INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 8 (2008), #A18

TILING PROOFS OF SOME FIBONACCI-LUCAS RELATIONS

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Received: 3/6/07, Revised: 1/7/08, Accepted: 4/18/08, Published: 5/8/08

Abstract

We provide tiling proofs for some relations between Fibonacci and Lucas numbers, as requested by Benjamin and Quinn in their text, *Proofs that Really Count.* Extending our arguments yields Gibonacci generalizations of these identities.

1. Introduction

Let F_n and L_n denote the Fibonacci and Lucas numbers defined, respectively, by $F_0 = 0$, $F_1 = 1$ with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 2$ and by $L_0 = 2$, $L_1 = 1$ with $L_n = L_{n-1} + L_{n-2}$ if $n \geqslant 2$. The following four relations can be shown using the Binet formulas for F_n and L_n and occur as (V85)–(V88) in Vajda [2]:

$$F_{(2k+3)t} = F_t \left[(-1)^{(k+1)t} + \sum_{i=0}^{k} (-1)^{it} L_{(2k+2-2i)t} \right] , \tag{1.1}$$

$$F_{(2k+2)t} = F_t \sum_{i=0}^{k} (-1)^{it} L_{(2k+1-2i)t} , \tag{1.2}$$

$$L_{(2k+3)t} = L_t \left[(-1)^{(k+1)(t+1)} + \sum_{i=0}^{k} (-1)^{i(t+1)} L_{(2k+2-2i)t} \right] , \tag{1.3}$$

and

$$F_{(2k+2)t} = L_t \sum_{i=0}^{k} (-1)^{i(t+1)} F_{(2k+1-2i)t} , \tag{1.4}$$

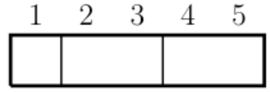
where k and t are nonnegative integers. Benjamin and Quinn request combinatorial proofs of (1.1)–(1.4) on page 145 of their text, *Proofs that Really Count* [1]. In this note, we provide tiling proofs for (1.1)–(1.4) as well as for a couple of closely related identities. The arguments can be extended to yield generalizations involving Gibonacci numbers. The tiling proofs we give will also supply a combinatorial insight into the various divisibility relations implicit in (1.1)–(1.4) within/between the Fibonacci and Lucas sequences.

The Fibonacci number F_{n+1} counts tilings of a board of length n with cells labeled 1, 2, ..., n using squares and dominos (termed *n-tilings*). The Lucas number L_n is given by the simple relation

$$L_n = F_{n+1} + F_{n-1}, \qquad n \geqslant 2\,. \tag{1.5}$$

From (1.5), we see that L_n counts n-tilings in which one may circle a domino covering cells $n-1$ and n, which we'll term *Lucas n-tilings*. (There are F_{n-1} Lucas n-tilings which end in a circled domino and F_{n+1} Lucas n-tilings which do not.)

We'll say that a tiling is *breakable at m* (as in [1, p. 3]) if cell m is covered by a square or by a second segment of a domino. The 5-tiling below is breakable at 0 (vacuously), 1, 3, and 5.



For $m \geqslant 0$, let $\mathfrak{F}_m$ and $\mathfrak{L}_m$ denote the sets consisting of m-tilings and of Lucas m-tilings, respectively.

2. Tiling Proofs

We first provide a combinatorial interpretation for (1.2) when $t \geqslant 2$ is even. By (1.5), we're to show

$$F_{(2k+2)t} = \sum_{i=0}^{k} \left(F_t F_{(2k+1-2i)t+1} + F_t F_{(2k+1-2i)t-1} \right)\,. \tag{2.1}$$

Given i, $0 \leqslant i \leqslant k$, let $\mathfrak{F}^i \subseteq \mathfrak{F}_{(2k+2)t-1}$ comprise those tilings starting with it consecutive dominos, but *not* with $(i+1)t$ consecutive dominos. There are clearly $F_t F_{(2k+1-2i)t-1}$ members of $\mathfrak{F}^i$ that are not breakable at $(2i+1)t$, as such tilings can be decomposed as $d^{it}\delta_1 d \delta_2$, where δ_1 and δ_2 are tilings of length $t-1$ and $(2k+1-2i)t-2$, respectively. So we need to show that there are $F_t F_{(2k+1-2i)t+1}$ members of $\mathfrak{F}^i$ that are breakable at $(2i+1)t$.

Suppose $\lambda \in \mathfrak{F}^i$ is breakable at $(2i+1)t$. Let α be the t-tiling obtained by taking all the pieces in λ covering cells $2it+1$ through $(2i+1)t$ and β be the $[(2k+1-2i)t-1]$-tiling obtained by taking all the pieces in λ past cell $(2i+1)t$. We place α above β, offsetting the tilings by shifting β to the right by one cell. (Tiling α covers cells 1 through t and β covers cells 2 through $(2k+1-2i)t$.)

Look for the first *fault line* passing through α and β. (A fault line occurs at cell j if both α and β are breakable at cell j.) Exchange all the pieces in α past the first fault line with all the pieces in β past it (i.e., swap tails as shown) to obtain $(A, B) \in \mathfrak{F}_{(2k+1-2i)t} \times \mathfrak{F}_{t-1}$.

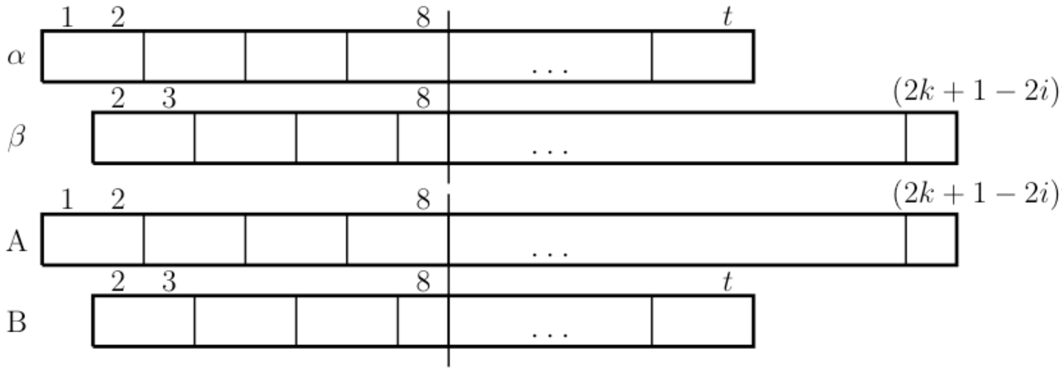


Figure 1: *Swapping tails past cell 8 converts* (α, β) *into* (A, B).

Note that $\lambda \in \mathfrak{F}^i$ implies that tails can always be swapped as it cannot be the case that α consists of and β starts with $\frac{t}{2}$ dominos. (If $i = k$, there is a single case in which tails of length zero are swapped.) Since $t - 1$ is odd, there is always at least one square in a tiling of length $t - 1$. Thus, the inverse on $\mathfrak{F}_{(2k+1-2i)t} \times \mathfrak{F}_{t-1}$ can always be defined, which completes the even case.

Now suppose $t \geqslant 1$ is odd. With the $\mathfrak{F}^i$ as above, let

$$\mathfrak{F}^i_{>} := \bigcup_{j=i+1}^{k} \mathfrak{F}^j, \qquad 0 \leqslant i \leqslant k\,; \tag{2.2}$$

note that $|\mathfrak{F}^i_{>}| = F_{(2k-2i)t}$, as members of $\mathfrak{F}^i_{>}$ must start with at least $(i + 1)t$ consecutive dominos. There are $F_t F_{(2k+1-2i)t-1}$ members of $\mathfrak{F}_{(2k+2)t-1}$ starting with it dominos that aren't breakable at $(2i + 1)t$. These tilings consist of the $F_{(2k-2i)t}$ members of $\mathfrak{F}^i_{>}$ (they aren't breakable at $(2i+1)t$ since t is odd) as well as the members of $\mathfrak{F}^i$ that aren't breakable at $(2i + 1)t$. Upon subtraction, there are then $F_t F_{(2k+1-2i)t-1} - F_{(2k-2i)t}$ members of $\mathfrak{F}^i$ that aren't breakable at $(2i + 1)t$.

If $\lambda \in \mathfrak{F}^i$ is breakable at $(2i + 1)t$, then use the correspondence above with $\mathfrak{F}_{(2k+1-2i)t} \times \mathfrak{F}_{t-1}$, noting that now the $F_{(2k-2i)t}$ ordered pairs whose first component starts with $\frac{t+1}{2}$ dominos and whose second component consists of $\frac{t-1}{2}$ dominos are missed. Thus, there are $F_t F_{(2k+1-2i)t+1} - F_{(2k-2i)t}$ members of $\mathfrak{F}^i$ that are breakable at $(2i + 1)t$. Upon adding the $F_t F_{(2k+1-2i)t-1} - F_{(2k-2i)t}$ members of $\mathfrak{F}^i$ that aren't breakable at $(2i + 1)t$, we see that there are $(F_t F_{(2k+1-2i)t+1} - F_{(2k-2i)t}) + (F_t F_{(2k+1-2i)t-1} - F_{(2k-2i)t}) = F_t L_{(2k+1-2i)t} - 2F_{(2k-2i)t}$ members of $\mathfrak{F}^i$ altogether, which implies $|\mathfrak{F}^i| + 2|\mathfrak{F}^i_{>}| = F_t L_{(2k+1-2i)t}$. Therefore,

$$\begin{aligned} F_t \sum_{i=0}^{k} (-1)^i L_{(2k+1-2i)t} &= \sum_{i=0}^{k} (-1)^i \left[|\mathfrak{F}^i| + 2|\mathfrak{F}^i_{>}| \right] \\ = \sum_{i=0}^{k} |\mathfrak{F}^i| &= |\mathfrak{F}_{(2k+2)t-1}| = F_{(2k+2)t}\,, \end{aligned}$$

where the second equality follows from a simple characteristic function argument: If $0 \leqslant j \leqslant k$ and $\lambda \in \mathfrak{F}^j$, then both the second and third sums count λ one time, the second since $(-1)^j + \sum_{i=0}^{j-1} 2 \cdot (-1)^i = 1$. This completes the odd case and the proof of (1.2). A slight modification of the preceding argument gives (1.1) as well.

We now turn to identity (1.3), first assuming $t \geqslant 1$ is odd. A similar proof will apply to (1.4). By (1.5), we're to show

$$L_{(2k+3)t} = L_t + \sum_{i=0}^{k} \left(F_{t+1} + F_{t-1}\right) L_{(2k+2-2i)t} \,. \tag{2.3}$$

Given i, $0 \leqslant i \leqslant k+1$, let $\mathfrak{L}^i \subseteq \mathfrak{L}_{(2k+3)t}$ comprise those tilings starting with it consecutive dominos, but *not* with $(i+1)t$ consecutive dominos. Clearly, $|\mathfrak{L}^{k+1}| = L_t$. If $0 \leqslant i \leqslant k$, then there are $F_{t+1}L_{(2k+2-2i)t}$ members of $\mathfrak{L}^i$ that are breakable at $(2i+1)t$ as well, since such tilings can be decomposed as $d^{it}\delta_1\delta_2$, where δ_1 is a regular tiling of length t and δ_2 is a Lucas tiling of length $(2k+2-2i)t$. So we need to show that there are $F_{t-1}L_{(2k+2-2i)t}$ members of $\mathfrak{L}^i$ that are not breakable at $(2i+1)t$.

Suppose $\lambda \in \mathfrak{L}^i$ is not breakable at $(2i+1)t$. Let α be the $(t-1)$-tiling obtained by taking all the pieces in λ covering cells $2it+1$ through $(2i+1)t-1$ and β be the Lucas $[(2k+2-2i)t-1]$-tiling obtained by taking all the pieces in λ past cell $(2i+1)t+1$. Set α above β, shifting β to the right by one cell, and exchange pieces past the first fault line to obtain $(A, B) \in \mathfrak{L}_{(2k+2-2i)t} \times \mathfrak{F}_{t-2}$. Note that this operation can always be performed since $\lambda \in \mathfrak{L}^i$ and can always be reversed since $t-2$ is odd.

If $t \geqslant 2$ is even, then let

$$\mathfrak{L}^i_> := \bigcup_{j=i+1}^{k+1} \mathfrak{L}^j, \qquad 0 \leqslant i \leqslant k \,, \tag{2.4}$$

where the $\mathfrak{L}^i$ are as above. Note that $|\mathfrak{L}^i_>| = L_{(2k+1-2i)t}$, since members of $\mathfrak{L}^i_>$ must start with at least $(i+1)t$ consecutive dominos. If $0 \leqslant i \leqslant k$, then there are $F_{t+1}L_{(2k+2-2i)t}$ members of $\mathfrak{L}_{(2k+3)t}$ which start with it consecutive dominos and are breakable at $(2i+1)t$. Since t is even, there are among these the $L_{(2k+1-2i)t}$ members of $\mathfrak{L}^i_>$. By subtraction, we get $F_{t+1}L_{(2k+2-2i)t} - L_{(2k+1-2i)t}$ members of $\mathfrak{L}^i$ that are breakable at $(2i+1)t$ if $0 \leqslant i \leqslant k$.

If $\lambda \in \mathfrak{L}^i$ is not breakable at $(2i+1)t$, then use the correspondence above with $\mathfrak{L}_{(2k+2-2i)t} \times \mathfrak{F}_{t-2}$, noting that now the ordered pairs whose second component consists of $\frac{t}{2}-1$ dominos and whose first component starts with $\frac{t}{2}$ dominos are missed. Thus, there are $F_{t-1}L_{(2k+2-2i)t} - L_{(2k+1-2i)t}$ members of $\mathfrak{L}^i$ that aren't breakable at $(2i+1)t$ and $(F_{t+1}L_{(2k+2-2i)t} - L_{(2k+1-2i)t}) + (F_{t-1}L_{(2k+2-2i)t} - L_{(2k+1-2i)t}) = L_tL_{(2k+2-2i)t} - 2L_{(2k+1-2i)t}$ members of $\mathfrak{L}^i$ in all. The rest of the proof goes much like the odd case of (1.2) above.

Two further identities can also be obtained from the arguments above. On page 146 of their text, Benjamin and Quinn request combinatorial proofs for Identities V93 and V94, which may be written as

$$F_{(2k+3)t} = (-1)^{(k+1)t} F_t \left[(2k+3) + 5 \sum_{i=1}^{k+1} (-1)^{it} F_{it}^2 \right] \tag{2.5}$$

and

$$F_{(2k+3)t} = (-1)^{(k+1)t} F_t \left[-(2k+1) + \sum_{i=1}^{k+1} (-1)^{it} L_{it}^2 \right] . \tag{2.6}$$

Replacing i by $k+1-i$ in (1.1) gives

$$F_{(2k+3)t} = (-1)^{(k+1)t} F_t \left[1 + \sum_{i=1}^{k+1} (-1)^{it} L_{2it} \right] , \tag{2.7}$$

and substituting

$$L_{2it} = 5F_{it}^2 + 2 \cdot (-1)^{it} \tag{2.8}$$

and

$$L_{2it} = L_{it}^2 - 2 \cdot (-1)^{it} \tag{2.9}$$

into (2.7) yields (2.5) and (2.6), respectively. The preceding argument for (1.1) and hence (2.7) can then be easily combined with the arguments for (2.8) and (2.9), which are known and involve tailswapping (see, e.g., Identities 36, 45, and 53 of [1]), to obtain combinatorial interpretations for (2.5) and (2.6), as desired.

3. Generalizations

Let $(G_n)_{n \geqslant 0}$ be the sequence given by $G_n = G_{n-1} + G_{n-2}$ if $n \geqslant 2$, where G_0 and G_1 are nonnegative integers (termed *Gibonacci numbers* [1, p. 17] as shorthand for *generalized Fibonacci numbers*). The G_n are seen to enumerate n-tilings in which a terminal square is assigned one of G_1 possible phases and a terminal domino is assigned one of G_0 possible phases (termed *phased n-tilings*). Note that G_n reduces to F_{n+1} when $G_0 = G_1 = 1$ and to L_n when $G_0 = 2$, $G_1 = 1$.

Reasoning as in the prior section with phased tilings instead of regular tilings yields the following (reindexed) generalizations of (1.1)–(1.4):

$$G_{(2k+3)t-1} = (-1)^{kt} \left[(-1)^t G_{t-1} + F_t \sum_{i=0}^{k} (-1)^{it} (G_{(2i+2)t} + G_{(2i+2)t-2}) \right] , \tag{3.1}$$

$$G_{(2k+2)t-1} = (-1)^{kt}\left[(-1)^t(G_1 - G_0) + F_t\sum_{i=0}^{k}(-1)^{it}(G_{(2i+1)t} + G_{(2i+1)t-2})\right], \tag{3.2}$$

$$G_{(2k+3)t} = (-1)^{k(t+1)}\left[(-1)^{t+1}G_t + L_t\sum_{i=0}^{k}(-1)^{i(t+1)}G_{(2i+2)t}\right], \tag{3.3}$$

and

$$G_{(2k+2)t-1} = (-1)^{k(t+1)}\left[(-1)^{t+1}(G_1 - G_0) + L_t\sum_{i=0}^{k}(-1)^{i(t+1)}G_{(2i+1)t-1}\right]. \tag{3.4}$$

Identities (3.1), (3.2), and (3.4) reduce to (1.1), (1.2), and (1.4), respectively, when $G_n = F_{n+1}$, and (3.3) reduces to (1.3) when $G_n = L_n$.

We outline the proof of (3.2), and leave the others as exercises for the interested reader. We adjust the argument above for (1.2), first assuming t is even. Let $\mathfrak{G}_m$ denote the set consisting of phased tilings of length m and let $\mathfrak{G}^i \subseteq \mathfrak{G}_{(2k+2)t-1}$ comprise those tilings starting with it consecutive dominos, but not with $(i+1)t$ consecutive dominos, $0 \leqslant i \leqslant k$. There are then $F_tG_{(2i+1)t-2}$ members of $\mathfrak{G}^{k-i}$ that are not breakable at $(2k-2i+1)t$ if $0 \leqslant i \leqslant k$ and, upon tailswapping as above, $F_tG_{(2i+1)t}$ members of $\mathfrak{G}^{k-i}$ that are breakable at $(2k-2i+1)t$ if $0 < i \leqslant k$.

If $i = 0$, we omit the G_1 cases in which $\lambda \in \mathfrak{G}^k$ ends in a phased square preceded by $t-1$ dominos, since here tailswapping doesn't actually move any tiles (i.e., the first and only fault occurs directly after cell t) and hence the phased square ending λ fails to be moved. Similarly, we must omit from consideration the G_0 members of $\mathfrak{G}_t \times \mathfrak{F}_{t-1}$ whose first coordinate is a phased tiling which contains no squares and whose second coordinate is a regular tiling which contains only a single square occurring at the end. Thus, there are $F_tG_t + (G_1 - G_0)$ members of $\mathfrak{G}^k$ that are breakable at $(2k+1)t$. Adding together all of the cases gives (3.2) when t is even. Similar adjustments apply when t is odd.

References

[1] A. Benjamin and J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof,* The Dolciani Mathematical Expositions, 27, Mathematical Association of America, 2003.

[2] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications,* John Wiley & Sons, Inc., 1989.