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# EXTREMAL ORDERS OF COMPOSITIONS OF CERTAIN ARITHMETICAL FUNCTIONS

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## Abstract

We study the exact extremal orders of compositions $f(g(n))$ of certain arithmetical functions, including the functions $\sigma(n)$, $\phi(n)$, $\sigma^*(n)$ and $\phi^*(n)$, representing the sum of divisors of $n$, Euler's function and their unitary analogues, respectively. Our results complete, generalize and refine known results.

## 1. Introduction

Let $\sigma(n)$, $\phi(n)$ and $\psi(n)$ denote – as usual – the sum of divisors of $n$, Euler's function and the Dedekind function, respectively, where $\psi(n) = n\prod_{p|n}(1+1/p)$.

Extremal orders of the composite functions $\sigma(\phi(n))$, $\phi(\sigma(n))$, $\sigma(\sigma(n))$, $\phi(\phi(n))$, $\phi(\psi(n))$, $\psi(\phi(n))$, $\psi(\psi(n))$ were investigated by L. Alaoglu and P. Erdős [1], A. Mąkowski and A. Schinzel [9], J. Sándor [10], F. Luca and C. Pomerance [7], J.-M. de Koninck and F. Luca [8], and others. For example, in paper [9] it is shown that

$$\liminf_{n\to\infty} \frac{\sigma(\sigma(n))}{n} = 1, \tag{1}$$

$$\limsup_{n\to\infty} \frac{\phi(\phi(n))}{n} = \frac{1}{2}, \tag{2}$$

while in paper [7] the result

$$\limsup_{n\to\infty} \frac{\sigma(\phi(n))}{n\log\log n} = e^{\gamma} \tag{3}$$

is proved, where $\gamma$ is Euler's constant.

It is the aim of the present paper to extend the study of exact extremal orders to other compositions $f(g(n))$ of arithmetical functions, considering also the functions $\sigma^*(n)$ and $\phi^*(n)$, representing the sum of unitary divisors of $n$ and the unitary Euler function, respectively. Recall that $d$ is a unitary divisor of $n$ if $d \mid n$ and $(d, n/d) = 1$. The functions $\sigma^*(n)$ and $\phi^*(n)$ are multiplicative and if $n = p_1^{a_1}\cdots p_r^{a_r}$ is the prime factorization of $n > 1$, then

$$\sigma^*(n) = (p_1^{a_1}+1)\cdots(p_r^{a_r}+1), \quad \phi^*(n) = (p_1^{a_1}-1)\cdots(p_r^{a_r}-1). \tag{4}$$

Note that $\sigma^*(n) = \sigma(n)$, $\phi^*(n) = \phi(n)$ for all squarefree $n$ and that for every $n \geq 1$,

$$\phi(n) \leq \phi^*(n) \leq n \leq \sigma^*(n) \leq \psi(n) \leq \sigma(n). \tag{5}$$

We give some general results which can be applied easily also for other special functions. Our results complete, generalize and refine known results. They are stated in Section 2, their proofs are given in Section 3. Some open problems are formulated in Section 4.

## 2. Main Results

**Theorem 1.** *Let $f$ be an arithmetical function. Assume that*

*(i) $f$ is integral valued and $f(n) \geq 1$ for every $n \geq 1$,*

*(ii) $f(n) \leq n$ for every sufficiently large $n$ ($n \geq n_0$),*

*(iii) $f(p) = p - 1$ for every sufficiently large prime $p$ ($p \geq p_0$).*

*Then*

$$\limsup_{n\to\infty} \frac{\sigma(f(n))}{n\log\log n} = \limsup_{n\to\infty} \frac{\sigma(f(n))}{f(n)\log\log f(n)} = e^{\gamma}, \tag{6}$$

$$\limsup_{n\to\infty} \frac{\psi(f(n))}{n\log\log n} = \limsup_{n\to\infty} \frac{\psi(f(n))}{f(n)\log\log f(n)} = \frac{6}{\pi^2}e^{\gamma}, \tag{7}$$

$$\limsup_{n\to\infty} \frac{\sigma(f(n))}{\phi(f(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\sigma(f(n))}{\phi(f(n))(\log\log f(n))^2} = e^{2\gamma}, \tag{8}$$

$$\limsup_{n\to\infty} \frac{\psi(f(n))}{\phi(f(n))(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi(f(n))}{\phi(f(n))(\log\log f(n))^2} = \frac{6}{\pi^2}e^{2\gamma}. \tag{9}$$

Theorem 1 can be applied for $f(n) = \phi(n)$ and $f(n) = \phi^*(n)$, the unitary Euler function. For example, (6) and (7) give

$$\limsup_{n\to\infty} \frac{\sigma(\phi^*(n))}{n\log\log n} = e^{\gamma}, \tag{10}$$

$$\limsup_{n\to\infty} \frac{\psi(\phi(n))}{n\log\log n} = \frac{6}{\pi^2}e^{\gamma}. \tag{11}$$

The weaker result $\limsup_{n\to\infty} \frac{\psi(\phi(n))}{n} = \infty$ is proved in [10].

Figure 1 is a plot of the functions $\sigma(\phi^*(n))$ and $e^{\gamma} n\log\log n$ for $10 \le n \le 10\,000$.

**Theorem 2.** *Let $g$ be an arithmetical function. Assume that*

*(i) $g$ is integral valued and $g(n) \ge 1$ for every $n \ge 1$,*

*(ii) $g(n) \ge n$ for every sufficiently large $n$ ($n \ge n_0$),*

*(iii) either $g(p) = p+1$ for every sufficiently large prime $p$ ($p \ge p_0$), or $g$ is multiplicative and $g(p) = p$ for every sufficiently large prime $p$ ($p \ge p_0$).*

*Then*

$$\liminf_{n\to\infty} \frac{\phi(g(n))\log\log n}{n} = \liminf_{n\to\infty} \frac{\phi(g(n))\log\log g(n)}{g(n)} = e^{-\gamma}. \tag{12}$$

Theorem 2 applies for $g(n) = \sigma(n), \sigma^*(n), \psi(n), \sigma^{(e)}(n)$, where $\sigma^{(e)}(n)$ represents the sum of exponential divisors of $n$. We have for example

$$\liminf_{n\to\infty} \frac{\phi(\sigma(n))\log\log n}{n} = e^{-\gamma}. \tag{13}$$

Remark that according to a result of L. Alaoglu and P. Erdős [1], $\lim_{n\to\infty} \frac{\phi(\sigma(n))}{n} = 0$ on a set of density 1.

Theorems 1 and 2 can be generalized as follows. If $f(n) \ge 1$ is an integer valued arithmetical function, let $f_k(n)$ denote its $k$-fold iterate, i.e., $f_0(n) = n$, $f_1(n) = f(n)$, ..., $f_k(n) = f(f_{k-1}(n))$.

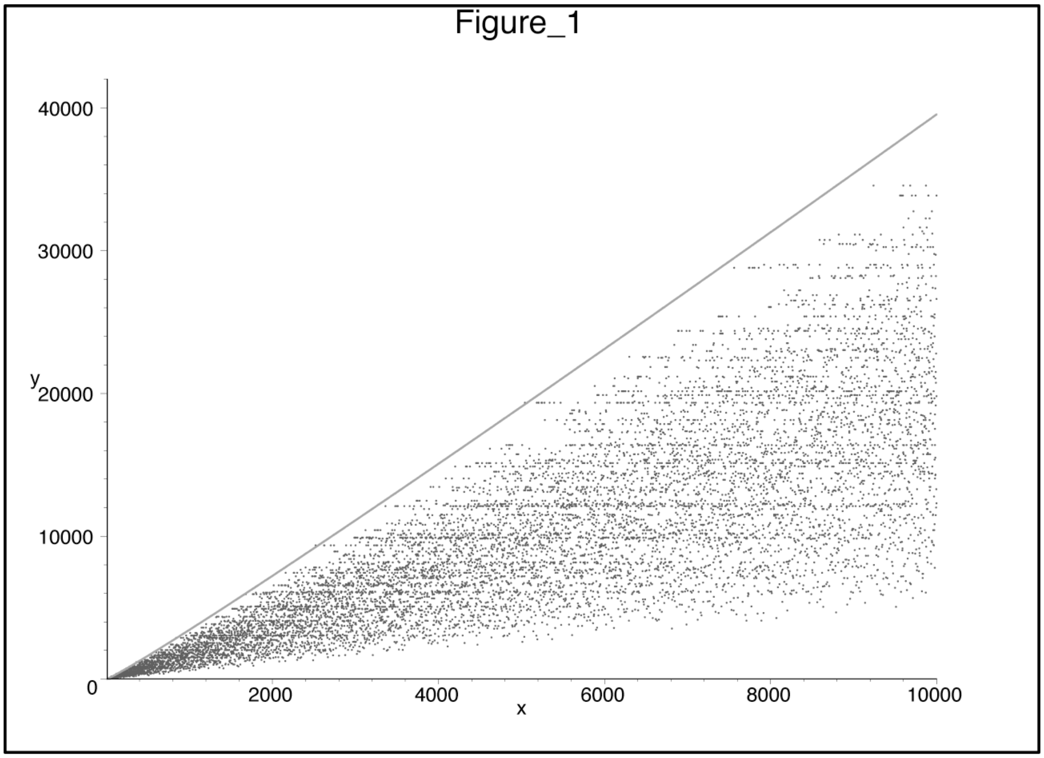


Figure 1: Plot of $\sigma(\phi^*(n))$ and $e^\gamma n \log\log n$ for $10 \le n \le 10\,000$

**Theorem 3.** *Let $f$ be an arithmetical function. Suppose that*

*(i) $f$ is integral valued and $1 \le f(n) \le n$ for every $n \ge 1$,*

*(ii) $f(p) = p - 1$ for every prime $p$,*

*(iii) for every $s, t \ge 1$ if $s \mid t$, then $f(s) \mid f(t)$.*

*Then for every $k \ge 0$,*

$$\limsup_{n\to\infty} \frac{\sigma(f_k(n))}{f_k(n)\log\log n} = e^\gamma. \tag{14}$$

Theorem 3 applies for $f(n) = \phi(n)$, $f(n) = (p_1 - 1)\cdots(p_r - 1)$, $f(n) = (p_1 - 1)^{a_1}\cdots(p_r - 1)^{a_r}$, where $n = p_1^{a_1}\cdots p_r^{a_r}$.

**Theorem 4.** *Let $g$ be an arithmetical function. Suppose that*

*(i) $g$ is integral valued and $g(n) \ge n$ for every $n \ge 1$,*

*(ii) $g(p) = p + 1$ for every prime $p$,*

*(iii) for every $s, t \geq 1$ if $s \mid t$, then $g(s) \mid g(t)$.*

*Then for every $k \geq 0$,*

$$\liminf_{n\to\infty} \frac{\phi(g_k(n)) \log\log n}{g_k(n)} = e^{-\gamma}. \tag{15}$$

Theorem 4 applies for $g(n) = \psi(n)$, $g(n) = (p_1+1)\cdots(p_r+1)$, $g(n) = (p_1+1)^{a_1}\cdots(p_r+1)^{a_r}$, where $n = p_1^{a_1}\cdots p_r^{a_r}$.

For $f(n) = \phi(n)$ and $g(n) = \psi(n)$ we have for every $k \geq 0$,

$$\limsup_{n\to\infty} \frac{\sigma(\phi_k(n))}{\phi_k(n)\log\log n} = e^{\gamma}, \tag{16}$$

$$\liminf_{n\to\infty} \frac{\phi(\psi_k(n))}{\psi_k(n)\log\log n} = e^{-\gamma}. \tag{17}$$

Compare Theorems 1–4 with the following deep results:

– for $k \geq 2$ the normal order of $\dfrac{\sigma_k(n)}{\sigma_{k-1}(n)}$ is $ke^{\gamma}\log\log\log n$, i.e. $\sigma_k(n) \sim ke^{\gamma}\sigma_{k-1}(n)\log\log\log n$ on a set of density 1, cf. P. Erdős [2],

– for $k \geq 1$ the normal order of $\dfrac{\phi_k(n)}{\phi_{k+1}(n)}$ is $ke^{\gamma}\log\log\log n$, proved by P. Erdős, A. Granville, C. Pomerance and C. Spiro [4].

– the normal order of $\dfrac{\phi(\sigma(n))}{\sigma(n)}$ is $e^{-\gamma}/\log\log\log n$ and the normal order of $\dfrac{\sigma(\phi(n))}{\phi(n)}$ is $e^{\gamma}\log\log\log n$, see L. Alaoglu and P. Erdős [1].

Note that the average orders of $\phi(n)/\phi_2(n)$ and $\phi_2(n)/\phi(n)$ were investigated by R. Warlimont [15].

**Theorem 5.** *Let $h(n)$ be an arithmetical function such that $n \leq h(n) \leq \sigma(n)$ for every sufficiently large $n$ ($n \geq n_0$). Then*

$$\liminf_{n\to\infty} \frac{h(\sigma(n))}{n} = 1. \tag{18}$$

For $h(n) = \sigma(n)$ this is formula (1), for $h(n) = \psi(n)$ it is due by J. Sándor [10], Theorem 3.30. Theorem 5 applies also for $h(n) = \sigma^*(n), \sigma^{(e)}(n)$.

**Theorem 6.**

$$\limsup_{n\to\infty} \frac{\phi(\phi^*(n))}{n} = \limsup_{n\to\infty} \frac{\phi^*(\phi(n))}{n} = \limsup_{n\to\infty} \frac{\phi^*(\phi^*(n))}{n} = 1. \tag{19}$$

Compare the results of (19) with (2).

Figure 2 is a plot of the functions $\phi^*(\phi(n))$ and $n$ for $1 \le n \le 10\,000$.

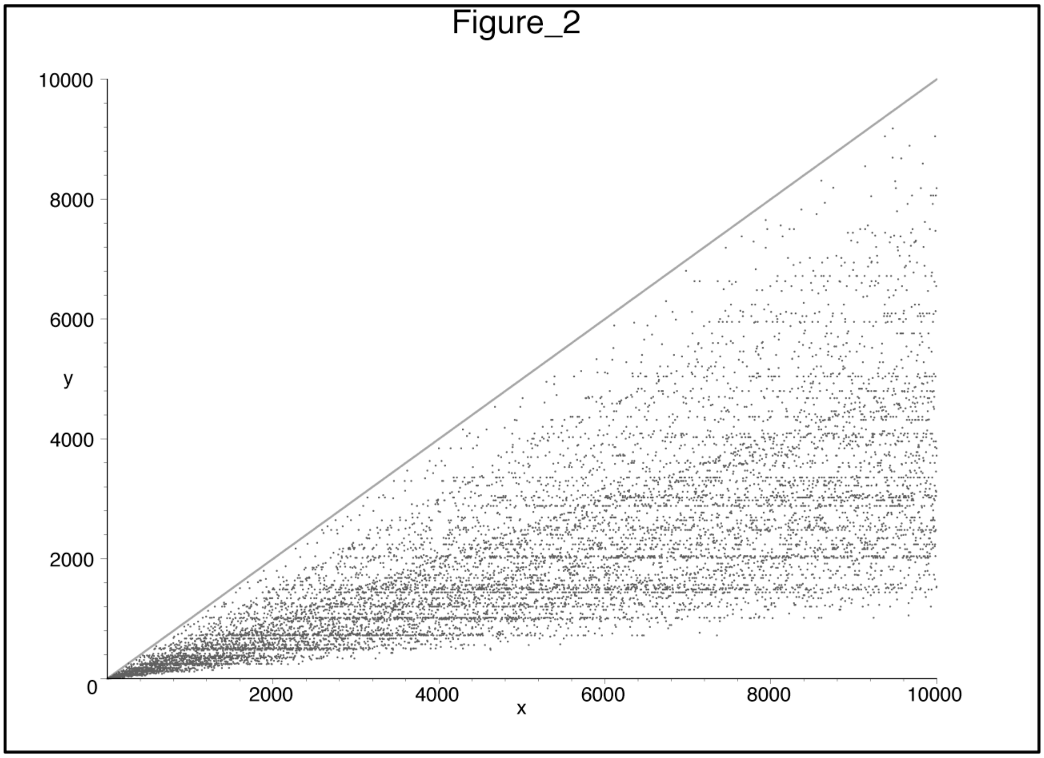


Figure 2: Plot of $\phi^*(\phi(n))$ and $n$ for $1 \le n \le 10\,000$

Concerning $\phi^*(\phi^*(n))$ and $\sigma^*(\phi^*(n))$ we also prove:

**Theorem 7.**

$$\liminf_{n\to\infty} \frac{\phi^*(\phi^*(n))}{\log n \log\log n} > 0. \tag{20}$$

**Theorem 8.**

$$\liminf_{n\to\infty} \frac{\sigma^*(\phi^*(n))}{n} \le \inf\left\{ \frac{\sigma^*(\phi^*(m/2))}{m/2} : 2 \mid m, m \ne 2^\ell, \ell \ge 2 \right\}, \tag{21}$$

$$\liminf_{n\to\infty} \frac{\sigma^*(\phi^*(n))}{n} \le \frac{1}{4} + \varepsilon, \tag{22}$$

*where* $\varepsilon = \dfrac{3}{4(2^{32}-1)} \approx 0.17 \cdot 10^{-9}$.

## 3. Proofs

The proofs of Theorems 1 and 2 are similar to the proof of (3) given in [7], using a simple argument based on Linnik's theorem, which states that if $(k,\ell)=1$, then there exists a prime $p$ such that $p \equiv \ell \pmod{k}$ and $p \ll k^c$, where $c$ is a constant (one can take $c \le 11$).

*Proof of Theorem 1.* To obtain the maximal orders of the functions $\sigma(n)/n$, $\psi(n)/n$, $\sigma(n)/\phi(n)$ and $\psi(n)/\phi(n)$, which are needed in the proof, we apply the following result of L. Tóth and E. Wirsing, see [13], Corollary 1:

If $F$ is a nonnegative real-valued multiplicative arithmetic function such that for each prime $p$,

a) $\rho(p) := \sup_{\nu \ge 0} F(p^\nu) \le (1-1/p)^{-1}$, and

b) there is an exponent $e_p = p^{o(1)}$ satisfying $F(p^{e_p}) \ge 1 + 1/p$,

then

$$\limsup_{n\to\infty} \frac{F(n)}{\log\log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right)\rho(p).$$

For $F(n) = \sigma(n)/n$ (with $\rho(p) = (1-1/p)^{-1}$, $e_p = 1$), $F(n) = \psi(n)/n$ (with $\rho(p) = 1 + 1/p$, $e_p = 1$), $F(n) = \sqrt{\sigma(n)/\phi(n)}$ (with $\rho(p) = (1-1/p)^{-1}$, $e_p = 1$) and $F(n) = \sqrt{\psi(n)/\phi(n)}$ (with $\rho(p) = \sqrt{(p+1)/(p-1)}$, $e_p = 1$), respectively, we obtain

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^\gamma, \tag{23}$$

$$\limsup_{n\to\infty} \frac{\psi(n)}{n\log\log n} = \frac{6}{\pi^2}e^\gamma, \tag{24}$$

$$\limsup_{n\to\infty} \frac{\sigma(n)}{\phi(n)(\log\log n)^2} = e^{2\gamma}, \tag{25}$$

$$\limsup_{n\to\infty} \frac{\psi(n)}{\phi(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}. \tag{26}$$

Note that (23) is the result of T. H. Gronwall [5], (26) is due to S. Wigert [16] and (25) is better than $\limsup_{n\to\infty} \sigma(n)/\phi(n) = \infty$ given in [11].

We now prove (6). Using assumption (ii),

$$\ell_f := \limsup_{n\to\infty} \frac{\sigma(f(n))}{n\log\log n} \le \ell'_f := \limsup_{n\to\infty} \frac{\sigma(f(n))}{f(n)\log\log f(n)} \le \limsup_{m\to\infty} \frac{\sigma(m)}{m\log\log m} = e^\gamma,$$

according to (23). For every $n$, let $p_n$ be the least prime such that $p_n \geq p_0$ and $p_n \equiv 1 \pmod{n}$. Here $n \mid p_n - 1$ and by Linnik's theorem $p_n \ll n^c$, so $\log\log p_n \sim \log\log n$. Hence, using condition (iii),

$$\frac{\sigma(f(p_n))}{p_n \log\log p_n} = \frac{\sigma(p_n-1)}{p_n \log\log p_n} \sim \frac{\sigma(p_n-1)}{(p_n-1)\log\log n} \geq \frac{\sigma(n)}{n \log\log n},$$

applying that if $s \mid t$, then $\sigma(s)/s = \sum_{d|s} 1/d \leq \sum_{d|t} 1/d = \sigma(t)/t$. We obtain that $\ell_f \geq e^{\gamma}$, therefore $e^{\gamma} \leq \ell_f \leq \ell'_f \leq e^{\gamma}$, that is $\ell_f = \ell'_f = e^{\gamma}$.

The proofs of (7), (8), (9). Analogous to the method of above taking into account (24), (25), (26) and that $s \mid t$ implies $\psi(s)/s \leq \psi(t)/t$, $\sigma(s)/\phi(s) \leq \sigma(t)/\phi(t)$, $\psi(s)/\phi(s) \leq \psi(t)/\phi(t)$. □

*Proof of Theorem 2.* This is similar to the proof of Theorem 1. We use a result of E. Landau [6],

$$\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma}. \tag{27}$$

By condition (ii) and using that the function $(\log\log x)/x$ is decreasing for $x \geq x_0$,

$$\begin{aligned}\ell_g := \liminf_{n\to\infty} \frac{\phi(g(n))\log\log n}{n} \geq \ell'_g &:= \liminf_{n\to\infty} \tfrac{\phi(g(n))\log\log g(n)}{g(n)} \\ &\geq \liminf_{m\to\infty} \tfrac{\phi(m)\log\log m}{m} = e^{-\gamma},\end{aligned}$$

according to (27).

Assume that $g(p) = p+1$ for every $p \geq p_0$. For every $n$, let $q_n$ be the least prime such that $q_n \geq p_0$ and $q_n \equiv -1 \pmod{n}$. Here $n \mid q_n + 1$ and by Linnik's theorem $\log\log q_n \sim \log\log n$. Hence

$$\frac{\phi(g(q_n))\log\log q_n}{q_n} = \frac{\phi(q_n+1)\log\log q_n}{q_n} \sim \frac{\phi(q_n+1)\log\log n}{q_n+1} \leq \frac{\phi(n)\log\log n}{n},$$

applying that if $s \mid t$, then $\phi(s)/s \geq \phi(t)/t$. We obtain that $e^{-\gamma} \geq \ell_g$, therefore $e^{-\gamma} \leq \ell'_g \leq \ell_g \leq e^{-\gamma}$, that is $\ell_g = \ell'_g = e^{-\gamma}$.

Now suppose that $g$ is multiplicative and $g(p) = p$ for every prime $p \geq p_0$. As it is known, in (27) the liminf is attained for $n = n_k = p_1 \cdots p_k$, the product of the first $k$ primes, when $k \to \infty$. Since $g(n_k) = g(p_1 \cdots p_k) = g(p_1)\cdots g(p_k) = p_1 \cdots p_k = n_k$, $\lim_{k\to\infty} \frac{\phi(g(n_k))\log\log n_k}{n_k} = \lim_{k\to\infty} \frac{\phi(n_k)\log\log n_k}{n_k} = e^{-\gamma}$. □

*Proof of Theorem 3.* By condition (i), $f_2(n) = f(f(n)) \leq f(n) \leq n$ and $f_k(n) \leq n$ for every $k \geq 0$. Therefore,

$$\ell_k := \limsup_{n\to\infty} \frac{\sigma(f_k(n))}{f_k(n)\log\log n} \leq \limsup_{n\to\infty} \frac{\sigma(f_k(n))}{f_k(n)\log\log f_k(n)} \leq \ell_0 := \limsup_{m\to\infty} \frac{\sigma(m)}{m\log\log m} = e^{\gamma},$$

by (23), for every $k \geq 0$.

By (iii), if $s \mid t$, then $f(s) \mid f(t)$, $f_2(s) \mid f_2(t)$ and $f_k(s) \mid f_k(t)$ for every $k \geq 0$. Now let $k \geq 1$. If $p_n$ is the least prime such that $p_n \equiv 1 \pmod{n}$, cf. proof of Theorem 1, then $n \mid p_n - 1$ and $f_{k-1}(n) \mid f_{k-1}(p_n - 1)$. Therefore, applying also (ii),

$$\frac{\sigma(f_k(p_n))}{f_k(p_n)\log\log p_n} \sim \frac{\sigma(f_{k-1}(p_n-1))}{f_{k-1}(p_n-1)\log\log n} \geq \frac{\sigma(f_{k-1}(n))}{f_{k-1}(n)\log\log n} = \ell_{k-1},$$

Hence $\ell_k \geq \ell_{k-1}$, and it follows $\ell_k \geq \ell_{k-1} \geq ... \geq \ell_0$, $\ell_0 \leq \ell_k \leq \ell_0$, $\ell_k = \ell_0 = e^{\gamma}$. □

*Proof of Theorem 4.* Similar to the proof of Theorem 3. By condition (i), $g_2(n) = g(g(n)) \geq g(n) \geq n$ and $g_k(n) \geq n$ for every $k \geq 0$. Therefore,

$$\begin{aligned} L_k := \liminf_{n\to\infty} \frac{\phi(g_k(n))\log\log n}{g_k(n)} &\geq \liminf_{n\to\infty} \tfrac{\phi(g_k(n))\log\log g_k(n)}{g_k(n)} \\ &\geq L_0 := \limsup_{m\to\infty} \tfrac{\phi(m)\log\log m}{m} = e^{-\gamma}, \end{aligned}$$

by (27), for every $k \geq 0$.

By (iii), if $s \mid t$, then $g(s) \mid g(t)$, $g_k(s) \mid g_k(t)$ for every $k \geq 0$. Now let $k \geq 1$. If $q_n$ is the least prime such that $q_n \equiv -1 \pmod{n}$, cf. proof of Theorem 2, then $n \mid q_n + 1$ and $g_{k-1}(n) \mid g_{k-1}(q_n + 1)$. Therefore, applying also (ii),

$$\frac{\phi(g_k(q_n))\log\log q_n}{g_k(q_n)} \sim \frac{\phi(g_{k-1}(q_n+1))\log\log n}{g_{k-1}(q_n+1)} \leq \frac{\phi(g_{k-1}(n))\log\log n}{g_{k-1}(n)} = L_{k-1},$$

Hence $L_k \leq L_{k-1}$, and it follows $L_k \leq L_{k-1} \leq ... \leq L_0$, $L_0 \leq L_k \leq L_0$, $L_k = L_0 = e^{-\gamma}$. □

*Proof of Theorem 5.* By $h(n) \geq n$ we have $h(\sigma(n)) \geq \sigma(n) \geq n$, $h(\sigma(n))/n \geq 1$ $(n \geq n_0)$. We use that for a fixed integer $a > 1$ and with $p$ prime, for $N(a,p) = \frac{a^p-1}{a-1}$ and for an arithmetical function satisfying $\phi(n) \leq F(n) \leq \sigma(n)$ $(n \geq n_0)$ one has

$$\lim_{p\to\infty} \frac{F(N(a,p))}{N(a,p)} = 1, \tag{28}$$

cf., for example, D. Suryanarayana [12].

For $p, q$ primes, $\sigma(q^{p-1}) = \frac{q^p-1}{q-1} = N(q,p)$. We obtain, using (28),

$$\frac{h(\sigma(q^{p-1}))}{q^{p-1}} = \frac{h(N(q,p))}{N(q,p))} \cdot \frac{q^p-1}{q^{p-1}(q-1)} \to \frac{q}{q-1}, \text{ as } p \to \infty,$$

where $\frac{q}{q-1} < 1 + \epsilon$ for each $\epsilon > 0$ if $q \geq q(\epsilon)$. □

*Proof of Theorem 6.* We have $\phi(n) \leq n$ and $\phi^*(n) \leq n$ for all $n \geq 1$, and hence $\phi(\phi^*(n)) \leq \phi^*(n) \leq n$. Similarly, $\phi^*(\phi^*(n)) \leq n$.

If $n = 2^p$, $p$ prime, then $\phi^*(n) = 2^p - 1$ and

$$\frac{\phi(\phi^*(n))}{n} = \frac{\phi(2^p-1)}{2^p} = \frac{\phi(2^p-1)}{2^p-1} \cdot \frac{2^p-1}{2^p} \to 1, \quad p \to \infty,$$

using (28) for $a = 2$ and $F(n) = \phi(n)$.

Similarly the relation holds for $\phi^*(\phi^*(n))$, using (28) for $F(n) = \phi^*(n)$.

For $\phi^*(\phi(n))$ this can not be applied and we need a special treatment.

Let $M = \prod_{p \le x} p^{a_p}$, where $a_p = \begin{cases} [2\log x], & \text{if } p < x^{1/2}, \\ 4, & \text{if } p \in [x^{1/2}, x] \end{cases}$ ($p$ prime).

Let $q$ be the least prime of the form $q \equiv M+1 \pmod{M^2}$. By Linnik's theorem one has $q \ll M^c$, where $c$ satisfies $c \le 11$.

Now, put $n = q$. Then $\phi(n) = q - 1 = M(1 + kM) = MN$ for some $k$. Thus $(M, N) = 1$, so $N$ is free of prime factors $\le x$. Since $\phi^*$ is multiplicative, $\dfrac{\phi^*(\phi(n))}{n} = \dfrac{\phi^*(M)}{M} \cdot \dfrac{\phi^*(N)}{N} \cdot \dfrac{MN}{1+MN}$. Here $\dfrac{MN}{1+MN} \to 1$, as $n = q \to \infty$, so it is sufficient to study $\dfrac{\phi^*(M)}{M}$ and $\dfrac{\phi^*(N)}{N}$. Clearly, $\dfrac{\phi^*(M)}{M} = \prod_{p \le x} \dfrac{p^{a_p}-1}{p^{a_p}} = \prod_{p \le x}\left(1 - \dfrac{1}{p^{a_p}}\right)$. If $p < x^{1/2}$, then $p^{a_p} \ge 2^{[2\log x]} > x$ for sufficiently large $x$. Otherwise, $p^{a_p} \ge (x^{1/2})^4 = x^2 > x$ again. So $p^{a_p} > x$ anyway, implying that

$$\frac{\varphi^*(M)}{M} > \left(1 - \frac{1}{x}\right)^{\pi(x)} = 1 + \mathcal{O}\left(\frac{1}{\log x}\right). \tag{29}$$

Remark that $M < \prod_{p < x^{1/2}} p^{2\log x} \cdot \prod_{p \le x} p^4 < \exp\big(\mathcal{O}(x^{1/2}\log x + x)\big) = \exp\big(\mathcal{O}(x)\big)$ by the well-known fact: $\prod_{p \le a} p = e^{\mathcal{O}(a)}$. From $q \ll M^{c'}$ and $M < \exp\big(\mathcal{O}(x)\big)$, by $N \ll M^{10}$ it follows also that

$$N < \exp\big(\mathcal{O}(x)\big). \tag{30}$$

Let now $N = \prod_{i=1}^{k} q_i^{b_i}$ be the prime factorization of $N$. We have $\log N = \sum_{i=1}^{k} b_i \log q_i > (\log x)\sum_{i=1}^{k} b_i$, as $q_i > x$ for all $1 \le i \le k$. Here $\sum_{i=1}^{k} b_i \ge k$, thus $k < \dfrac{\log N}{\log x} \ll \dfrac{x}{\log x}$ by (30). Thus

$$\frac{\phi^*(N)}{N} = \prod_{i=1}^{k}\left(1 - \frac{1}{q_i^{b_i}}\right) > \left(1 - \frac{1}{x}\right)^{k} \ge \left(1 - \frac{1}{x}\right)^{\mathcal{O}(x/\log x)} = 1 + \mathcal{O}\left(\frac{1}{\log x}\right). \tag{31}$$

By (29) and (31), $\dfrac{\phi^*(\phi(n))}{n} > 1+\mathcal{O}\left(\dfrac{1}{\log x}\right)$ for sufficiently large $n$. As $n \ll \exp\big(\mathcal{O}(x)\big)$, we get $\log n \ll x$, so $\dfrac{\phi^*(\phi(n))}{n} \to 1$, as $n = q \to \infty$.

As $\dfrac{\phi^*(\phi(n))}{n} \leq \dfrac{\phi(n)}{n} \leq 1$, the proof is ready. □

*Proof of Theorem 7.* For all $n \geq 1$, $\phi^*(n) \geq P(n) - 1$, where $P(n)$ is the greatest prime factor of $n$. Let $n = 2^p$, $p$ prime, then $\phi^*(\phi^*(n)) = \phi^*(2^p - 1) \geq P(2^p - 1) - 1$. Now we use the following result of P. Erdős and T. N. Shorey [3]: $P(2^p - 1) \geq cp\log p$ for every prime $p$, where $c > 0$ is an absolute constant, and obtain

$$\frac{\phi^*(\phi^*(n))}{\log n \log\log n} \geq \frac{cp\log p - 1}{p\log 2(\log p + \log\log 2)} \to \frac{c}{\log 2}, \quad p \to \infty, \tag{32}$$

and the result follows. □

*Proof of Theorem 8.* To prove (21), remark that if $2 \mid m$ and $m \neq 2^\ell$ $(\ell \geq 2)$, then $m/2$ is not a power of 2, so $\phi^*(m/2)$ will be even (having at least an odd prime divisor). Since $2 \mid \phi^*(m/2)$, one can write $\sigma^*(2\phi^*(m/2)) < 2\sigma^*(\phi^*(m/2))$. Let $p$ be a sufficiently large prime $(p > p_0)$, then $(p, m/2) = 1$ and obtain

$$\frac{\sigma^*(\phi^*(mp/2))}{mp/2} = \frac{\sigma^*((p-1)\phi^*(m/2))}{mp/2} \leq$$

$$\leq \frac{\sigma^*((p-1)/2)\sigma^*(2\phi^*(m/2))}{mp/2} \leq \frac{\sigma^*((p-1)/2)}{p/2} \cdot \frac{\sigma^*(\phi^*(m/2))}{m/2}$$

by the above remark.

It is known that $\dfrac{F((p-1)/2)}{(p-1)/2} \to 1$, as $p \to \infty$, for $F(n) = \sigma(n)$, see [9] and it follows that it holds also for $F(n) = \sigma^*(n)$ and obtain (21).

Now for (22) let $m = 4(2^{32} - 1) = 4F_0F_1F_2F_3F_4$ be 4 times the product of the known Fermat primes. Then $\phi^*(m/2) = \phi^*(2F_0F_1F_2F_3F_4) = 2^{1+2+4+8+16} = 2^{31}$, $\dfrac{\sigma^*(\phi^*(m/2))}{m/2} = \dfrac{2^{31}+1}{2(2^{32}-1)} = \dfrac{1}{4} + \varepsilon$, with the given value of $\varepsilon$. □

## 4. Open Problems

**Problem 1.** Are the results of Theorem 1 valid if $f(n) \leq n$ for each $n \geq n_0$ and $f(p) = p$ for each prime $p \geq p_0$?

Let $n = p_1^{\nu_1} \cdots p_r^{\nu_r} > 1$ be an integer. An integer $a$ is called regular (mod $n$) if there is an integer $x$ such that $a^2x \equiv a \pmod{n}$. Let $\varrho(n)$ denote the number of regular integers $a$ (mod

$n$) such that $1 \le a \le n$. Here $\varrho(n) = (\phi(p_1^{\nu_1}) + 1) \cdots (\phi(p_r^{\nu_r}) + 1)$, in particular $\varrho(p) = p$ for every prime $p$, cf. L. Tóth [14].

Does Theorem 1 hold for $f(n) = \varrho(n)$?

**Problem 2.** The method of proof of Theorems 1–4 does not work in the cases of $\sigma^*(\phi(n))$ and $\sigma^*(\phi^*(n))$, for example. We have

$$\limsup_{n\to\infty} \frac{\sigma^*(\phi(n))}{n \log\log n} \le \limsup_{n\to\infty} \frac{\sigma^*(\phi(n))}{\phi(n) \log\log \phi(n)} \le \limsup_{n\to\infty} \frac{\sigma^*(n)}{n \log\log n} = \frac{6}{\pi^2} e^{\gamma},$$

cf. [13], but the second part of the proof can not be applied, because $n \mid m$ does not imply $\sigma^*(n)/n \le \sigma^*(m)/m$.

What are the maximal orders $\sigma^*(\phi(n))$ and $\sigma^*(\phi^*(n))$?

Figure 3 is a plot of the function $\sigma^*(\phi(n))$ for $1 \le n \le 10\,000$.

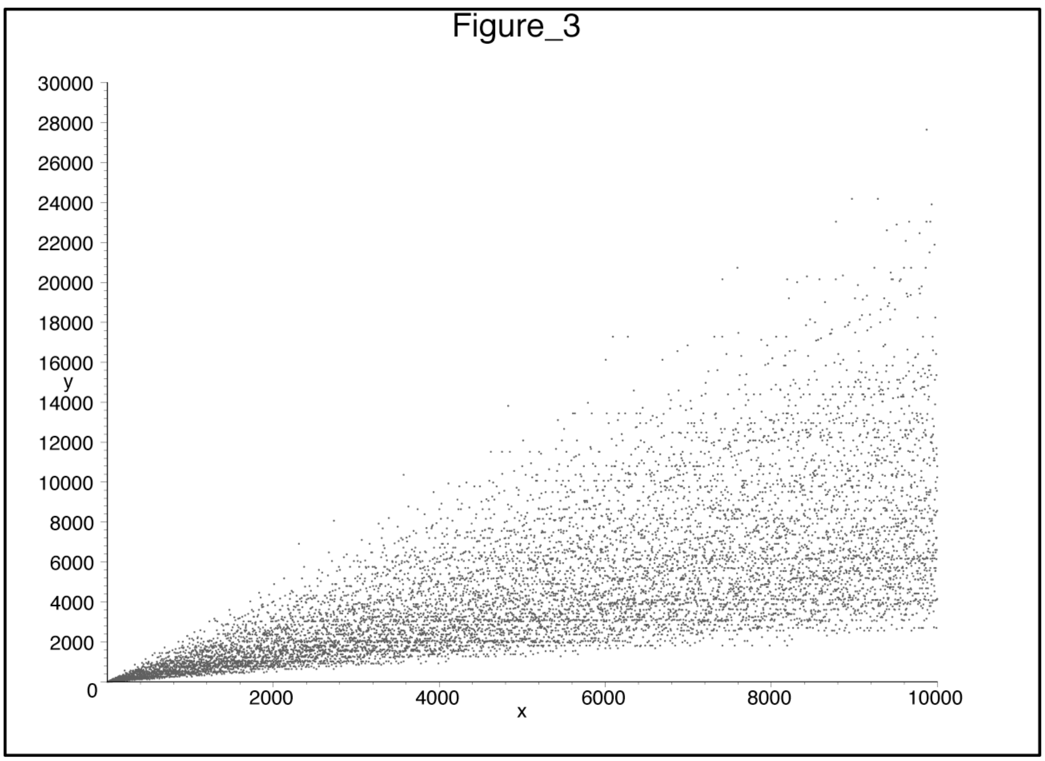


Figure 3: Plot of $\sigma^*(\phi(n))$ for $1 \le n \le 10\,000$

**Problem 3.** Note that

$$\limsup_{n\to\infty} \frac{\sigma^*(\sigma(n))}{n} = \limsup_{n\to\infty} \frac{\sigma(\sigma^*(n))}{n} = \limsup_{n\to\infty} \frac{\sigma^*(\sigma^*(n))}{n} = \infty,$$

since for $n = n_k = p_1 \cdots p_k$ (the product of the first $k$ primes),

$$\frac{\sigma^*(\sigma(n_k))}{n_k} \geq \frac{\sigma(n_k)}{n_k} = (1+1/p_1)\cdots(1+1/p_k) \to \infty, \quad k \to \infty;$$

similarly, the other relations hold.

What are the maximal orders of $\sigma(\sigma^*(n))$, $\sigma^*(\sigma(n))$, $\sigma^*(\sigma^*(n))$?

**Problem 4.** Also,

$$\liminf_{n\to\infty} \frac{\phi(\phi^*(n))}{n} = \liminf_{n\to\infty} \frac{\phi^*(\phi(n))}{n} = \liminf_{n\to\infty} \frac{\phi^*(\phi^*(n))}{n} = 0,$$

which follow at once by taking $n = n_k = p_1 \cdots p_k$. Here $\phi^*(\phi(n_k)) = \phi^*((p_1-1)\cdots(p_k-1)) \leq (p_1-1)\cdots(p_k-1)-1$, and hence

$$\frac{\phi^*(\phi(n_k))}{n_k} \leq \frac{(p_1-1)\cdots(p_k-1)-1}{p_1\cdots p_k} < \left(1-\frac{1}{p_1}\right)\cdots\left(1-\frac{1}{p_k}\right) \to 0, \quad k \to \infty,$$

and similarly for the other relations.

What are the minimal orders of $\phi(\phi^*(n))$, $\phi^*(\phi(n))$, $\phi^*(\phi^*(n))$?

## 5. Maple Notes

The plots were produced using Maple. The functions $\sigma^*(n)$ and $\phi^*(n)$ were generated by the following procedures:

```
sigmastar:= proc(n) local x, i: x:= 1: for i from 1 to nops(ifactors(n)[ 2 ]) do
p_i:=ifactors(n)[2][i][1]: a_i:=ifactors(n)[2][i][2];
x := x*(1+p_i^(a_i)): od: RETURN(x) end; # sum of unitary divisors

phistar:= proc(n) local x, i: x:= 1: for i from 1 to nops(ifactors(n)[ 2 ]) do
p_i:=ifactors(n)[2][i][1]: a_i:=ifactors(n)[2][i][2];
x := x*(p_i^(a_i)-1): od: RETURN(x) end; # unitary Euler function
```

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