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DISTANCE IN THE AFFINE BUILDINGS OF SL_n AND Sp_n

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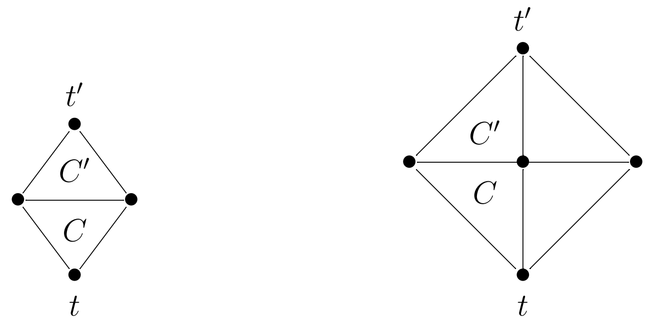
Abstract

For a local field K and $n \geq 2$, let Ξ_n and Δ_n denote the affine buildings naturally associated to the special linear and symplectic groups $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively. We relate the number of vertices in Ξ_n ($n \geq 3$) close (i.e., gallery distance 1) to a given vertex in Ξ_n to the number of chambers in Ξ_n containing the given vertex, proving a conjecture of Schwartz and Shemanske. We then consider the special vertices in Δ_n ($n \geq 2$) close to a given special vertex in Δ_n (all the vertices in Ξ_n are special) and establish analogues of our results for Δ_n.

1. Introduction

A building is a finite-dimensional simplicial complex in which any two of its chambers (maximal simplices) can be connected by a gallery. In other words, if Δ is a building, then for any chambers $C, D \in \Delta$, there is a sequence $C = C_0, C_1, \ldots, C_m = D$ of chambers in Δ such that C_i and C_{i+1} are adjacent (share a codimension-one face) for all $0 \leq i \leq m-1$; in this case, the number m is the length of the gallery $C_0, \ldots, C_m$. The combinatorial distance between C and D is the minimal length of a gallery in Δ connecting C and D (see [1, p. 14]). Following [1, p. 15], define the *distance* between any non-empty simplices $A, B \in \Delta$ to be the minimal length of a gallery in Δ starting at a chamber containing A and ending at a chamber containing B (cf. [6, p. 125]). Then the vertices $t, t' \in \Delta$ are distance one apart or *close* if and only if there are adjacent chambers $C, C' \in \Delta$ such that $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$ (the simplex shared by C and C'); i.e., if and only if t and t' are in adjacent chambers in Δ but not a common one (cf. [6, p. 127]). Figures 1(a) and 1(b) show close vertices in the affine buildings naturally associated to $\mathrm{SL}_3(K)$ and $\mathrm{Sp}_2(K)$, respectively, for any local field K. Note that if Δ is a building and $t, t' \in \Delta$ are close vertices, then as vertices in the underlying graph of Δ, t and t' are not graph distance 1 apart but are always graph distance 2 apart.

Let K be a local field with valuation ring $\mathcal{O}$, uniformizer π, and residue field $k \cong \mathbb{F}_q$, and let Ξ_n denote the affine building naturally associated to $\mathrm{SL}_n(K)$. Schwartz and Shemanske



(a) Two close vertices in Ξ_3. (b) Two close vertices in Δ_2.

Figure 1: Examples of close vertices.

[6, Theorem 3.3] show that for all $n \geq 3$, the number ω_n of vertices in Ξ_n close to a given vertex in Ξ_n is the number of right cosets of $\mathrm{GL}_n(\mathcal{O})$ in $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \dots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$; i.e., the Hecke operator $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \dots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$ acts as a generalized adjacency operator on Ξ_n. They also conjecture that for all $n \geq 3$, $q \cdot r_n = r_{n-2}\, \omega_n$, where r_n is the number of chambers in Ξ_n containing a given vertex, with $r_1 := 1$ (see the remark following [6, Proposition 3.4]).

In Section 2, we prove Schwartz and Shemanske's conjecture in two ways. Our first approach is via module theory. More precisely, we use the description of the chambers in Ξ_n in terms of lattices in an n-dimensional K-vector space (see, for example, [5, p. 115]) to obtain an explicit formula for ω_n (Proposition 2.1); together with Schwartz and Shemanske's formula for r_n [6, Proposition 2.4], this proves Theorem 2.1. Our second approach is through combinatorics (Theorem 2.2). Specifically, we show that if $t, t' \in \Xi_n$ are close vertices, then there is a one-to-one correspondence between the galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' and the chambers in the spherical $A_{n-3}(k)$ building. This gives an explanation for the relationship between ω_n and r_n in terms of the structure of Ξ_n. In Section 3, we consider the special vertices in the affine building Δ_n naturally associated to $\mathrm{Sp}_n(K)$ ($n \geq 2$) close to a given special vertex in Δ_n (all the vertices in Ξ_n are special). Using the fact that Δ_n is a subcomplex of Ξ_{2n}, we adapt the proofs of the results for close vertices in Ξ_{2n} to prove analogues for Δ_n. In particular, we establish analogues of [6, Theorem 3.3] and Theorem 2.1 (Theorems 3.1 and 3.2, respectively) and a partial analogue of Theorem 2.2 (Proposition 3.11). Note that while every vertex in Ξ_{2n} is special, only two vertices in each chamber in Δ_n are special; hence, our analysis for Δ_n requires more care than that needed for Ξ_{2n}.

After proving Theorems 2.1 and 3.2, we learned that the formulas in Propositions 2.1 and 3.9 are both special cases of a result of Parkinson [4, Theorem 5.15] and that the formula in Proposition 2.1 also follows from a result of Cartwright [2, Lemma 2.2]. We view the buildings Ξ_n and Δ_n as combinatorial objects naturally associated to $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively, and make use of the lattice descriptions of these buildings (see [3] and [5]). As a result, our methods require little more than the definition of a building—namely, some module theory. In contrast to our approach, Cartwright views Ξ_n in terms of hyperplanes, affine transformations, and convex hulls, and Parkinson considers buildings via root systems and Poincaré polynomials of Weyl groups. The numbers ω_n and $\omega(\Delta_n)$ that we use are special cases of Parkinson's N_λ, which he uses to define vertex set averaging operators on arbitrary locally finite, regular affine buildings and whose formula he uses to prove results about those operators.

I thank Paul Garrett for the idea behind the proof of Proposition 2.1, and hence that of Proposition 3.9. Finally, the results contained here form part of my doctoral thesis, which I wrote under the guidance of Thomas R. Shemanske.

2. Close Vertices in the Affine Building Ξ_n of $\mathrm{SL}_n(K)$

From now on, K is a local field with discrete valuation "ord," valuation ring $\mathcal{O}$, uniformizer π, and residue field $k \cong \mathbb{F}_q$. For any finite-dimensional K-vector space V, define a *lattice* in V to be a free $\mathcal{O}$-submodule of V of rank $\dim_K V$, with two lattices L and L' in V *homothetic* if $L' = \alpha L$ for some $\alpha \in K^\times$; write $[L]$ for the homothety class of the lattice L.

The affine building Ξ_n naturally associated to $\mathrm{SL}_n(K)$ can be modeled as an $(n-1)$-dimensional simplicial complex as follows (see [5, p. 115]). Let V be an n-dimensional K-vector space. Then a *vertex* in Ξ_n is a homothety class of lattices in V, and two vertices $t, t' \in \Xi_n$ are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that $\pi L \subseteq L' \subseteq L$; i.e., such that $L'/\pi L$ is a k-subspace of $L/\pi L$. Thus, a *chamber* (maximal simplex) in Ξ_n has n vertices $t_0, \dots, t_{n-1}$ with representatives $L_i \in t_i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ and $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n-1$. From now on, write that a chamber in Ξ_n corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ only when the lattices $L_0, \dots, L_{n-1}$ satisfy the conditions in the last sentence.

For the rest of this section, $n \geq 3$. Let $t \in \Xi_n$ be a vertex with representative L. Then a chamber $C \in \Xi_n$ containing t corresponds to a chain of the form

$$\pi L \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1} \overset{q}{\subsetneq} L \tag{1}$$

(cf. [3, p. 323]). The codimension-one face in C not containing t thus corresponds to the chain

$$L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1},$$

and a vertex in Ξ_n is close to t if it has a representative $M \neq L$ such that

$$\pi M \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_{n-1} \overset{q}{\subsetneq} M. \tag{2}$$

Given the lattices L_1 and L_{n-1}, the possible L and M satisfy $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1}L_1$. On the other hand, if $t, t' \in \Xi_n$ are close vertices, then there must be representatives $L \in t$ and

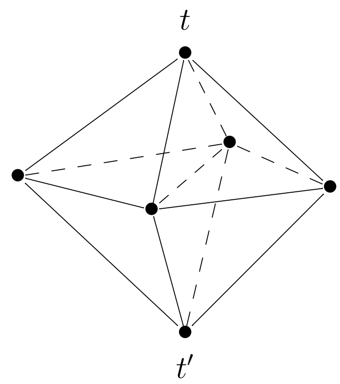


Figure 2: Two close vertices in Ξ_4.

$M \in t'$ and lattices $L_1, \ldots, L_{n-1}$ as in (1) such that $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1}L_1$. Recall that if M_1 and M_2 are free, rank n, $\mathcal{O}$-modules with $M_1 \subseteq M_2$, then $M_1 \subseteq M' \subseteq M_2$ implies M' is also a free, rank n, $\mathcal{O}$-module. Thus, both $L \cap M$ and $L + M$ are lattices in V. Furthermore, $L \neq M$ and $[L : L_{n-1}] = q = [M : L_{n-1}]$ imply $L \cap M = L_{n-1}$ and $L + M = \pi^{-1}L_1$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. In other words, if t and t' are close vertices in Ξ_n, there may be two (or more) pairs of adjacent chambers C and C' in Ξ_n with $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$ (see Figure 2). We return to this later.

Before we count the number of vertices in Ξ_n close to a given vertex $t \in \Xi_n$, we make a few observations. Fix a representative $L \in t$. Since $L/\pi L \cong k^n$, the Correspondence Theorem and the fact that any $\mathcal{O}$-submodule of L containing πL is a lattice in V imply that the number of L_1 is the number of 1-dimensional k-subspaces of $L/\pi L$. Similarly, given L_1 as above, the number of lattices L_{n-1} with $L_1 \subsetneq L_{n-1} \subsetneq L$ and $[L : L_{n-1}] = q$ is the number of $(n-2)$-dimensional k-subspaces of $L/L_1 \cong k^{n-1}$. Finally, given L_1 and L_{n-1} as above, the number of lattices $M \neq L$ such that $L_{n-1} \subsetneq M \subsetneq \pi^{-1}L_1$ is one less than the number of non-trivial, proper k-subspaces of $\pi^{-1}L_1/L_{n-1} \cong k^2$.

Proposition 2.1. If $t \in \Xi_n$ is a vertex, then the number ω_n of vertices in Ξ_n close to t is

$$\frac{q^n - 1}{q - 1} \cdot \frac{q^{n-1} - 1}{q - 1} \cdot q$$

(independent of t).

Proof. This follows from the preceding comments, duality, and the fact that the number of 1-dimensional subspaces of $\mathbb{F}_q^m$ is exactly $(q^m - 1)/(q - 1)$. □

Corollary 2.1. The number of right cosets of $\mathrm{GL}_n(\mathcal{O})$ in $\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O})$ is $((q^n - 1)(q^{n-1} - 1) \cdot q)/(q - 1)^2$.

Proof. This follows from [6, Theorem 3.3] and the last proposition. □

Let r_n be the number of chambers in Ξ_n containing a vertex $t \in \Xi_n$. Then [6, Proposition 2.4] and the last proposition establish the conjecture following Proposition 3.4 of [6]:

Theorem 2.1. For all $n \geq 3$, $q \cdot r_n = r_{n-2}\, \omega_n$, where $r_1 = 1$.

We now use the structure of Ξ_n to give a combinatorial proof for the relationship given in Theorem 2.1. Fix a vertex $t \in \Xi_n$. Then we can try to count the number of vertices in Ξ_n close to t by counting the number of galleries (in Ξ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. By definition, there are r_n chambers $C \in \Xi_n$ containing t. Since a chamber in Ξ_n adjacent to C and not containing t must contain the codimension-one face in C not containing t, [3, p. 324] implies that there are q chambers in Ξ_n adjacent to C not containing t; hence, there are exactly $r_n \cdot q$ galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber does not contain t. On the other hand, if $t' \in \Xi_n$ is a vertex close to t, we count t' more than once if there is more than one gallery of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' (see Figure 2); hence, $\omega_n = (r_n \cdot q)/m(t, t')$, where $m(t, t')$ is the number of galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t'.

To determine $m(t, t')$, fix the following notation for the rest of this section. For close vertices $t, t' \in \Xi_n$, let $L \in t$, $M \in t'$ be representatives such that there are lattices $L_1, \ldots, L_{n-1}$ as in (1) and (2). Recall that $L_1 = \pi(L + M)$ and $L_{n-1} = L \cap M$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. Since any gallery C, C' in Ξ_n such that $C = \{t, [L_1], \ldots, [L_{n-1}]\}$ and $C' = \{t', [L_1], \ldots, [L_{n-1}]\}$ satisfies $C \cap C' = \{[L_1], \ldots, [L_{n-1}]\}$, each gallery in Ξ_n counted by $m(t, t')$ is uniquely determined by the lattices $L_2, \ldots, L_{n-2}$. Define two vertices in Ξ_n to be *adjacent* if they are distinct and incident.

Proposition 2.2. Let $t, t' \in \Xi_n$ be adjacent vertices. If $L \in t$, then there is a unique representative $L' \in t'$ such that $\pi L \subsetneq L' \subsetneq L$.

Proof. Since t and t' are incident and $t \neq t'$, there are representatives $M \in t$ and $M' \in t'$ such that $\pi M \subsetneq M' \subsetneq M$. Moreover, M and L are homothetic, so $L = \alpha M$ for some $\alpha \in K^\times$; hence, $\pi L \subsetneq \alpha M' \subsetneq L$. Let $L' = \alpha M'$. If $L'' \in t'$ such that $\pi L \subsetneq L'' \subsetneq L$, let $\beta \in K^\times$ such that $L'' = \beta L'$. Suppose $\operatorname{ord}(\beta) = m$. Then $\pi L \subsetneq L' \subsetneq L$ implies $\pi^{m+1} L \subsetneq L'' \subsetneq \pi^m L$ and $L = \pi^m L$; i.e., $L'' = L'$. $\square$

Consider the set of vertices in Ξ_n that are adjacent to $t, t', [L + M]$, and $[L \cap M]$ (in the case $n = 3$, this set is empty), and define two such vertices to be incident if they are incident as vertices in Ξ_n. Let $\Xi_n^c(t, t')$ be the set consisting of

- the empty set,
- all vertices in Ξ_n adjacent to $t, t', [L + M]$, and $[L \cap M]$, and
- all finite sets A of vertices in Ξ_n adjacent to $t, t', [L + M]$, and $[L \cap M]$ such that any two vertices in A are adjacent.

Then $\Xi_n^c(t, t')$ is a simplicial complex. In particular, $\Xi_n^c(t, t')$ is a subcomplex of Ξ_n.

Lemma 2.1. If $\emptyset \neq A \in \Xi_n^c(t,t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$. In particular, A has at most $n-3$ vertices.

Proof. We proceed by induction on i. If $i = 0$, then A adjacent to $[L \cap M]$ implies A has a unique representative M_1 such that $\pi(L \cap M) \subsetneq M_1 \subsetneq L \cap M$ by Proposition 2.2. Then by [3, p. 322], either $M_1 \subsetneq \pi(L+M)$ or $M_1 \supsetneq \pi(L+M)$. In the second case, we are done, so assume $M_1 \subsetneq \pi(L+M)$. Then $\pi(L \cap M) \subsetneq M_1 \subsetneq \pi(L+M)$. On the other hand, $\pi(L \cap M) \subsetneq \pi L \subsetneq \pi(L+M)$ and $[\pi(L+M) : \pi(L \cap M)] = q^2$. Since A is adjacent to t, [3, p. 322] implies that either $M_1 \subsetneq \pi L$ or $M_1 \supsetneq \pi L$. Thus, either $\pi(L \cap M) \subsetneq M_1 \subsetneq \pi L \subsetneq \pi(L+M)$ or $\pi(L \cap M) \subsetneq \pi L \subsetneq M_1 \subsetneq \pi(L+M)$, which is impossible given the previous index computation.

Now suppose $0 \leq i \leq n-5$ and that the claim holds for any i-simplex in $\Xi_n^c(t,t')$. Let $A \in \Xi_n^c(t,t')$ be an $(i+1)$-simplex and $x \in A$ a vertex. Then the i-simplex $A - \{x\}$ corresponds to a chain of lattices $M_1' \subsetneq \cdots \subsetneq M_{i+1}'$ such that $\pi(L+M) \subsetneq M_1' \subsetneq \cdots \subsetneq M_{i+1}' \subsetneq L \cap M$. By the last paragraph, x has a representative M' such that $\pi(L+M) \subsetneq M' \subsetneq L \cap M$. If $M' \subsetneq M_1'$, set $M_1 = M'$ and $M_j = M_{j-1}'$ for all $2 \leq j \leq i+2$. Otherwise, $M' \supsetneq M_1'$ by [3, p. 322]. Let $j \in \{1, \ldots, i+1\}$ be maximal such that $M' \supsetneq M_j'$. If $j = i+1$, set $M_\ell = M_\ell'$ for all $1 \leq \ell \leq i+1$ and $M_{i+2} = M'$. Setting $M_\ell = M_\ell'$ for all $1 \leq \ell \leq j$, $M_{j+1} = M'$, and $M_\ell = M_{\ell-1}'$ for all $j+2 \leq \ell \leq i+2$ finishes the proof if $j \neq i+1$. Finally, note that if the claim holds for $i \geq n-3$, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$, contradicting the fact that $[L \cap M : \pi(L+M)] = q^{n-2}$. □

Write $\Xi_n^s(k)$ for the spherical $A_n(k)$ building described in [5, p. 4].

Proposition 2.3. For any close vertices $t, t' \in \Xi_n$, $\Xi_n^c(t,t')$ is isomorphic (as a poset) to $\Xi_{n-3}^s(k)$ (independent of t and t'), where $\Xi_0^s(k) = \emptyset$.

Proof. Let $L \in t, M \in t'$ be as in the paragraph preceding Proposition 2.2, and let $\Xi_{n-3}^s(k)$ be the spherical $A_{n-3}(k)$ building with simplices the empty set, together with the nested sequences of non-trivial, proper k-subspaces of $(L \cap M)/\pi(L+M)$. Then by the Correspondence Theorem and the last lemma, there is a bijection between the i-simplices in $\Xi_n^c(t,t')$ and the i-simplices in $\Xi_{n-3}^s(k)$ for all i. Since this bijection preserves the partial order (face) relation, it is a poset isomorphism. □

Theorem 2.2. If $t, t' \in \Xi_n$ are close vertices, then $m(t,t') = r_{n-2}$ (independent of t and t'). In particular, $\omega_n = (r_n \cdot q)/r_{n-2}$.

Proof. By the last proposition and previous comments, $m(t,t')$ is the number of chambers in $\Xi_{n-3}^s(k)$. The proof now follows from [6, Proposition 2.4]. □

3. Close Vertices in the Affine Building Δ_n of $\mathrm{Sp}_n(K)$

Let Δ_n denote the affine building naturally associated to $\mathrm{Sp}_n(K)$. Then Δ_n is a subcomplex of Ξ_{2n}, and there is a natural embedding of Δ_n in Ξ_{2n}. As we will see, this embedding allows us to derive information about Δ_n and to prove results for Δ_n by adapting the proofs of the analogous results for Ξ_{2n}. As noted in the introduction, while all the vertices in Ξ_{2n} are special, only two vertices in each chamber in Δ_n are special. Consequently, the Sp_n case requires more care than that needed in the last section. We start by looking at properties of Δ_n that we need to consider close vertices in Δ_n.

3.1 The Building Δ_n

The building Δ_n can be modeled as an n-dimensional simplicial complex as follows (see [3, pp. 336 – 337]). Fix a $2n$-dimensional K-vector space V endowed with a non-degenerate, alternating bilinear form $\langle\cdot,\cdot\rangle$, and recall that a subspace U of V is *totally isotropic* if $\langle u, u'\rangle = 0$ for all $u, u' \in U$. A lattice L in V is *primitive* if $\langle L, L\rangle \subseteq \mathcal{O}$ and $\langle\cdot,\cdot\rangle$ induces a non-degenerate, alternating k-bilinear form on $L/\pi L$. Then a *vertex* in Δ_n is a homothety class of lattices in V with a representative L such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$; equivalently, $L/\pi L_0$ is a totally isotropic k-subspace of $L_0/\pi L_0$. Two vertices $t, t' \in \Delta_n$ are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that there is a primitive lattice L_0 with $\langle L, L\rangle \subseteq \pi\mathcal{O}$, $\langle L', L'\rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq L \subseteq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subseteq L \subseteq L_0$. Thus, a *chamber* in Δ_n has $n+1$ vertices $t_0, \ldots, t_n$ with representatives $L_i \in t_i$ such that L_0 is primitive, $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \le i \le n$, and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. From now on, write that a chamber in Δ_n corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ only when the lattices $L_0, \ldots, L_n$ satisfy the conditions in the last sentence.

Recall that a basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V is *symplectic* if $\langle u_i, w_j\rangle = \delta_{ij}$ (Kronecker delta) and $\langle u_i, u_j\rangle = 0 = \langle w_i, w_j\rangle$ for all i, j. If a 2-dimensional, totally isotropic subspace U of V is a hyperbolic plane, then a frame is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines (1-dimensional K-subspaces) in V such that

1. $\lambda_i^1 + \lambda_i^2$ is a hyperbolic plane for all $1 \le i \le n$,
2. $\lambda_i^1 + \lambda_i^2$ is orthogonal to $\lambda_j^1 + \lambda_j^2$ for all $i \ne j$, and
3. $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$.

A vertex $t \in \Delta_n$ lies in the apartment specified by the frame $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ if for any representative $L \in t$, there are lattices M_i^j in λ_i^j for all i, j such that $L = M_1^1 + M_1^2 + \cdots + M_n^1 + M_n^2$. The following lemma is easily established.

Lemma 3.1.

1. Every symplectic basis for V specifies an apartment of Δ_n.
2. If Σ is an apartment of Δ_n, there is a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for V such that every vertex in Σ has the form

$$[\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n]$$

for some $a_i, b_i \in \mathbb{Z}$.

Remark. A frame specifying an apartment of Δ_n also specifies an apartment of Ξ_{2n} (see [3, p. 323]). In particular, a symplectic basis for V specifies an apartment of Ξ_{2n}.

Since π is fixed, if $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis for V, follow [7, p. 3411] and write $(a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ for the lattice $\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$ and $[a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}}$ for its homothety class. Then the lattice $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ is primitive if and only if $a_i + b_i = 0$ for all i by [7, p. 3411], and $[L]$ is a *special* vertex in Δ_n if and only if $a_i + b_i = \mu$ is constant for all i by [7, Corollary 3.4]. Note that by [7, p. 3412], a chamber in Δ_n has exactly two special vertices.

Lemma 3.2. Let $t \in \Delta_n$ be a vertex with a primitive representative L, and let Σ be an apartment of Δ_n containing t. Then there is a symplectic basis $\mathcal{B}$ for V specifying Σ as in Lemma 3.1 such that $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$.

Proof. This follows from Lemma 3.1 and [7, p. 3411]. □

Let $t \in \Delta_n$ be a vertex. Then the link of t in Δ_n, denoted $\mathrm{lk}_{\Delta_n}t$, is a building (see [1, Proposition IV.1.3]) that is isomorphic (as a poset) to the subposet of Δ_n consisting of those simplices containing t by [1, p. 31]. In particular, if $A \in \Delta_n$ is a codimension-one simplex containing t and $A' \in \mathrm{lk}_{\Delta_n}t$ is the codimension-one simplex corresponding to A, then the number of chambers in Δ_n containing A is the number of chambers in $\mathrm{lk}_{\Delta_n}t$ containing A'. Note that if t is *special*, then [8, p. 35] implies $\mathrm{lk}_{\Delta_n}t$ is isomorphic to the spherical $C_n(k)$ building $\Delta_n^s(k)$ described in [5, pp. 5 – 6].

Proposition 3.1. Every special vertex in Δ_n is contained in exactly $r(\Delta_n) = \prod_{m=1}^{n}((q^{2m} - 1)/(q-1))$ chambers in Δ_n.

Proof. Let $t \in \Delta_n$ be a special vertex. By the preceding comments and [5, pp. 5 – 6], it suffices to count the number of maximal flags of non-trivial, totally isotropic subspaces of a $2n$-dimensional k-vector space endowed with a non-degenerate, alternating bilinear form. An obvious modification of the proof of [6, Proposition 2.4] finishes the proof. □

Remark. The number $r(\Delta_n)$ in the last proposition corresponds to the number r_n given in [6, Proposition 2.4]. Since $\mathrm{Sp}_1(K) = \mathrm{SL}_2(K)$, set $r(\Delta_1) = q + 1$ for completeness.

Proposition 3.2. If $A \in \Delta_n$ is a codimension-one simplex, then A is contained in exactly $q+1$ chambers in Δ_n.

Proof. Let t be a special vertex in A and A' the codimension-one simplex in $\mathrm{lk}_{\Delta_n} t$ corresponding to A. By the comments preceding the last proposition, it suffices to count the number of chambers in $\Delta_n^s(k)$ containing A'. A case-by-case analysis finishes the proof. $\square$

We now use the fact that Δ_n is a subcomplex of Ξ_{2n} to derive information about Δ_n. For a vertex $t \in \Xi_{2n}$ with representative $L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_{2n}$ and $g \in \mathrm{GL}_{2n}(K)$, define $gt = [\mathcal{O}(gv_1) + \cdots + \mathcal{O}(gv_{2n})]$. Then $\mathrm{GL}_{2n}(K)$ acts transitively on the lattices in V.

Let

$$J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) \text{ and } \mathrm{GSp}_n(K) = \left\{g \in M_{2n}(K) : g^t J_n g = \nu(g) J_n \text{ for some } \nu(g) \in K^\times\right\},$$

so that $\mathrm{Sp}_n(K)$ consists of the matrices $g \in \mathrm{GSp}_n(K)$ with $\nu(g) = 1$. Alternatively, abuse notation and think of $\mathrm{GSp}_n(K)$ as

$$\{g \in \mathrm{GL}_K(V) : \forall\ v_1, v_2 \in V,\ \exists\ \nu(g) \in K^\times \text{ such that } \langle gv_1, gv_2\rangle = \nu(g)\langle v_1, v_2\rangle\}.$$

If $g \in \mathrm{GL}_{2n}(K)$ and $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ is a basis for V, write $g\mathcal{B}$ for $\{gv_1, \ldots, gv_{2n}\}$.

Lemma 3.3. The group $\mathrm{Sp}_n(K)$ acts on the set of primitive lattices in V.

Proof. Let L be a primitive lattice in V, and let Σ be an apartment of Δ_n containing $[L]$ and $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 3.1. Then $L = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ by [7, p. 3411]; hence, for $g \in \mathrm{Sp}_n(K)$, $g\mathcal{B}$ a symplectic basis for V implies that gL is primitive. $\square$

For the rest of this section, let $\mathcal{B}_0 = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be the standard symplectic basis for V ($f_i = e_{n+i}$ for all i), $L_0 = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0}$, and $t_0 = [L_0]$. Following [5, p. 116], assign *types* to the vertices in Ξ_{2n} as follows: assign type 0 to t_0 and type $\mathrm{ord}(\det g) \bmod 2n$ to any other vertex $t = [L] \in \Xi_{2n}$, where $g \in \mathrm{GL}_{2n}(K)$ such that $L = gL_0$. This induces a labelling on the vertices in Δ_n. For the rest of this section, let C_0 be the chamber in Δ_n whose vertices are the homothety classes of the lattices

$$L_0 = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0}, L_1 = (0, 1, \ldots, 1; 1, \ldots, 1)_{\mathcal{B}_0}, \ldots, L_n = (0, \ldots, 0; 1, \ldots, 1)_{\mathcal{B}_0}. \quad (3)$$

Note that $[L_i]$ has type $2n - i$ for all $1 \le i \le n$. Recall that since Δ_n is the affine building naturally associated to $\mathrm{Sp}_n(K)$, $\mathrm{Sp}_n(K)$ acts on the vertices in Δ_n in a type-preserving manner and also acts transitively on the chambers in Δ_n.

Proposition 3.3. If $t \in \Delta_n$ is a vertex, then t has type i for some $i \equiv n, \ldots, 2n \mod 2n$.

Proof. By the preceding comments, it suffices to show that for all $0 \le j \le n$, $[L_j]$ (as in (3)) has type i for some $i \equiv n, \ldots, 2n \mod 2n$, which we already observed. $\square$

We now use types to characterize the vertices in Δ_n with a primitive representative, as well as those that are special.

Proposition 3.4. A vertex in Δ_n has a primitive representative if and only if it has type 0.

Proof. Let $t \in \Delta_n$ be a type 0 vertex and $C \in \Delta_n$ a chamber containing t. Choose $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$. Then $gL_0 \in t$. Since L_0 is primitive, Lemma 3.3 implies that gL_0 is primitive. Conversely, let $t \in \Delta_n$ be a vertex with a primitive representative L, and let $C \in \Delta_n$ be a chamber containing t. Let $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$. Then $gL = \pi^m L_j$ for some $0 \le j \le n$ and some $m \in \mathbb{Z}$. If $L_j = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}_0}$ as in (3), then $gL = (a_1 + m, \ldots, a_n + m; b_1 + m, \ldots, b_n + m)_{\mathcal{B}_0}$. But gL primitive (by Lemma 3.3) implies that $a_i + b_i = -2m$ for all i. By (3), $m = 0$ and $gt = [L_0]$; hence, t has type 0. □

Proposition 3.5. A vertex in Δ_n is special if and only if it has type 0 or n.

Proof. Let $t \in \Delta_n$ be a type 0 (resp., type n) vertex, and let $C \in \Delta_n$ be a chamber containing t. If $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$, then $t = g[L_0]$ (resp., $t = g[L_n]$), and t is special by [7, Corollary 3.4]. Conversely, let $t \in \Delta_n$ be a special vertex. Let $C \in \Delta_n$ be a chamber containing t, Σ an apartment of Δ_n containing C, and $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 3.1. By [7, Corollary 3.4], $t = [a_1, \ldots, a_n; \mu - a_1, \ldots, \mu - a_n]_{\mathcal{B}}$ for some $\mu \in \mathbb{Z}$. If $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$, then $gt = [L_i]$ for some $0 \le i \le n$; hence, gt special, [7, Corollary 3.4], and (3) imply $i = 0$ or $i = n$, and t has type 0 or n. □

We now consider the action of $\mathrm{GSp}_n(K)$ on the vertices in Ξ_{2n}.

Proposition 3.6. If $[L]$ is a type i vertex in Ξ_{2n}, then for any $g \in \mathrm{GL}_{2n}(K)$, the vertex $g[L] \in \Xi_{2n}$ has type $i + \mathrm{ord}(\det g) \mod 2n$.

Proof. Since $[L]$ has type i, we can write $L = g_i L_0$, where $g_i \in \mathrm{GL}_{2n}(K)$ with $\mathrm{ord}(\det g_i) \equiv i \mod 2n$. Then $g[L]$ has type $\mathrm{ord}(\det(gg_i)) \mod 2n \equiv i + \mathrm{ord}(\det g) \mod 2n$. □

Corollary 3.1. If $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$, then g maps a non-special vertex in Δ_n to a vertex in Ξ_{2n} that is not in Δ_n.

Proof. First note that $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$ implies $\mathrm{ord}(\det g) \equiv n \mod 2n$. If t is a non-special vertex in Δ_n, then t has type i for some $n + 1 \le i \le 2n - 1$ by Propositions 3.3 and 3.5. Thus, the last proposition implies gt has type $i + n \mod 2n \in \{1, \ldots, n - 1\}$. Proposition 3.3 finishes the proof. □

3.2 The Building Δ_n in the Building Ξ_{2n}

Let $C \in \Delta_n$ be a chamber corresponding to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. Let Σ be an apartment of Δ_n containing C, $\mathcal{B}$ a symplectic basis for V specifying Σ as in Lemma 3.1, and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} specified by $\mathcal{B}$. Let $D \in \widetilde{\Sigma}$ be any chamber containing C. Then D corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ for some lattices $L_{n+1}, \ldots, L_{2n-1}$ in V. For $0 \le j \le 2n - 1$, write

$$L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}.$$

Lemma 3.4. The two special vertices in C are $[L_0]$ and $[L_n]$.

Proof. The fact that $[L_0]$ is special follows from [7, Corollary 3.4] and [7, p. 3411]. To see that $[L_n]$ is special, note that if L_j represents a special vertex in C for $1 \leq j \leq n$, then $a_i^{(j)} + b_i^{(j)} = \mu$ for all i (by [7, Corollary 3.4]), where $\mu \in \{1, 2\}$ (since $\langle L_j, L_j \rangle \subseteq \pi\mathcal{O}$). But $\mu = 2$ implies $L_j = \pi L_0$, which is impossible. Thus, $a_i^{(j)} + b_i^{(j)} = 1$ for all i and $L_j/\pi L_0 \cong k^n$; hence, $j = n$. □

For $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ a symplectic basis for V and $g \in \mathrm{GSp}_n(K)$, let

$$\mathcal{B}_g := \{\nu(g)^{-1}gu_1, \ldots, \nu(g)^{-1}gu_n, gw_1, \ldots, gw_n\}.$$

Note that $\mathcal{B}_g$ is a symplectic basis for V; hence, $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ and $\mathrm{ord}(\nu(g)) = m$ imply $gL = (a_1 + m, \ldots, a_n + m; b_1, \ldots, b_n)_{\mathcal{B}_g}$.

Proposition 3.7. The group $\mathrm{GSp}_n(K)$ acts transitively on the special vertices in Δ_n.

Proof. Note that if $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n, then [7, Proposition 3.3] implies that the action is transitive. We thus show that $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n. Let $t \in \Delta_n$ be a special vertex and $L \in t$ a representative such that there is a primitive lattice L_0 with $\langle L, L \rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$. Let Σ be an apartment of Δ_n containing t and $[L_0]$, and let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 3.1. Then [7, p. 3411], the last lemma, and [7, Corollary 3.4] imply

$$L_0 = (c_1, \ldots, c_n; -c_1, \ldots, -c_n)_{\mathcal{B}} \qquad \text{and} \qquad L = (a_1, \ldots, a_n; \mu - a_1, \ldots, \mu - a_n)_{\mathcal{B}},$$

where $\mu \in \{1, 2\}$. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) = m$. Since $gt = [a_1 + m, \ldots, a_n + m; \mu - a_1, \ldots, \mu - a_n]_{\mathcal{B}_g}$, [7, Corollary 3.4] implies that it suffices to show gt is a vertex in Δ_n. First suppose $m \equiv 0 \mod 2$, say $m = 2r$. Then $\pi^{-r}gL_0$ is primitive, $\langle \pi^{-r}gL, \pi^{-r}gL \rangle \subseteq \pi\mathcal{O}$, and $\pi^{-r}g(\pi L_0) \subseteq \pi^{-r}gL \subseteq \pi^{-r}gL_0$; i.e., gt is a vertex in Δ_n. Now suppose $m = 2r + 1$. If $\mu = 1$, then $\pi^{-r-1}gL$ is primitive and gt is a vertex in Δ_n. Otherwise, $\mu = 2$, and $\langle \pi^{-r-1}gL, \pi^{-r-1}gL \rangle \subseteq \pi\mathcal{O}$. Let $\pi M_0 = (a_1 + r, \ldots, a_n + r; \mu - a_1 - r, \ldots, \mu - a_n - r)_{\mathcal{B}_g}$. Then M_0 is primitive and $\pi M_0 \subseteq \pi^{-r-1}gL \subseteq M_0$; i.e., gt is a vertex in Δ_n. Thus, $\mathrm{GSp}_n(K)$ acts on the special vertices in Δ_n. □

Note that by Propositions 3.4 and 3.5, $[L_n]$ has type n. Then by Proposition 3.3, the type of $[L_j]$ is in $\{n+1, \ldots, 2n-1\}$ for all $1 \leq j \leq n-1$ and the type of $[L_i]$ is in $\{1, \ldots, n-1\}$ for all $n+1 \leq i \leq 2n-1$.

Lemma 3.5. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. If $L_0, L_n, \ldots, L_{2n-1}$ are lattices in V as above, then the vertices $g[L_n], \ldots, g[L_{2n-1}], g[L_0]$ in Ξ_{2n} are the vertices in a chamber in Δ_n.

Proof. Write $\mathrm{ord}(\nu(g)) = 2r + 1$. Then Lemma 3.4 and [7, p. 3411] imply that $L'_n = \pi^{-(r+1)}gL_n$ is primitive (see the proof of Proposition 3.7). Furthermore, if $L'_j = \pi^{-r}gL_j$ for $j = 0, n+1, \ldots, 2n-1$, then $\pi L'_n \subsetneq L'_{n+1} \subsetneq \cdots \subsetneq L'_{2n-1} \subsetneq L'_0 \subsetneq L'_n$ and $\langle L'_j, L'_j \rangle \subseteq \pi\mathcal{O}$ for $j = 0, n+1, \ldots, 2n-1$; i.e., $[L'_n], \ldots, [L'_{2n-1}], [L'_0]$ are the vertices in a chamber in Δ_n. □

Lemma 3.6. Let Σ be an apartment of Δ_n and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} such that $\mathcal{B}$ a symplectic basis for V specifying Σ implies $\mathcal{B}$ specifies $\widetilde{\Sigma}$. If C, C' is a gallery in Σ, then there is a gallery D, D' in $\widetilde{\Sigma}$ such that D (resp., D') contains C (resp., C') and $C \neq C'$ implies $D \neq D'$.

Remark. More generally, if $C_0, \ldots, C_m$ is a gallery in Δ_n, then there is a gallery $D_0, \ldots, D_\ell$ in Ξ_{2n} and integers $0 \leq i_0 < \cdots < i_m \leq \ell$ such that D_j contains C_0 for all $0 \leq j \leq i_0$ and D_j contains C_r for all $i_{r-1} < j \leq i_r$ and all $1 \leq r \leq m$.

Proof of Lemma 3.6. If $C = C'$, set $D = D'$, where $D \in \widetilde{\Sigma}$ is a chamber containing C. Now suppose $C \neq C'$, with C corresponding to the chain

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0. \tag{4}$$

Let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 3.1, and let $0 \leq j \leq n$ such that $C \cap C'$ corresponds to (4) with L_j deleted if $1 \leq j \leq n$ or with both πL_0 and L_0 deleted if $j = 0$. Note that if t' is the vertex in C' not in C, then t' has a representative L' such that C' corresponds to (4) with L_j replaced by L'.

If $1 \leq j \leq n-1$, then the result in [7, p. 3411], Lemma 3.4, and (4) imply $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ and $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$, where $a_i + 1 \geq b_i \geq a_i$ for all i. For $1 \leq i \leq n$, let $a_{n+i} = -a_i$ and $b_{n+i} = 1 - b_i$. Let $\{i_1, \ldots, i_n\}$ be the n values of i such that $b_i = a_i + 1$, and for $1 \leq r \leq n-1$, set $L_{n+r} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_\ell = b_\ell - 1 = a_\ell$ if $\ell \in \{i_1, \ldots, i_r\}$ and $c_\ell = b_\ell$ otherwise. Then $L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, and letting $D \in \widetilde{\Sigma}$ (resp., $D' \in \widetilde{\Sigma}$) be the simplex with vertices the vertices in C (resp., the vertices in C'), together with $[L_{n+1}], \ldots, [L_{2n-1}]$ finishes the proof in this case

If $j = n$, write $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$, $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$, and $L' = (b'_1, \ldots, b'_n; 1-b'_1, \ldots, 1-b'_n)_{\mathcal{B}}$. Note that $a_i + 1 \geq b_i, b'_i \geq a_i$ for all i and $b_i \neq b'_i$ for at least one value of i. Let $L_{n+1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \min\{b_i, b'_i\}$ and $c_{n+i} = \min\{1-b_i, 1-b'_i\}$ for $1 \leq i \leq n$. Then $L_{n+1} = L_n + L'$, so $L_n, L' \subsetneq L_{n+1}$ and $[L_{n+1} : L_n] = q = [L_{n+1} : L']$. An obvious modification of the second half of the last paragraph finishes the proof in this case.

Finally, if $j = 0$, write $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$, $L' = (a'_1, \ldots, a'_n; -a'_1, \ldots, -a'_n)_{\mathcal{B}}$, and $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$. Note that $a_i + 1, a'_i + 1 \geq b_i \geq a_i, a'_i$ for all i and $a_i \neq a'_i$ for at least one value of i. Let $L_{2n-1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \max\{a_i, a'_i\}$ and $c_{n+i} = \max\{-a_i, -a'_i\}$ for $1 \leq i \leq n$. Then $L_{2n-1} = L_0 \cap L'$, so $L_{2n-1} \subsetneq L_0, L'$ and $[L_0 : L_{2n-1}] = q = [L' : L_{2n-1}]$. An obvious modification of the second half of the first paragraph finishes the proof in this case. □

It will turn out to be convenient to first prove results about the type 0 vertices in Δ_n and to then use the transitive action of $\mathrm{GSp}_n(K)$ on the special vertices in Δ_n (see Proposition 3.7) to deduce the same results about the type n vertices in Δ_n. For $g \in \mathrm{GL}_{2n}(K)$ and a chamber $C \in \Xi_{2n}$, abuse notation and write gC for the image of the vertices in C under the action of g.

Proposition 3.8. The group $\mathrm{GL}_{2n}(K)$ (resp., $\mathrm{GSp}_n(K)$) maps a gallery in Ξ_{2n} of length m to a gallery in Ξ_{2n} of length m. In particular, if $C \neq C'$ are adjacent chambers in Ξ_{2n} and $g \in \mathrm{GL}_{2n}(K)$ (resp., $g \in \mathrm{GSp}_n(K)$), then $gC \neq gC'$ are adjacent chambers in Ξ_{2n}.

Proof. Let $C_0, \ldots, C_m$ be a gallery in Ξ_{2n}, and let $g \in \mathrm{GL}_{2n}(K)$. If $m = 0$ and C_0 corresponds to the chain $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, then $g(\pi L_0) \subsetneq gL_1 \subsetneq \cdots \subsetneq gL_{2n-1} \subsetneq gL_0$; i.e., gC_0 is a chamber in Ξ_{2n}. If $m = 1$ and $C_0 = C_1$, then gC_0, gC_1 is a gallery in Ξ_{2n}, so suppose $C_0 \neq C_1$. Let $t_0, \ldots, t_{2n-1}$ (resp., $x_0, \ldots, x_{2n-1}$) be the vertices in C_0 (resp., in C_1), and let $0 \leq j \leq 2n-1$ such that $t_j \neq x_j$. For $0 \leq i \leq 2n-1$, let $L_i \in t_i$ (resp., let $M_i \in x_i$) such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ (resp., $\pi M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{2n-1} \subsetneq M_0$) corresponds to C_0 (resp., to C_1). Then $g(\pi L_0) \subsetneq gL_1 \subsetneq \cdots \subsetneq gL_{2n-1} \subsetneq gL_0$ (resp., $g(\pi M_0) \subsetneq gM_1 \subsetneq \cdots \subsetneq gM_{2n-1} \subsetneq gM_0$). Since $t_i = x_i$ implies $gt_i = gx_i$, gC_0, gC_1 is a gallery in Ξ_{2n}. The fact that $gC_0 \neq gC_1$ follows from the fact that $gx_j \neq gt_j$. The proof for $m \geq 2$ follows from the fact that gC_i, gC_{i+1} is a gallery in Ξ_{2n} for all $0 \leq i \leq m-1$. $\square$

3.3 Counting Close Vertices in Δ_n

Let $\Gamma = \mathrm{Sp}_n(\mathcal{O})$, and note that the analogues of the results in section 4.1 of [7] hold if $\mathrm{GSp}_n(K)$ acts on the lattices in V on the left (rather than on the right). The following is an analogue of Theorem 3.3 of [6] for the special vertices in Δ_n.

Theorem 3.1. If $t \in \Delta_n$ is a special vertex, then the number of vertices in Δ_n close to t is the number of left cosets of Γ in

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. First note that by Proposition 3.5, a special vertex in Δ_n has type either 0 or n. Let $t \in \Delta_n$ be a special vertex and $t' \in \Delta_n$ a vertex close to t. Then there are adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, but $t, t' \notin C \cap C'$. Let Σ be an apartment of Δ_n containing C and C'. If t has type 0, then by Lemma 3.2, we may assume that relative to some symplectic basis $\mathcal{B}$ for V specifying Σ, $t = [0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}} \in C_0$, where $C_0 \in \Sigma$ is the chamber with vertices $[0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}}, [0, 1, \ldots, 1; 1, \ldots, 1]_{\mathcal{B}}, \ldots, [0, \ldots, 0; 1, \ldots, 1]_{\mathcal{B}}$. A straightforward modification of the fourth and fifth paragraphs of the proof of [6, Theorem 3.3] using the reflections defined in [7, p. 3411] finishes the proof in this case.

Now suppose t has type n, and let $\mathcal{B}$ be a symplectic basis for V specifying Σ as in Lemma 3.1. Let $\widetilde{\Sigma}$ be the apartment of Ξ_{2n} specified by $\mathcal{B}$, and let $D, D' \in \widetilde{\Sigma}$ be adjacent chambers with C in D, C' in D', and $D \neq D'$ as in Lemma 3.6. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. Then by Proposition 3.6, gt has type 0. By Lemma 3.5, gD (resp., gD') contains a chamber $C_1 \in \Delta_n$ (resp., a chamber $C_1' \in \Delta_n$) with $gt \in C_1$ (resp., with $gt' \in C_1'$). Furthermore, $gD \neq gD'$ are adjacent chambers in Ξ_{2n} and $gt, gt' \notin gD \cap gD'$ by Proposition 3.8; i.e., gt and gt' are close vertices in Δ_n. Finally, if S_t and S_{gt} are the sets

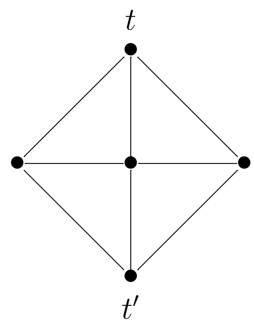


Figure 3: Two close special vertices, both of type 0, in Δ_2.

of vertices in Δ_n close to t and gt, respectively, then $\mathrm{Card}(S_t) = \mathrm{Card}(S_{gt})$, and the last paragraph finishes the proof. $\square$

Remark. The analogues of the results in [7, Section 4.1] also hold if $\mathrm{Sp}_n(\mathcal{O})$ and $\mathrm{GSp}_n^S(K)$ are replaced by $\mathrm{GSp}_n(\mathcal{O}) = \mathrm{GL}_{2n}(\mathcal{O}) \cap \mathrm{GSp}_n(K)$ and $\mathrm{GSp}_n(K)$, respectively, and with $\mathrm{GSp}_n(K)$ acting on the left rather than on the right. In addition, the analogue of the above theorem holds with $\Gamma = \mathrm{GSp}_n(\mathcal{O})$; hence, so does Corollary 3.2.

We now count the number of vertices in Δ_n close to a given special vertex $t \in \Delta_n$. By Proposition 3.5 and Theorem 3.1, it suffices to assume t has type 0. By Proposition 3.4, t has a primitive representative L, so a chamber $C \in \Delta_n$ containing t corresponds to a chain of the form

$$\pi L \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n \overset{q^n}{\subsetneq} L. \tag{5}$$

The codimension-one face in C not containing t thus corresponds to the chain

$$L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n,$$

and a vertex in Δ_n is close to t if it has a primitive representative $M \neq L$ such that

$$\pi M \overset{q}{\subsetneq} L_1 \overset{q}{\subsetneq} \cdots \overset{q}{\subsetneq} L_n \overset{q^n}{\subsetneq} M. \tag{6}$$

Given the lattice L_1, the possible L and M satisfy $L \neq M \subsetneq \pi^{-1}L_1$ with $[\pi^{-1}L_1 : L] = q = [\pi^{-1}L_1 : M]$ and both L and M primitive. On the other hand, if $t, t' \in \Delta_n$ are close type 0 vertices, then there must be primitive representatives $L \in t$ and $M \in t'$ and lattices $L_1, \ldots, L_n$ as in (5) such that $L \neq M \subsetneq \pi^{-1}L_1$. The same argument as in Section 2 shows that $\pi^{-1}L_1 = L + M$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $2 \leq i \leq n$ and the chains $\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L$ and $\pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$ correspond to chambers in Δ_n. In other words (as in the case of Ξ_n), if $t, t' \in \Delta_n$ are close type 0 vertices, there may be more than one pair of adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$ (see Figure 3). We return to this later.

Before we count the number of vertices in Δ_n close to t, we make a few observations similar to those preceding Proposition 2.1. Fix a primitive representative $L \in t$. Then

$L/\pi L \cong k^{2n}$ is endowed with a non-degenerate, alternating k-bilinear form. Moreover, the Correspondence Theorem, the fact that any $\mathcal{O}$-submodule of L containing πL is a lattice in V, and the fact that every 1-dimensional k-subspace of $L/\pi L$ is totally isotropic imply that the number of L_1 is the number of 1-dimensional k-subspaces of $L/\pi L$. Given L_1, let $C \in \Delta_n$ be a chamber containing $[L_1]$ and t, and let A be the codimension-one face in C not containing t. Then the number of primitive lattices $M \neq L$ in V such that $M \subsetneq \pi^{-1}L_1$ and $[\pi^{-1}L_1 : M] = q$ is one less than the number of chambers in Δ_n containing A.

Proposition 3.9. If $t \in \Delta_n$ is a special vertex, then the number $\omega(\Delta_n)$ of vertices in Δ_n close to t is

$$\frac{q^{2n}-1}{q-1} \cdot q$$

(independent of t).

Proof. This follows from the preceding comments, the fact that the number of 1-dimensional subspaces of $\mathbb{F}_q^m$ is exactly $(q^m-1)/(q-1)$, and Proposition 3.2. $\square$

Corollary 3.2. The number of left cosets of $\Gamma = \mathrm{Sp}_n(\mathcal{O})$ in

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma$$

is $((q^{2n}-1) \cdot q)/(q-1)$.

Proof. This follows from Theorem 3.1 and the last proposition. $\square$

Proposition 3.1 and the last proposition prove the following analogue of Theorem 2.1.

Theorem 3.2. Let $r(\Delta_n)$ be the number of chambers in Δ_n containing a given special vertex (as in Proposition 3.1) and $\omega(\Delta_n)$ the number of vertices in Δ_n close to a given special vertex in Δ_n (as in Proposition 3.9). Then for all $n \geq 2$, $q \cdot r(\Delta_n) = r(\Delta_{n-1})\, \omega(\Delta_n)$, where $r(\Delta_1) = q + 1$.

When the given vertex in Δ_n has type 0, we can also give a combinatorial proof of Theorem 3.2. As in Section 2, if $t \in \Delta_n$ is a fixed type 0 vertex, then we can try to count the number of vertices in Δ_n close to t by counting the number of galleries (in Δ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. An argument analogous to that in Section 2 shows that if $t' \in \Delta_n$ is a vertex close to t, then $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/m(\Delta_n, t, t')$, where $m(\Delta_n, t, t')$ is the number of galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t'.

To determine $m(\Delta_n, t, t')$, fix the following notation for the rest of this section. For close special vertices $t, t' \in \Delta_n$ with t of type 0, let $L \in t$, $M \in t'$ be primitive representatives (by Proposition 3.4) such that there are lattices $L_1, \ldots, L_n$ as in (5) and (6) with $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $1 \leq i \leq n$. Recall that $L_1 = \pi(L + M)$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $2 \leq i \leq n$ and the chains

$$\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L \qquad \text{and} \qquad \pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$$

correspond to chambers in Δ_n. As in Section 2, each gallery in Δ_n counted by $m(\Delta_n, t, t')$ is uniquely determined by $L_2, \ldots, L_n$. Define two vertices in Δ_n to be *adjacent* if they are distinct and incident.

Lemma 3.7. Let $t, t' \in \Delta_n$ be adjacent vertices such that t has a primitive representative L. Then t' has a unique representative L' such that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$.

Proof. Since t and t' are adjacent vertices in Ξ_{2n}, by Proposition 2.2, t' has a unique representative L' such that $\pi L \subsetneq L' \subsetneq L$. It thus suffices to show that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$. But t and t' incident vertices in Δ_n with $t \neq t'$ implies they have representatives $M \in t$ and $M' \in t'$ such that there is a primitive lattice L_0 with $\langle M, M\rangle \subseteq \pi\mathcal{O}$, $\langle M', M'\rangle \subseteq \pi\mathcal{O}$, and either $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$ or $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$. Suppose $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$ (resp., $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$). Then M and πL (resp., M and L) homothetic implies $\pi L = \pi^r M$ (resp., $L = \pi^r M$) for some $r \in \mathbb{Z}$; hence, $\pi L \subsetneq \pi^r M' \subsetneq L$. Let $L' = \pi^r M'$. Since L is primitive, $\langle \pi^{r-1}M, \pi^{r-1}M\rangle \subseteq \mathcal{O}$ (resp., $\langle \pi^r M, \pi^r M\rangle \subseteq \mathcal{O}$). On the other hand, $\langle \pi^{r-1}M, \pi^{r-1}M\rangle \subseteq \pi^{2(r-1)+1}\mathcal{O}$ (resp., $\langle \pi^r M, \pi^r M\rangle \subseteq \pi^{2r+1}\mathcal{O}$), so $r \in \mathbb{Z}^+$ (resp., $r \in \mathbb{Z}^{\geq 0}$) and $\langle L', L'\rangle \subseteq \pi\mathcal{O}$. □

Consider the set of vertices in Δ_n that are adjacent to t, t', and $[L + M]$, and define two such vertices to be incident if they are incident as vertices in Δ_n. Let $\Delta_n^c(t, t')$ be the set consisting of

- the empty set,
- all vertices in Δ_n adjacent to t, t', and $[L + M]$, and
- all finite sets A of vertices in Δ_n adjacent to t, t', and $[L + M]$ such that any two vertices in A are adjacent.

Then $\Delta_n^c(t, t')$ is a simplicial complex. In particular, $\Delta_n^c(t, t')$ is a subcomplex of Δ_n.

Lemma 3.8. If $\emptyset \neq A \in \Delta_n^c(t, t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$ and $\pi(L + M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$. In particular, A has at most $n - 1$ vertices.

Proof. As in the proof of Lemma 2.1, we proceed by induction on i. If $i = 0$, then L primitive, A adjacent to t, and Lemma 3.7 imply A has a unique representative M_1 such that $\langle M_1, M_1\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq M_1 \subsetneq L$. Since A and $[L + M]$ are adjacent vertices in Ξ_{2n}, either $M_1 \subsetneq \pi(L + M)$ or $M_1 \supsetneq \pi(L + M)$ by [3, p. 322]. But $M_1 \subsetneq \pi(L + M)$ means $\pi L \subsetneq M_1 \subsetneq \pi(L + M)$, which is impossible since $[\pi(L + M) : \pi L] = q$; hence, $M_1 \supsetneq \pi(L + M)$. Then A and t' adjacent vertices in Ξ_{2n} and [3, p. 322] imply that either $M_1 \subsetneq M$ or $M_1 \supsetneq M$. Since $M_1 \supsetneq M$ means $M \subsetneq M_1 \subsetneq L$, which contradicts the fact that $[M : \pi(L + M)] = [L : \pi(L + M)]$, $M_1 \subsetneq M$ and $M_1 \subseteq L \cap M$. Moreover, $\langle M_1, M_1\rangle \subseteq \pi\mathcal{O}$ implies $M_1/\pi L$ is a totally isotropic k-subspace of $L/\pi L$ and $[M_1 : \pi L] \leq q^n$. The fact that $[L \cap M : \pi L] = q^{2n-1}$ finishes the proof in this case.

Recall that $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $L/\pi L$. Then with respect to this induced bilinear form, $(L \cap M)/\pi L$ is the orthogonal complement of $\pi(L + M)/\pi L$ in $L/\pi L$. In addition, $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form on $(L \cap M)/\pi(L + M) \cong k^{2(n-1)}$, and there is a bijection between nested sequences $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ of totally isotropic k-subspaces of $(L \cap M)/\pi(L + M)$ and chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of $L \cap M$ containing $\pi(L + M)$ with $\langle M_j, M_j \rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i + 1$. An obvious modification of the second paragraph of the proof of Lemma 2.1 finishes the proof. □

Recall that $\Delta_n^s(k)$ denotes the spherical $C_n(k)$ building described in [5, pp. 5 – 6].

Proposition 3.10. For any close special vertices $t, t' \in \Delta_n$ with t of type 0, $\Delta_n^c(t, t')$ is isomorphic (as a poset) to $\Delta_{n-1}^s(k)$ (independent of t and t' with t of type 0).

Proof. Let $L \in t, M \in t'$ be primitive representatives as in the paragraph preceding Lemma 3.7, and let $\Delta_{n-1}^s(k)$ be the spherical $C_{n-1}(k)$ building with simplices the empty set, together with the nested sequences of non-trivial, totally isotropic k-subspaces of $(L \cap M)/\pi(L + M)$. Then the last lemma implies that there is a bijection between the i-simplices in $\Delta_n^c(t, t')$ and the i-simplices in $\Delta_{n-1}^s(k)$ for all i. Since this bijection preserves the partial order (face) relation, it is a poset isomorphism. □

Proposition 3.11. If $t, t' \in \Delta_n$ are close special vertices with t of type 0, then $m(\Delta_n, t, t') = r(\Delta_{n-1})$ (independent of t and t'). In particular, $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/r(\Delta_{n-1})$.

Proof. The proof is an obvious modification of the proof of Theorem 2.2. □

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