# THE GAME OF CUTBLOCK

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**Abstract**

The game of Cutblock is the three-player variant of Cutcake, a classical combinatorial game. Even though to determine the solution of Cutcake is trivial, solving Cutblock is challenging because of the identification of *queer* games, i.e., games where no player has a winning strategy. A classification of the instances of Cutblock is presented.

## 1. Introduction

The game of Cutblock (a three-player version of Cutcake [1]) was introduced by Propp in [5]. Every instance of this game is defined as a set of blocks of integer side-lengths, with edges parallel to the $x-$, $y-$, and $z-$axes. We use $[l, c, r]$ to indicate an $l$ by $c$ by $r$ block. A legal move for Left is to divide one of the blocks into two blocks of integer side-length by means of a single cut perpendicular to the $x-$axis; analogously, we define the legal moves for Center and Right. Players take turns making legal moves in cyclic fashion (..., Left, Center, Right, Left, Center, Right, ...). When one of the three players is unable to move, he/she leaves the game and the remaining players continue in alternation until one of them is unable to move. Then that player leaves the game and the remaining player is the winner.

In the game of Cutcake the outcome of an $l$ by $r$ rectangle depends on the dimensions of $l$ and $r$ as shown in Table 1. We briefly recall two conjectures introduced by Propp [5] concerning the outcome of Cutblock.

**Conjecture 1.** $[n, n, n]$ is always a win for the player who makes his/her first move last, under cyclic play.

**Conjecture 2.** The winner of $[l, c, r]$ is determined by whichever of $\lfloor \log_2 l \rfloor$, $\lfloor \log_2 c \rfloor$, and $\lfloor \log_2 r \rfloor$ is largest, with tied comparisons being decided by which of the tied players makes his or her first move last.

Table 1: Outcome Classes in Cutcake.

| | Left starts | Right starts |
|---|---|---|
| $\lfloor \log_2 l \rfloor > \lfloor \log_2 r \rfloor$ | Left wins | Left wins |
| $\lfloor \log_2 l \rfloor < \lfloor \log_2 r \rfloor$ | Right wins | Right wins |
| $\lfloor \log_2 l \rfloor = \lfloor \log_2 r \rfloor$ | Right wins | Left wins |

The previous conjectures are supported by data for $1 \leq l, c, r \leq 3$.

## 2. Three-player Partizan Games

For the sake of self-containment, we recall the basic definitions and main results of a theory for three-player partizan games [2]. Such a theory is an extension of Conway's theory of partizan games [3] and, as a consequence, it is both a theory of games and a theory of numbers.

**Definition 3.** If $L, C, R$ are any three sets of numbers previously defined and

1. no element of $L$ is $\geq_L$ any element of $C \cup R$, and
2. no element of $C$ is $\geq_C$ any element of $L \cup R$, and
3. no element of $R$ is $\geq_R$ any element of $L \cup C$,

then $\{L|C|R\}$ is a number. All numbers are constructed in this way.

This definition for numbers is based on the definition and comparison operators for games given in the following two definitions.

**Definition 4.** If $L, C, R$ are any three sets of games previously defined then $\{L|C|R\}$ is a game. All games are constructed in this way.

**Definition 5.** Let $x = \{L|C|R\}$ and $y = \{L'|C'|R'\}$ be two games.

- $x \geq_L y$ iff ($y \geq_L$ no $x^C$, $y \geq_L$ no $x^R$ and no $y^L \geq_L x$),
- $x \leq_L y$ iff $y \geq_L x$,
- $x \geq_C y$ iff ($y \geq_C$ no $x^L$, $y \geq_C$ no $x^R$ and no $y^C \geq_C x$),
- $x \leq_C y$ iff $y \geq_C x$,
- $x \geq_R y$ iff ($y \geq_R$ no $x^L$, $y \geq_R$ no $x^C$ and no $y^R \geq_R x$),

- $x \leq_R y$ iff $y \geq_R x$,

where $x^L$, $x^C$, $x^R$ are respectively the typical elements of $L$, $C$, $R$ and $y^L$, $y^C$, $y^R$ are respectively the typical elements of $L'$, $C'$, $R'$.

We write

- $x \ngeq_L y$ to mean that $x \geq_L y$ does not hold,
- $x \ngeq_C y$ to mean that $x \geq_C y$ does not hold,
- $x \ngeq_R y$ to mean that $x \geq_R y$ does not hold.

**Definition 6.** We say that

- $x =_L y$ iff ($x \geq_L y$ and $y \geq_L x$),
- $x >_L y$ iff ($x \geq_L y$ and $y \ngeq_L x$),
- $x <_L y$ iff $y >_L x$,
- $x =_C y$ iff ($x \geq_C y$ and $y \geq_C x$),
- $x >_C y$ iff ($x \geq_C y$ and $y \ngeq_C x$),
- $x <_C y$ iff $y >_C x$,
- $x =_R y$ iff ($x \geq_R y$ and $y \geq_R x$),
- $x >_R y$ iff ($x \geq_R y$ and $y \ngeq_R x$),
- $x <_R y$ iff $y >_R x$,
- $x = y$ iff ($x =_L y$, $x =_C y$, and $x =_R y$),
- $x$ is identical to $y$ ($x \equiv y$) iff their left, center and right sets are identical, i.e., if every $x^L$ is identical to some $y^L$, every $x^C$ is identical to some $y^C$, every $x^R$ is identical to some $y^R$, and vice versa.

All the given definitions are inductive, so that to decide whether $x \geq_L y$ we check the pairs $(x^C, y)$, $(x^R, y)$, and $(x, y^L)$.

**Theorem 7.** For any number $x$

- $x^L <_L x, x <_L x^C, x <_L x^R$,
- $x^C <_C x, x <_C x^L, x <_C x^R$,

Table 2: Outcome Classes for Numbers.

| Class | Short notation | Left starts | Center starts | Right starts |
|---|---|---|---|---|
| $=_L, =_C, =_R$ | $=$ | Right wins | Left wins | Center wins |
| $>_L, <_C, <_R$ | $>_L$ | Left wins | Left wins | Left wins |
| $<_L, >_C, <_R$ | $>_C$ | Center wins | Center wins | Center wins |
| $<_L, <_C, >_R$ | $>_R$ | Right wins | Right wins | Right wins |
| $=_L, =_C, <_R$ | $=_{LC}$ | Center wins | Left wins | Center wins |
| $=_L, <_C, =_R$ | $=_{LR}$ | Right wins | Left wins | Left wins |
| $<_L, =_C, =_R$ | $=_{CR}$ | Right wins | Right wins | Center wins |
| $=_L, <_C, <_R$ | $<_{CR}$ | ? | Left wins | Left wins |
| $<_L, =_C, <_R$ | $<_{LR}$ | Center wins | ? | Center wins |
| $<_L, <_C, =_R$ | $<_{LC}$ | Right wins | Right wins | ? |
| $<_L, <_C, <_R$ | $<$ | ? | ? | ? |

Table 3: Outcomes of Sums of Numbers.

| | $=$ | $>_L$ | $>_C$ | $>_R$ | $=_{LC}$ | $=_{LR}$ | $=_{CR}$ | $<_{CR}$ | $<_{LR}$ | $<_{LC}$ | $<$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $=$ | $=$ | $>_L$ | $>_C$ | $>_R$ | $=_{LC}$ | $=_{LR}$ | $=_{CR}$ | $<_{CR}$ | $<_{LR}$ | $<_{LC}$ | $<$ |
| $>_L$ | $>_L$ | $>_L$ | ? | ? | $>_L$ | $>_L$ | ? | $>_L$ | ? | ? | ? |
| $>_C$ | $>_C$ | ? | $>_C$ | ? | $>_C$ | ? | $>_C$ | ? | $>_C$ | ? | ? |
| $>_R$ | $>_R$ | ? | ? | $>_R$ | ? | $>_R$ | $>_R$ | ? | ? | $>_R$ | ? |
| $=_{LC}$ | $=_{LC}$ | $>_L$ | $>_C$ | ? | $=_{LC}$ | $<_{CR}$ | $<_{LR}$ | $<_{CR}$ | $<_{LR}$ | $<$ | $<$ |
| $=_{LR}$ | $=_{LR}$ | $>_L$ | ? | $>_R$ | $<_{CR}$ | $=_{LR}$ | $<_{LC}$ | $<_{CR}$ | $<$ | $<_{LC}$ | $<$ |
| $=_{CR}$ | $=_{CR}$ | ? | $>_C$ | $>_R$ | $<_{LR}$ | $<_{LC}$ | $=_{CR}$ | $<$ | $<_{LR}$ | $<_{LC}$ | $<$ |
| $<_{CR}$ | $<_{CR}$ | $>_L$ | ? | ? | $<_{CR}$ | $<_{CR}$ | $<$ | $<_{CR}$ | $<$ | $<$ | $<$ |
| $<_{LR}$ | $<_{LR}$ | ? | $>_C$ | ? | $<_{LR}$ | $<$ | $<_{LR}$ | $<$ | $<_{LR}$ | $<$ | $<$ |
| $<_{LC}$ | $<_{LC}$ | ? | ? | $>_R$ | $<$ | $<_{LC}$ | $<_{LC}$ | $<$ | $<$ | $<_{LC}$ | $<$ |
| $<$ | $<$ | ? | ? | ? | $<$ | $<$ | $<$ | $<$ | $<$ | $<$ | $<$ |

- $x^R <_R x, x <_R x^L, x <_R x^C$.

Numbers are totally ordered with respect to $\geq_L$, $\geq_C$, and $\geq_R$, but games are partially-ordered, i.e., there exist games $x$ and $y$ for which we have neither $x \geq_L y$ nor $y \geq_L x$.

All numbers can be classified into 11 outcome classes as shown in Table 2.

**Definition 8.** We define the sum of two numbers as follows:

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}.$$

Table 3 shows all the possible cases concerning the sum of two numbers. The entries '?' are unrestricted and indicate that we can have more than one outcome, e.g., if $x = \{1_L | \ | \ \} =$

$2_L$ and $y = 1_C$ then $x + y >_L 0$ but if $x = 1_L$ and $y = 1_C$ then $x + y =_{LC} 0$.

For further details, please refer to [2].

## 3. Classifying the Instances of Cutblock

**Theorem 9.** Let $G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$ be a general instance of Cutblock. Then, $G$ is a number.

*Proof.* Let $G = \{G^L|G^C|G^R\}$ be a general instance of Cutblock. By inductive hypothesis, $G^L$, $G^C$, and $G^R$ are numbers; moreover, for every pair of options $G^L$ and $G^C$, we can distinguish two different sub-cases:

1. If $G^L$ cut the $i$-th block into $[l_{i_1}, c_i, r_i] + [l_{i_2}, c_i, r_i]$ and $G^C$ cut the $i$-th block into $[l_i, c_{i_1}, r_i] + [l_i, c_{i_2}, r_i]$, then there exists a center option of $G^L$ ($G^{LC}$) that cut $[l_{i_1}, c_i, r_i]$ into $[l_{i_1}, c_{i_1}, r_i] + [l_{i_1}, c_{i_2}, r_i]$, and there exists a center option of $G^{LC}$ ($G^{LCC}$) that cut $[l_{i_2}, c_i, r_i]$ into $[l_{i_2}, c_{i_1}, r_i] + [l_{i_2}, c_{i_2}, r_i]$. Symmetrically, there exists a left option of $G^C$ ($G^{CL}$) that cut $[l_i, c_{i_1}, r_i]$ into $[l_{i_1}, c_{i_1}, r_i] + [l_{i_2}, c_{i_1}, r_i]$, and a left option of $G^{CL}$ ($G^{CLL}$) that cut $[l_i, c_{i_2}, r_i]$ into $[l_{i_1}, c_{i_2}, r_i] + [l_{i_2}, c_{i_2}, r_i]$. As a result, $G^{LCC} \equiv G^{CLL}$ and we have
$$G^L <_L G^{LC} <_L G^{LCC} \equiv G^{CLL} <_L G^{CL} <_L G^C \Rightarrow G^L <_L G^C$$
2. If $G^L$ cut the $i$-th block into $[l_{i_1}, c_i, r_i] + [l_{i_2}, c_i, r_i]$ and $G^C$ cut the $j$-th block into $[l_j, c_{j_1}, r_j] + [l_j, c_{j_2}, r_j]$, then
$$G^L <_L G^{LC} \equiv G^{CL} <_L G^C \Rightarrow G^L <_L G^C$$

In the same way, we prove that $G^L <_L G^R$, $G^C <_C G^L$, $G^C <_C G^R$, $G^R <_R G^L$, and $G^R <_C G^C$.

**Example 10.** Let $G = [5, 5, 5]$ be a block of Cutblock. Let $G^L = [2, 5, 5] + [3, 5, 5]$ and $G^C = [5, 1, 5] + [5, 4, 5]$ be, respectively, a left and a center option. We observe that
$$\begin{aligned} [2,5,5] + [3,5,5] &<_L [2,1,5] + [2,4,5] + [3,5,5] \\ &<_L [2,1,5] + [2,4,5] + [3,1,5] + [3,4,5] \\ &\equiv [2,1,5] + [3,1,5] + [2,4,5] + [3,4,5] \\ &<_L [2,1,5] + [3,1,5] + [5,4,5] \\ &<_L [5,1,5] + [5,4,5] \end{aligned}$$

**Theorem 11.** In the game of Cutblock

1. $G = [1, 1, 1] = 0$

2. $G = [l, 1, 1] >_L 0, l > 1$
3. $G = [1, c, 1] >_C 0, c > 1$
4. $G = [1, 1, r] >_R 0, r > 1$

*Proof.*

1. Trivial.
2. We observe that $G^L = [\lceil l/2 \rceil, c, r] + [\lfloor l/2 \rfloor, c, r] \geq_L 0$ by inductive hypothesis and $G >_L 0$.
3. Analogous to (2).
4. Analogous to (2).

**Theorem 12.** In the game of Cutblock, $[l, c, r]$:

1. If $\lfloor \log_2 l \rfloor = \lfloor \log_2 c \rfloor$ then $G = [l, c, 1] =_{LC} 0$,
2. If $\lfloor \log_2 l \rfloor > \lfloor \log_2 c \rfloor$ then $G = [l, c, 1] >_L 0$,
3. If $\lfloor \log_2 l \rfloor < \lfloor \log_2 c \rfloor$ then $G = [l, c, 1] >_C 0$,
4. If $\lfloor \log_2 l \rfloor = \lfloor \log_2 r \rfloor$ then $G = [l, 1, r] =_{LR} 0$,
5. If $\lfloor \log_2 l \rfloor > \lfloor \log_2 r \rfloor$ then $G = [l, 1, r] >_L 0$,
6. If $\lfloor \log_2 l \rfloor < \lfloor \log_2 r \rfloor$ then $G = [l, 1, r] >_R 0$,
7. If $\lfloor \log_2 c \rfloor = \lfloor \log_2 r \rfloor$ then $G = [1, c, r] =_{CR} 0$,
8. If $\lfloor \log_2 c \rfloor > \lfloor \log_2 r \rfloor$ then $G = [1, c, r] >_C 0$,
9. If $\lfloor \log_2 c \rfloor < \lfloor \log_2 r \rfloor$ then $G = [1, c, r] >_R 0$,

where $l, c, r > 1$.

*Proof.*

1. A generic left option is represented by $[l_1, c, 1] + [l_2, c, 1]$ where $l_1 + l_2 = l$, $l_1 > 0$, and $l_2 > 0$. Let's suppose, without loss of generality, that $l_1 \geq l_2$. We have two cases: $\lfloor \log_2 l_1 \rfloor = \lfloor \log_2 l \rfloor$ or $\lfloor \log_2 l_1 \rfloor < \lfloor \log_2 l \rfloor$. (In both cases $\lfloor \log_2 l_2 \rfloor < \lfloor \log_2 l \rfloor$.) In the first case, by inductive hypothesis, we have $[l_1, c, 1] =_{LC} 0$ and $[l_2, c, 1] >_C 0$; in the second case we have $[l_1, c, 1] >_C 0$ and $[l_2, c, 1] >_C 0$, therefore, in both cases, we have $G^L >_C 0$.
   By similar reasoning we can prove that $G^C >_L 0$, and therefore $G =_{LC} 0$.

2. We observe that there exists at least a left option $G^L = [l_1, c, 1] + [l_2, c, 1]$, where $l_1 = \lceil l/2 \rceil$ and $l_2 = \lfloor l/2 \rfloor$, such that $G^L >_L 0$ or $G^L =_{LC} 0$. In both cases we have $G >_L 0$.

3. Analogous to (2).

The other 6 cases can be proved analogously.

**Lemma 13.** If $x <_{CR} 0$, $w < 0$, $y >_C 0$, and $z >_R 0$ are numbers, then

1. $x + y <_L 0$,
2. $x + z <_L 0$,
3. $w + y <_L 0$,
4. $w + z <_L 0$.

*Proof.*

1. We recall that

$$x + y = \{x^L + y, x + y^L | x^C + y, x + y^C | x^R + y, x + y^R\}.$$

   By hypothesis there exists at least a center option of $y$ such that $y^C \geq_C 0$. If $y^C >_C 0$ then $x + y^C <_L 0$ by inductive hypothesis, otherwise, if $y^C =_C 0$ then $x + y^C \leq_L 0$. In both cases it follows that $x + y <_L 0$.

2. Similar to step 1 above.
3. Similar to step 1 above.
4. Similar to step 1 above.

**Theorem 14.** Let $G = [l, c, r]$ be a block of Cutblock where $l, c, r > 1$. If

- $\lfloor \log_2 l \rfloor < \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$ and
- $\lfloor \log_2 c \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 r \rfloor$ and
- $\lfloor \log_2 r \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 c \rfloor$

then $G < 0$, else one of the following 6 cases occurs:

1. If $\lfloor \log_2 l \rfloor > \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$, then $G >_L 0$, or

2. If $\lfloor \log_2 l \rfloor = \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$, then $G <_{CR} 0$, or
3. If $\lfloor \log_2 c \rfloor > \lfloor \log_2 l \rfloor + \lfloor \log_2 r \rfloor$, then $G >_C 0$, or
4. If $\lfloor \log_2 c \rfloor = \lfloor \log_2 l \rfloor + \lfloor \log_2 r \rfloor$, then $G <_{LR} 0$, or
5. If $\lfloor \log_2 r \rfloor > \lfloor \log_2 l \rfloor + \lfloor \log_2 c \rfloor$, then $G >_R 0$, or
6. If $\lfloor \log_2 r \rfloor = \lfloor \log_2 l \rfloor + \lfloor \log_2 c \rfloor$, then $G <_{LC} 0$.

*Proof.* Let's assume that $\lfloor \log_2 l \rfloor < \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$, $\lfloor \log_2 c \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 r \rfloor$, and $\lfloor \log_2 r \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 c \rfloor$. Let's consider the left option $G^L = [l_1, c, r] + [l_2, c, r]$, where $l_1 = \lceil l/2 \rceil$ and $l_2 = \lfloor l/2 \rfloor$. We have three sub-cases:

- $l = 2$. In this case $\log_2 l = 1$ and $\log_2 c = \log_2 r$, therefore $G^L =_{CR} 0$, as shown in the previous theorem.
- $l = 3$. Even in this case, $\log_2 l = 1$ and $\log_2 c = \log_2 r$, therefore $[2, c, r] < 0$ by inductive hypothesis, and $[1, c, r] =_{CR}$ as shown in the previous theorem. It follows that $G^L < 0$.
- $l \geq 4$. If $\lfloor \log_2 l_1 \rfloor = \lfloor \log_2 l \rfloor$, then $[l_1, c, r] < 0$ by inductive hypothesis, else if $\lfloor \log_2 l_1 \rfloor = \lfloor \log_2 l \rfloor - 1$ then:

$$[l_1, c, r] \begin{cases} <_{LR} 0 & \text{if } \lfloor \log_2 c \rfloor = \lfloor \log_2 l_1 \rfloor + \lfloor \log_2 r \rfloor, \\ <_{LC} 0 & \text{if } \lfloor \log_2 r \rfloor = \lfloor \log_2 l_1 \rfloor + \lfloor \log_2 c \rfloor, \\ < 0 & \text{otherwise.} \end{cases}$$

  Analogously, $\lfloor \log_2 l_2 \rfloor = \lfloor \log_2 l \rfloor - 1$, therefore

$$[l_2, c, r] \begin{cases} <_{LR} 0 & \text{if } \lfloor \log_2 c \rfloor = \lfloor \log_2 l_2 \rfloor + \lfloor \log_2 r \rfloor, \\ <_{LC} 0 & \text{if } \lfloor \log_2 r \rfloor = \lfloor \log_2 l_2 \rfloor + \lfloor \log_2 c \rfloor, \\ < 0 & \text{otherwise.} \end{cases}$$

  Therefore $G^L$ is $<_{LR} 0$, $<_{LC} 0$, or $< 0$.

It follows that for each of the 3 aforementioned sub-cases, there exists at least one left option $G^L \leq_C 0$ and $G^L \leq_R 0$ therefore $G <_C 0$ and $G <_R 0$. Analogously, we can prove that $G <_L 0$ ($G <_R 0$) considering $G^C = [l, c_1, r] + [l, c_2, r]$, where $c_1 = \lceil c/2 \rceil$ and $c_2 = \lfloor c/2 \rfloor$, therefore $G < 0$.

Now, let's suppose that the conditions shown in Theorem 14 $\lfloor \log_2 l \rfloor < \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$, $\lfloor \log_2 c \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 r \rfloor$, and $\lfloor \log_2 r \rfloor < \lfloor \log_2 l \rfloor + \lfloor \log_2 c \rfloor$ are not true; it follows that only one of the subsequent 6 cases can be true.

Table 4: Outcomes of Sums of Numbers.

| $[l_1, c, r]$ | $[l_2, c, r]$ | $[l_1, c, r] + [l_2, c, r]$ |
|---|---|---|
| $<_{CR}$ | $>_C$ | $<_L$ (By Lemma 13) |
| $<_{CR}$ | $>_R$ | $<_L$ (By Lemma 13) |
| $<_{CR}$ | $=_{CR}$ | $<$ |
| $<_{CR}$ | $<_{LR}$ | $<$ |
| $<_{CR}$ | $<_{LC}$ | $<$ |
| $<_{CR}$ | $<$ | $<$ |
| $<$ | $>_C$ | $<_L$ (By Lemma 13) |
| $<$ | $>_R$ | $<_L$ (By Lemma 13) |
| $<$ | $=_{CR}$ | $<$ |
| $<$ | $<_{LR}$ | $<$ |
| $<$ | $<_{LC}$ | $<$ |
| $<$ | $<$ | $<$ |

1. Therefore, there exists at least a left option $G^L = [l_1, c, r] + [l_2, c, r]$, where $l_1 = \lceil l/2 \rceil$ and $l_2 = \lfloor l/2 \rfloor$, such that by inductive hypothesis either $G^L >_L 0$ or $G^L <_{CR} 0$. In both cases we have $G >_L 0$.

2. For every center option $[l, c_1, r] + [l, c_2, r]$ (assuming $c_1 \geq c_2$), we have either $\lfloor \log_2 l \rfloor = \lfloor \log_2 c_1 \rfloor + \lfloor \log_2 r \rfloor$ or $\lfloor \log_2 l \rfloor > \lfloor \log_2 c_1 \rfloor + \lfloor \log_2 r \rfloor$. Moreover, $\lfloor \log_2 l \rfloor > \lfloor \log_2 c_2 \rfloor + \lfloor \log_2 r \rfloor$. Therefore, by inductive hypothesis, $G^C >_L 0$. In the same way, we prove that $G^R >_L 0$. Let's consider a generic left option $G^L = [l_1, c, r] + [l_2, c, r]$ where $l_1 \geq l_2$. As shown in Table 4, $G^L$ is always $<_L 0$, and therefore $G <_{CR} 0$.

We can prove the other 4 cases analogously.

**Theorem 15.** Let $G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$ be a general instance of Cutblock. If $\lfloor \log_2 l_i \rfloor \leq \lfloor \log_2 c_i \rfloor$ for all $1 \leq i \leq n$, and Left has to play, then Left does not have a winning strategy.

*Proof.* Let's suppose that Left plays in the i-th block $[l_i, c_i, r_i]$. After his/her move we have two new sub-blocks, $[l_{i_1}, c_i, r_i]$ and $[l_{i_2}, c_i, r_i]$. Without loss of generality, we assume $l_{i_1} \geq l_{i_2}$, therefore $\lfloor \log_2 l_{i_2} \rfloor < \lfloor \log_2 l_i \rfloor$. It follows that Center can play in $[l_{i_2}, c_i, r_i]$ resulting in $[l_{i_2}, c_{i_1}, r_i]$ and $[l_{i_2}, c_{i_2}, r_i]$. If Center chooses $c_{i_1} = \lceil c_i/2 \rceil$, and $c_{i_2} = \lfloor c_i/2 \rfloor$ then $\lfloor \log_2 l_{i_2} \rfloor \leq \lfloor \log_2 c_{i_1} \rfloor$ and $\lfloor \log_2 l_{i_2} \rfloor \leq \lfloor \log_2 c_{i_2} \rfloor$. The whole process is shown in Fig. 1.

Without loss of generality, let's suppose that Right divides the block $[l_1, c_1, r_1]$ into $[l_1, c_1, r_{1_1}] + [l_1, c_1, r_{1_2}]$. We call the instance obtained so far $G'$. At this point, Left has to play again, and we observe that in every block $[l, c, r] \in G'$ we have $\lfloor \log_2 l \rfloor \leq \lfloor \log_2 c \rfloor$, therefore, by inductive hypothesis, Left does not have a winning strategy.

The following theorem can be proven in the same way.

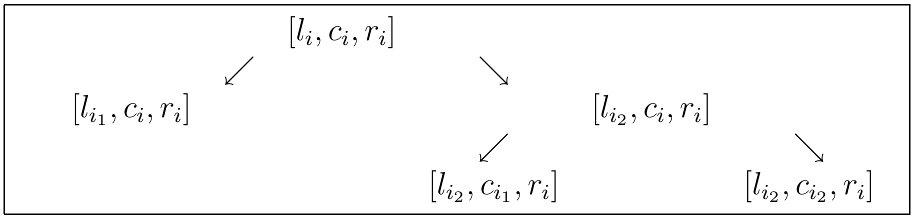


Figure 1: The new blocks created by Left and Center.

Table 5: Outcome classes for $G \leq_{CR} 0$, $G \leq_{LR} 0$, and $G \leq_{LC} 0$.

| | Left starts | Center starts | Right starts |
|---|---|---|---|
| $G <_{CR} 0$ | Left wins/Queer | Left wins | Left wins |
| $G <_{LR} 0$ | Center wins | Center wins/Queer | Center wins |
| $G <_{LC} 0$ | Right wins | Right wins | Right wins/Queer |

**Theorem 16.** Let $G = [l_1, c_1, r_1] + \ldots + [l_i, c_i, r_i] + \ldots + [l_n, c_n, r_n]$ be a general instance of Cutblock. If $\lfloor \log_2 l_i \rfloor \leq \lfloor \log_2 r_i \rfloor$ for all $1 \leq i \leq n$, and Left has to play, then Left does not have a winning strategy.

Analogously, we can get the same results for Center and Right.

The previous theorems give us some further information about the outcome of the game $G = [l, c, r] <_{CR} 0$ when Left makes the first move. In this case, $\lfloor \log_2 l \rfloor = \lfloor \log_2 c \rfloor + \lfloor \log_2 r \rfloor$, and when Left starts the game he/she can divide $G$ into $[l_1, c, r]$ and $[l_2, c, r]$, where $l_1 = \lceil l/2 \rceil$ and $l_2 = \lfloor l/2 \rfloor$. Now, Center has to play, but we observe that $\lfloor \log_2 c \rfloor \leq \lfloor \log_2 l_1 \rfloor$ and $\lfloor \log_2 c \rfloor \leq \lfloor \log_2 l_2 \rfloor$, therefore Center does not have a winning strategy. Moreover, $\lfloor \log_2 r \rfloor$ is less than or equal to $\lfloor \log_2 l_2 \rfloor$ ($\lfloor \log_2 l_1 \rfloor$), and Center's move cannot affect this relation, therefore neither does Right have a winning strategy. Analogous reasoning holds for $G = [l, c, r] <_{LR} 0$, $G = [l, c, r] <_{LC} 0$, and $G = [l, c, r] < 0$, as shown in Tables 5 and 6.

Table 6: Outcome classes for $G < 0$.

| $G < 0$ | Left starts | Center starts | Right starts |
|---|---|---|---|
| $L > C$, $L > R$ | Left wins/Queer | Left wins/Queer | Left wins/Queer |
| $C > L$, $C > R$ | Center wins/Queer | Center wins/Queer | Center wins/Queer |
| $R > L$, $R > C$ | Right wins/Queer | Right wins/Queer | Right wins/Queer |
| $L = C$, $L > R$ | Center wins/Queer | Left wins/Queer | Center wins/Queer |
| $L = R$, $L > C$ | Right wins/Queer | Left wins/Queer | Left wins/Queer |
| $C = R$, $C > L$ | Right wins/Queer | Right wins/Queer | Center wins/Queer |
| $L = C$, $L = R$ | Right wins/Queer | Left wins/Queer | Center wins/Queer |

$L = \lfloor \log_2 l \rfloor$, $C = \lfloor \log_2 c \rfloor$, $R = \lfloor \log_2 r \rfloor$.

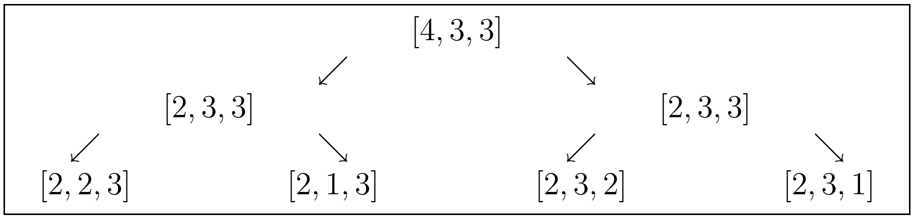


Figure 2: The game $[4, 3, 3]$ after three moves.

We briefly recall the definition of *queer* game introduced by Propp [4]:

**Definition 17.** A position in a three-player combinatorial game is called queer if no player can force a win.

## 4. $[4, 3, 3]$ is a Queer Game

Let's consider the game $G = [4, 3, 3]$. We observe that $\lfloor \log_2 4 \rfloor = \lfloor \log_2 3 \rfloor + \lfloor \log_2 3 \rfloor$, therefore $G <_{CR} 0$. When Center or Right makes the first move, Left always has a winning strategy. When Left makes the first move, we know by previous theorems that neither Center nor Right has a winning strategy; therefore, we have two possible cases: either Left, according to the Conjecture 2, has a winning strategy, or $G$ is a queer game. We show that Left has no winning strategy.

In the beginning, Left has two possible moves: $[4, 3, 3] \to [3, 3, 3] + [1, 3, 3]$ and $[4, 3, 3] \to [2, 3, 3] + [2, 3, 3]$. In the first case, Center and Right can make 4 moves each in $[1, 3, 3]$, and Left can make at most two moves in $[3, 3, 3]$, therefore Left cannot win the game. In the second case, Center and Right can move as shown in Fig. 2 resulting in the instance [2,2,3]+[2,1,3]+[2,3,2]+[2,3,1]. Now, Left has 4 possible moves:

1. $[2, 2, 3] \to [1, 2, 3] + [1, 2, 3]$. In this case, Center and Right can make respectively 4 and 6 moves in $[1, 2, 3] + [1, 2, 3]$. Left can make at most 3 moves in the other blocks therefore he/she has no winning strategy.

2. $[2, 1, 3] \to [1, 1, 3] + [1, 1, 3]$. In this case, Center plays $[2, 2, 3] \to [2, 1, 3] + [2, 1, 3]$ and Right plays $[1, 1, 3] \to [1, 1, 2] + [1, 1, 1]$. Therefore, the instance becomes $[2, 1, 3] + [2, 1, 3] + [1, 1, 2] + [1, 1, 1] + [1, 1, 3] + [2, 3, 2] + [2, 3, 1]$ and you can easily check that Left has no winning strategy.

3. $[2, 3, 2] \to [1, 3, 2] + [1, 3, 2]$. Analogous to the first case.

4. $[2, 3, 1] \to [1, 3, 1] + [1, 3, 1]$. In this case, Center plays $[1, 3, 1] \to [1, 2, 1] + [1, 1, 1]$ and Right plays $[2, 3, 2] \to [2, 3, 1] + [2, 3, 1]$. Therefore, the instance becomes $[2, 2, 3] +$

$[2,1,3]+[2,3,1]+[2,3,1]+[1,1,2]+[1,1,1]+[1,1,3]$, and again you can easily check that Left has no winning strategy.

## 5. Conclusions and Future Work

In conclusion, a classification of the instances of Cutblock is presented. It is not trivial to establish the winner for a generic block $[l, c, r]$, because the generalization of the rule used in the game of Cutcake does not work for all instances. Future work will concern the resolution of the following open problems:

- Is Conjecture 1 true?
- It is amazing to observe that in the game of Cutblock, both $[4, 3, 3]$ and $[4, 2, 2]$ are $<_{CR} 0$, but in $[4, 2, 2]$ Left still has a winning strategy when he/she makes the first move. Is it possible to establish a criteria to identify queer games?
- A general instance of Cutblock is defined as $G = G_1 \cup G_2 \cup \ldots G_n$ where every $G_i = [l_i, c_i, r_i]$ is an arbitrary block for all $1 \le i \le n$. Which is the complexity of $G$?

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