5) holds. While it is known that

$$R(S_{00}^{n+1} \cup S_{10}^{n+1}, \frac{1}{3}P_{n+1}) = R(S_{01}^{n+1} \cup S_{11}^{n+1}, \frac{1}{3}P_{n+1}) = K_2^n$$

and for each vertex $v$ in $R(S_{00}^{n+1} \cup S_{10}^{n+1}, \frac{1}{3}P_{n+1})$ there is exactly one vertex in $R(S_{01}^{n+1} \cup S_{11}^{n+1}, \frac{1}{3}P_{n+1})$ to which $v$ is connected in $R(S_{00}^{n+1} \cup S_{01}^{n+1} \cup S_{10}^{n+1} \cup S_{11}^{n+1}, \frac{1}{3}P_{n+1})$, it is not necessarily the case that $S_{00}^{n+1} \cup S_{01}^{n+1} \cup S_{10}^{n+1} \cup S_{11}^{n+1} = K_2^{n+1}$. We verify that there exists $y \in S_{0i}^{n+1}$ and $z \in S_{1i}^{n+1}$ with $\gcd(|y-z|, \frac{1}{3}P_{n+1}) = 1$. Then since $y' \in S_{0j}^{n+1}$ and $z' \in S_{1j}^{n+1}$ such that $i \neq j$, $\gcd(|y'-z|, \frac{1}{3}P_{n+1}) = 1$ and $\gcd(|y-z'|, \frac{1}{3}P_{n+1}) = 1$, it follows that $\gcd(|y'-z'|, \frac{1}{3}P_{n+1}) = 1$, creating a cycle on four vertices in the graph with vertices labeled with $y$, $z$, $y'$ and $z'$.

Next, let $y$, $z$, $y'$ and $z'$ be as above. If $y(\mod \frac{1}{3}P_n) \in S_{0k}^n$, then $z'(\mod \frac{1}{3}P_n) \in S_{1l}^n$ where $k \neq l$. We note that if $x \in S_{1k}^n$ such that $\gcd(|x-y|, \frac{1}{3}P_n) = 1$, then there exists a $w \in S_{1j}^{n+1}$ such that $w \equiv x(\mod \frac{1}{3}P_n)$ and $w \equiv [2(1+i) + k \bmod 4](\mod p_{n+1})$, where the vertex labeled $w$ is adjacent to the one labelled $y$. As a result of property (3), the vertex labeled $w$ is the only one with a label from $S_{1j}^{n+1}$ that is adjacent to the vertex labeled $y$, so $w = z'$. By a similar proof, we also know that if $y'(\mod \frac{1}{3}P_n) \in S_{0k'}^n$, then $z(\mod \frac{1}{3}P_n) \in S_{1l'}^n$ where $k' \neq l'$. Now we are left with two cases: $k = k'$ or $k = l'$.

In either case, $y'$ is not congruent to $z'$ modulo $p_{n+1}$ because $y' \in S_{0j}^{n+1}$ and $z' \in S_{1j}^{n+1}$. Hence we need only prove the vertices labeled $y'(\mod \frac{1}{3}P_n)$ and $z'(\mod \frac{1}{3}P_n)$ are adjacent in $K_2^n$. The case where $k = k'$, follows from properties (4) and (5). Now suppose that $k = l'$. Then $y'(\mod \frac{1}{3}P_n) \in S_{0l}^n$ and $z'(\mod \frac{1}{3}P_n) \in S_{1k}^n$. However $y \equiv 2i + k(\mod p_{n+1})$ and

$$z' \equiv [2(1+j) + k \bmod 4] \equiv [2(1 + (1+i) \bmod 2) + k \bmod 4] \equiv 2i + k \equiv y(\mod p_{n+1}),$$

would contradict our assumption that the vertices labeled $y$ and $z'$ are adjacent in $K_2^{n+1}$. Thus property (5) holds for $n+1$ and, for $n > 2$,

$$R(S_{00}^n \cup S_{01}^n \cup S_{10}^n \cup S_{11}^n, \frac{1}{3}P_n) = K_2^n.$$

We note that we have obtained a new result involving the product dimension of hypercubes.

**Corollary 4.** *pdim*$(K_2^n) \leq n-1$ *for* $n > 2$.

*Proof.* This follows from the fact that there is a one-to-one correspondence between the coordinate representation of a label of a vertex in a representation modulo $\frac{1}{3}P_n$ with a product representation over $n-1$ coordinates. □

## 7. Conclusion and Open Problems

Little is known about representation numbers for multipartite graphs, including the most basic cases involving trees and complete bipartite graphs. A collection of open problems involving representations modulo $n$ are presented in [11] and [5]. We state additional problems below.

## 8. Problems

(i) In Theorem 8, we give a bound for the representation number of a graph with one edge. The key idea in the theorem is start with two vectors $a_1$ and $a_2$ and generate a large set of coordinate vectors $a_3, \ldots, a_n$ which have at least one coordinate in the same position as $a_1$ and $a_2$ and also mutually share at least one coordinate in the same position. We obtained our set of vectors $a_3, \ldots, a_n$ so that they agreed all on the same coordinate. We conjecture that this is optimal. If this is proved, $\text{rep}(K_2 \cup tK_1)$ will be precisely determined.

(ii) In Theorem 1, we provide a result involving the representation number of a star. We pose the problem of determining $\text{rep}(K_{m,n})$.

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