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# THE DYING RABBIT PROBLEM REVISITED

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**Abstract**

In this paper we study a generalization of the Fibonacci sequence in which rabbits are mortal and take more that two months to become mature. In particular we give a general recurrence relation for these sequences (improving work by Hoggat and Lind) and we calculate explicitly their general term (extending work by Miller). In passing, and as a technical requirement, we also study the behavior of the positive real roots of the characteristic polynomial of the considered sequences.

## 1. Introduction

Fibonacci numbers arose in the answer to a problem proposed by Leonardo of Pisa who asked for the number of rabbits at the $n^{th}$ month if there is one pair of rabbits at the $0^{th}$ month which becomes mature one month later and that breeds another pair in each of the succeeding months, and if these new pairs breed again in the second month following birth. It can be easily proved by induction that the number of pairs of rabbits at the $n^{th}$ month is given by $f_n$, with $f_n$ satisfying the recurrence relation:

$$f_0 = f_1 = 1;$$

$$f_n = f_{n-1} + f_{n-2}, \text{ for every } n \geq 2.$$

It is not the point here to state any of the many properties of these numbers (see [9] for a good account of them); nevertheless we will recall that if $r_1 < r_2$ are the roots of the polynomial $g(x) = x^2 - x - 1$ then we can see that:

$$f_n = \frac{r_1^n}{r_1 - r_2} + \frac{r_2^n}{r_2 - r_1}. \tag{1}$$

In [7] the $k$-generalized Fibonacci numbers $f_n^{(k)}$ are defined as follows:

$$f_n^{(k)} = 1 \qquad \text{for every } 0 \leq n \leq k-1;$$

$$f_n^{(k)} = \sum_{i=1}^{k} f_{n-i}^{(k)} \qquad \text{for every } n \geq k.$$

In this paper Miles proves, among other results, that if $r_1, \ldots, r_k$ are the (distinct) roots of $g_k(x) = x^k - x^{k-1} - \cdots - x - 1$ then:

$$f_n^{(k)} = \sum_{i=1}^{k} \left( \prod_{i \neq j,\ 1 \leq j \leq k} (r_i - r_j)^{-1} \right) r_i^n, \tag{2}$$

which, of course, reduces to (1) if we set $k = 2$.

Later, in [6], Hoggat and Lind consider the so-called "dying rabbit problem", previously introduced in [1] and studied in [2] or [4], which consists of letting rabbits die.

The goal of this paper is to give a new look at the dying rabbit problem. In the second section we study a family of polynomials, focusing on the behavior of their positive roots. Although motivated by technical requirements, this study turns out to be of intrinsic interest. In the third section we will find a general recurrence relation for the sequence arising in this problem (which is given in [6] for only some particular cases) and we will deduce an explicit formula (which also generalizes the work by Miles) for the total number of live pairs at the $n^{\text{th}}$ time point. Finally, in an appendix, we give a procedure written using `Maple`® to calculate terms of the considered sequences.

## 2. A Family of Polynomials and Their Roots

Given natural numbers $h, k \geq 1$ we define the following polynomial:

$$g_{k,h}(x) = x^{k+h-1} - x^{k-1} - \cdots - x - 1.$$

In this section we will study the behavior of the roots of this polynomial in terms of $k$ and $h$. In particular we will be interested in the unique positive real root of $g_{k,h}(x)$. We will also study the polynomial $f_{k,h}(x) = (x-1)g_{k,h}(x) = x^{k+h} - x^{k+h-1} - x^k + 1$. These polynomials are closely related to those defined in [7] and [8]. In fact, if $h = 1$ they coincide.

**Proposition 1.** *If $k > 1$ the polynomial $g_{k,h}(x)$ has a unique positive real root $\alpha_{k,h}$ which lies in the interval $(1, 2)$.*

*Proof.* Apply Descartes' rule of signs and observe that $g_{k,h}(1) < 0 < g_{k,h}(2)$. □

**Remark.** Note that $\alpha_{1,h} = 1$ for all $h \geq 1$.

As a consequence of the previous proposition we have the following technical result.

**Lemma 2.** *The following hold:*

(a) *The real number $y \geq 0$ satisfies $y > \alpha_{k,h}$ if and only if $g_{k,h}(y) > 0$.*

(b) *The real number* $y \geq 0$ *satisfies* $1 < y < \alpha_{k,h}$ *if and only if* $f_{k,h}(y) < 0$.

c) *The polynomial* $g_{k,h}(x)$ *has no complex root with modulus in the interval* $(1, \alpha_{k,h})$.

*Proof.* Parts (a) and (b) are a direct consequence of the previous proposition. For part (c), if $g_{k,h}(w) = 0 = f_{k,h}(w)$ it follows that $|w^{k+h-1} + w^k| = |w^{k+h} + 1| \leq |w|^{k+h} + 1$ which cannot be true if $|w| \in (1, \alpha_{k,h})$. □

Now, the following result shows that the sequences $\varphi = \{\alpha_{k,h}\}_{k\geq 1}$ and $\psi = \{\alpha_{k,h}\}_{h\geq 1}$ are monotone.

**Proposition 3.** *The following hold:*

(a) *The sequence* $\varphi$ *is strictly increasing.*

(b) *The sequence* $\psi$ *is strictly decreasing for* $k > 1$ *and constant for* $k = 1$.

*Proof.* We start with (a). By definition we know that $g_{k+1,h}(\alpha_{k+1,h}) = 0$. Now, we have that

$$\alpha_{k+1,h}^{k+h-1} = \frac{\alpha_{k+1,h}^{k+h}}{\alpha_{k+1,h}} = \frac{\alpha_{k+1,h}^{k} + \alpha_{k+1,h}^{k-1} + \cdots + \alpha_{k+1,h} + 1}{\alpha_{k+1,h}} > \alpha_{k+1,h}^{k-1} + \cdots + \alpha_{k+1,h} + 1,$$

so $g_{k,h}(\alpha_{k+1,h}) > 0$ and the result follows from the previous lemma.

Moving onto (b), again by definition we have $g_{k,h}(\alpha_{k,h}) = 0$ and we can write $g_{k,h+1}(\alpha_{k,h}) = \alpha_{k,h}^{k+h} - \alpha_{k,h}^{k-1} - \cdots - \alpha_{k,h} - 1 = \alpha_{k,h}^{k+h} - \alpha_{k,h}^{k+h-1} = \alpha_{k,h}^{k+h-1}(\alpha_{k,h} - 1) \geq 0$, with the equality holding if and only if $k = 1$. An application of Lemma 2 completes the proof. □

Observe that since $\alpha_{k,h}$ is in the segment $(1, 2)$, the sequences $\varphi$ and $\psi$ are bounded and therefore they are convergent. Let $\alpha_h$ denote the unique positive root of the polynomial $p_h(x) = x^h - x^{h-1} - 1$.

**Proposition 4.** *The sequences* $\varphi$ *and* $\psi$ *converge to* $\alpha_h$ *and 1, respectively.*

*Proof.* Let us fix $h \geq 1$. Then for any $k \geq 2$ we have $\alpha_{k,h}^{k+h-1} = 1 + \alpha_{k,h} + \cdots + \alpha_{k,h}^{k-1} = \frac{\alpha_{k,h}^{k} - 1}{\alpha_{k,h} - 1}$ and thus $\alpha_{k,h}^{h} - \alpha_{k,h}^{h-1} - 1 = \frac{-1}{\alpha_{k,h}^{k}}$. Now, as we know that $\alpha_h = \lim_{k\to\infty} \alpha_{k,h} > 1$ it is enough to take limits in the previous equality to obtain the result.

Now let us fix $k \geq 1$. Then for any $h \geq 2$ we have $\alpha_{k,h}^{k+h-1} = 1 + \alpha_{k,h} + \cdots + \alpha_{k,h}^{k-1}$ so, we obtain the equality $\log \alpha_{k,h} = \frac{\log(1 + \alpha_{k,h} + \cdots + \alpha_{k,h}^{k-1})}{k+h-1}$. Finally, writing $\beta_k = \lim\limits_{h\to\infty} \alpha_{k,h}$ and taking limits in the previous expression we arrive at $\log \beta_k = 0$ for every $k \geq 1$ and the proof is complete. □

The previous propositions can be summarized in the following diagram:

$$
\begin{array}{ccccccccccccccc}
\alpha_{1,1} & < & \alpha_{2,1} & < & \alpha_{3,1} & < & \alpha_{4,1} & < & \dots & < & \alpha_{k,1} & < & \dots & \to & \alpha_1 \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\alpha_{1,2} & < & \alpha_{2,2} & < & \alpha_{3,2} & < & \alpha_{4,2} & < & \dots & < & \alpha_{k,2} & < & \dots & \to & \alpha_2 \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\alpha_{1,3} & < & \alpha_{2,3} & < & \alpha_{3,3} & < & \alpha_{4,3} & < & \dots & < & \alpha_{k,3} & < & \dots & \to & \alpha_3 \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\alpha_{1,4} & < & \alpha_{2,4} & < & \alpha_{3,4} & < & \alpha_{4,4} & < & \dots & < & \alpha_{k,4} & < & \dots & \to & \alpha_4 \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\alpha_{1,5} & < & \alpha_{2,5} & < & \alpha_{3,5} & < & \alpha_{4,5} & < & \dots & < & \alpha_{k,5} & < & \dots & \to & \alpha_5 \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \ddots & & \vdots \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\alpha_{1,h} & < & \alpha_{2,h} & < & \alpha_{3,h} & < & \alpha_{4,h} & < & \dots & < & \alpha_{k,h} & < & \dots & \to & \alpha_h \\
\shortparallel & & \vee & & \vee & & \vee & & & & \vee & & & & \vee \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots & & & & \vdots \\
\shortparallel & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
1 & = & 1 & = & 1 & = & 1 & = & \dots & = & 1 & = & \dots & \to & 1
\end{array}
$$

For the rest of the section we will assume that $k > 1$. Before we go on, we introduce a result by Cauchy (see [3]) which will be useful in what follows. This result gives a bound on the modulus of the roots of a polynomial with complex coefficients. Let $n$ be a natural number and let $a_0, \dots, a_{n-1}$ be complex numbers not all equal to zero. For a complex polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ let $\widetilde{f}$ be the real polynomial $\widetilde{f}(x) = x^n - |a_{n-1}|x^{n-1} - \cdots - |a_1|x - |a_0|$. It is easy to see that $\widetilde{f}$ has a unique positive root $\gamma(\widetilde{f})$ (it exists since $f(0) < 0$ and $\lim_{x\to\infty} \widetilde{f}(x) = \infty$ and is unique by Descartes' rule again). Let $Z(f)$ denote the set of complex roots of $f$.

**Theorem 5.** *For every $w \in Z(f)$, the relation $|w| \leq \gamma(\widetilde{f})$ holds.*

**Corollary 6.** *For every $w \in Z(g_{k,h})$, the relation $|w| \leq \alpha_{k,h}$ holds.*

*Proof.* Since $\widetilde{g}_{k,h}(x) = g_{k,h}(x)$, it is enough to apply the previous theorem. □

Now, we can refine the Corollary 6 to see that the root $\alpha_{k,h}$ has the largest modulus among all roots of $g_{k,h}$ in the following way.

**Proposition 7.** *For every $w \in Z(g_{k,h})$, the equalities $|w| = \alpha_{k,h}$ and $w = \alpha_{k,h}$ are equivalent.*

*Proof.* Let $w \in Z(g_{k,h})$ be such that $|w| = \alpha_{k,h}$. Then $g_{k,h}(w) = g_{k,h}(|w|) = 0$ so we have that $w^{k+h-1} = w^{k-1} + \cdots + w + 1$ and $|w|^{k+h-1} = |w|^{k-1} + \cdots + |w| + 1$ and it follows that $|w^{k-1} + \cdots + w + 1| = |w|^{k-1} + \cdots + |w| + 1$ which, in particular, implies that $w$ is real. Now, since the only real roots of $g_{k,h}$ are $\alpha_{k,h}$ and $-1$ (only if $k$ is odd and $h$ is even) and since $|w| = \alpha_{k,h} > 1$, it follows that $w = \alpha_{k,h}$ as claimed. □

We will finish this section with the following proposition which will be of great technical importance in the next section.

**Proposition 5.** *All the roots of $g_{k,h}$ are distinct.*

*Proof.* We will show that $g_{k,h}(x)$ and $g'_{k,h}(x)$ have no common root. First observe that if $w$ is such a common root, then $w \neq 0, 1$ and it is also a common root of $f_{k,h}(x)$ and $\frac{f'_{k,h}(x)}{x^{k-1}} = (k+h)x^h - (k+h-1)x^{h-1} - k$. From $g_{k,h}(w) = 0$ it follows that $w^{k+h-2} = \frac{w^k-1}{w^2-w}$. Thus, $0 = (w^2 - w)g'_{k,h}(w) = hw^k + w^{k-1} + \cdots + w - (k+h-1)$ and consequently $w$ is a root of $r(x) = hx^k + x^{k-1} + \cdots + x - (k+h-1)$.

Now, if $|w| < 1$ we have that $k + h - 1 = |hw^k + w^{k-1} + \cdots + w| \leq h|w|^k + |w|^{k-1} + \cdots + |w| < k + h - 1$. This is a contradiction and it implies that $|w| \geq 1$.

Let us suppose that $|w| = 1$. Then, since $f_{k,h}(w) = w^{k+h} - w^{k+h-1} - w^k + 1 = 0$ it follows that $|w^k - 1| = |w - 1|$ which implies that either $w^k = w$ or $w^k = w^{-1}$. If $w^k = w$ then $w^h = 1$ and from $(k + h)w^h - (k + h - 1)w^{h-1} - k = 0$ we get that $w$ is rational. Also, if $w^k = w^{-1}$ then $w^{h-1} = -1$ and again $w$ must be rational. Since the only real root of $g_{k,h}(x)$ is $\alpha_{k,h}$ and it lies in the interval $(1, 2)$ we have a contradiction and $|w| > 1$.

Finally from Lemma 1 c) it follows that the only root of $g_{k,h}(x)$ which has modulus strictly bigger than 1 is $\alpha_{k,h}$ and since it can be easily verified that it is not a root of $g'_{k,h}(x)$, the claim follows. □

## 3. The Dying Rabbit Sequence

As we mentioned in the introduction, we are interested in generalizing the Fibonacci sequence by considering that rabbits become mature $h$ months after their birth and that they die $k$ months after their matureness. Throughout this section we will assume $k, h \geq 2$, For the case $h = 1$ we refer to Miles' paper [7] and the case $k = 1$ is trivial. We will denote by $C_n^{(k,h)}$ the number of couples of rabbits at the $n^{\text{th}}$ month. Obviously we have:

$$C_0^{(k,h)} = \cdots = C_{h-1}^{(k,h)} = 1.$$

Now let us denote by $C_n^{(h)}$ the recurrence sequence defined by:

$$C_0^{(h)} = \cdots = C_{h-1}^{(h)} = 1,\ C_n^{(h)} = C_{n-1}^{(h)} + C_{n-h}^{(h)} \text{ for every } n \geq h.$$

**Proposition 9.** *We have*

$$C_n^{(k,h)} = \begin{cases} C_n^{(h)}, & \text{if } 0 \le n \le k+h-2; \\ C_{n-h}^{(k,h)} + C_{n-h-1}^{(k,h)} + \cdots + C_{n-k-h+1}^{(k,h)}, & \text{if } n > k+h-2. \end{cases}$$

*Proof.* If $0 \le n \le h-1$ it is clear since the only couple of rabbits is the initial one.

If $h \le n \le k+h-2$ no rabbits have died yet so the number of couples at the $n^{\text{th}}$ month is the sum of the couples at the preceding month, $C_{n-1}^{(k,h)}$, and those produced by the couples which are mature at that point; i.e., $C_{n-h}^{(k,h)}$.

Finally, if $n > k+h-2$ the number of rabbits at the $n^{th}$ month can be computed as the sum of all the preceding couples except those which are not mature yet ($C_{n-j}^{(k,h)}$ with $1 \le j \le h-1$) and those which have died ($C_{n-j}^{(k,h)}$ with $j > k+h-1$). □

**Examples.** (See the appendix.)

- If $k = 3$ and $h = 2$, the beginning terms of $C_n^{(3,2)}$ are:

$$1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots$$

- If $k = 7$ and $h = 4$, then the beginning terms of $C_n^{(7,4)}$ are:

$$1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 13, 17, 23, 32, \ldots$$

**Remark.** If we consider $C_n^{(k,h)}$ for $0 \le n \le k+h-1$ as initial conditions, then it is clear that the characteristic polynomial of the recurrence sequence $C_n^{(k,h)}$ is precisely the polynomial $g_{k,h}(x)$ studied in the previous section. For instance, if $k = 3$ and $h = 2$ the recurrence relation defining our sequence is $C_n^{(3,2)} = C_{n-2}^{(3,2)} + C_{n-3}^{(3,2)} + C_{n-4}^{(3,2)}$ whose characteristic polynomial is easily seen to be $x^4 - x^2 - x - 1 = g_{3,2}(x)$.

If we denote by $r_1, r_2, \ldots, r_{k+h-1}$ the (distinct) complex roots of $g_{k,h}(x)$ it follows from Section 2 and from well-known facts from the theory of recurrence sequences that there exist constants $a_1, a_2, \ldots, a_{k+h-1}$ such that:

$$C_n^{(k,h)} = a_1 r_1^n + a_2 r_2^n + \cdots + a_{k+h-1} r_{k+h-1}^n,$$

where we can suppose that $r_1 = \alpha_{k,h}$. In particular we can calculate those constants solving the system of linear equations given by:

$$\sum_{i=1}^{k+h-1} a_i r_i^l = C_l^{(h)}, \quad 0 \le l \le k+h-2, \tag{3}$$

which can be expressed in matrix notation as follows:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_{k+h-1} \\ \dots & \vdots & \ddots & \vdots \\ r_1^{k+h-2} & r_2^{k+h-2} & \cdots & r_{k+h-1}^{k+h-2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k+h-1} \end{pmatrix} = \begin{pmatrix} C_0^{(h)} \\ C_1^{(h)} \\ \vdots \\ C_{k+h-2}^{(h)} \end{pmatrix}$$

and which has a unique solution because all the roots $r_i$ are distinct.

To solve this system of equations we will use Cramer's rule. Recall that if we put

$$V = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_{k+h-1} \\ \dots & \vdots & \ddots & \vdots \\ r_1^{k+h-2} & r_2^{k+h-2} & \cdots & r_{k+h-1}^{k+h-2} \end{vmatrix} = \prod_{k+h-1 \geq i > j \geq 1} (r_i - r_j),$$

$$D_n = \begin{vmatrix} 1 & \dots & 1 & C_0 & 1 & \dots & 1 \\ r_1 & \dots & r_{n-1} & C_1 & r_{n+1} & \dots & r_{k+h-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^{k+h-2} & \dots & r_{n-1}^{k+h-2} & C_{k+h-2} & r_{n+1}^{k+h-1} & \dots & r_{k+h-1}^{k+h-2} \end{vmatrix}$$

then $a_n = \dfrac{D_n}{V}$. So it is enough to find $D_n$ for $n = 1, \dots, k+h-1$. We will work out the case $n = 1$ completely, the other cases being analogous. Also note that we have replaced the values $C_j^{(h)}$ ($j = 0, \dots, k+h-2$) by arbitrary constants $C_j$ ($j = 0, \dots, k+h-2$), in order to admit sequences satisfying the same recurrence equation but with different initial conditions.

To compute $D_1$ we first need the following generalization of Vandermonde determinant which can be found in [5, Lemma 2.1].

**Lemma 10.** *If $e_m$ is the $m^{\text{th}}$ elementary symmetric polynomial in the variables $\{x_1, \dots, x_n\}$, then:*

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ x_1 & \dots & x_{n-1} & x_n \\ \vdots & \ddots & \vdots & \vdots \\ \widehat{x_1^l} & \dots & \widehat{x_{n-1}^l} & \widehat{x_n^l} \\ \vdots & \ddots & \vdots & \vdots \\ x_1^n & \dots & x_{n-1}^n & x_n^n \end{vmatrix} = \left( \prod_{n \geq i > j \geq 1} (x_i - x_j) \right) e_{n-l}(x_1, \dots, x_n).$$

If we apply this lemma and we expand the determinant $D_1$ by its first column we obtain:

$$D_1 = \sum_{l=0}^{k+h-2} (-1)^l C_l \left( \prod_{k+h-1 \geq i > j \geq 2} (r_i - r_j) \right) e_{k+h-2-l}(r_2, \dots, r_{k+h-1})$$

and, consequently:

$$a_1 = \frac{D_1}{V} = \frac{1}{\displaystyle\prod_{k+h-1\geq i\geq 2}(r_i - r_1)} \sum_{l=0}^{k+h-2} (-1)^l C_l e_{k+h-2-l}(r_2, \ldots, r_{k+h-1}). \quad (4)$$

We are now interested in computing the values $e_j(r_2, \ldots, r_{k+h-1})$ for $0 \leq j \leq k+h-2$. By Cardano's formulae and taking into account that $r_1, \ldots, r_{k+h-1}$ are the roots of $g_{k,h}(x)$ we have that:

$$e_0(r_1, \ldots, r_{k+h-1}) = 1;$$
$$e_1(r_1, \ldots, r_{k+h-1}) = \cdots = e_{h-1}(r_1, \ldots, r_{k+h-1}) = 0;$$
$$e_s(r_1, \ldots, r_{k+h-1}) = (-1)^{s+1} \quad \text{for every } h \leq s \leq k+h-1.$$

On the other hand, the following lemma is easy to prove.

**Lemma 11.** $e_t(x_2, \ldots, x_n) = \displaystyle\sum_{i=1}^{n-t} (-1)^{i+1} \frac{e_{t+i}(x_1, \ldots, x_n)}{x_1^i}$ *for every* $0 \leq t < n$.

We put this together to obtain:

$$e_0(r_2, \ldots, r_{k+h-1}) = 1;$$
$$e_s(r_2, \ldots, r_{k+h-1}) = (-1)^s \sum_{i=1}^{k} \frac{1}{r_1^{i+h-1-s}} \quad \text{for every } 1 \leq s \leq h-1;$$
$$e_s(r_2, \ldots, r_{k+h-1}) = (-1)^s \sum_{i=1}^{k+h-1-s} \frac{1}{r_1^i} \quad \text{for every } h \leq s \leq k+h-2.$$

Summing the geometric series we get:

$$e_0(r_2, \ldots, r_{k+h-1}) = 1;$$
$$e_s(r_2, \ldots, r_{k+h-1}) = (-1)^s \frac{r_1^k - 1}{r_1^{k+h-1-s}(r_1 - 1)} \quad \text{for every } 1 \leq s \leq h-1;$$
$$e_s(r_2, \ldots, r_{k+h-1}) = (-1)^s \frac{r_1^{k+h-1-s} - 1}{r_1^{k+h-1-s}(r_1 - 1)} \quad \text{for every } h \leq s \leq k+h-2.$$

Finally if we substitute in (4) we get:

$$a_1 = \frac{(-1)^{k+h}}{\displaystyle\prod_{k+h-1\geq i>2}(r_i - r_1)} \left[ \sum_{l=0}^{k-2} C_l \frac{r_1^{l+1} - 1}{r_1^{l+1}(r_1 - 1)} + \sum_{l=k-1}^{k+h-3} C_l \frac{r_1^k - 1}{r_1^{l+1}(r_1 - 1)} + C_{k+h-2} \right].$$

Reasoning in a similar way and taking into account the symmetry of the $e_s$ we can calculate $a_n$ for every $1 \leq n \leq k+h-1$. In fact:

$$a_n = \frac{(-1)^{k+h+n-1}}{\prod_{i>n}(r_i - r_n)\prod_{n>j}(r_n - r_j)} \left[ \sum_{l=0}^{k-2} C_l \frac{r_n^{l+1}-1}{r_n^{l+1}(r_n-1)} + \sum_{l=k-1}^{k+h-3} C_l \frac{r_n^k - 1}{r_n^{l+1}(r_n-1)} + C_{k+h-2} \right].$$

**Remark.** It is interesting to observe that $a_1 \neq 0$. As a consequence and recalling that $|r_i| < |r_1|$ for all $i \geq 2$ we have that $\lim_{n\to\infty} \frac{C_{n+1}^{(k,h)}}{C_n^{(k,h)}} = r_1 = \alpha_{k,h}$. This generalizes the fact that $\frac{f_{n+1}}{f_n} = \Phi$ where $f_n$ is the $n^{\text{th}}$ Fibonacci number and $\Phi$ is the golden section (note that $\alpha_{2,1} = \Phi$).

**Example.** (Padovan sequence). Recall that the so-called Padovan sequence is defined by $P_0 = P_1 = P_2 = 1$ and

$$P_n = P_{n-2} + P_{n-3}, \quad \text{for every } n \geq 3.$$

Thus, it is clear that in our notation $P_n = C_n^{(2,2)}$ with the initial conditions $C_0 = C_1 = C_2 = 1$. So we can apply our previous results to obtain:

$$P_n = \frac{r_1^2 + r_1 + 1}{2r_1 + 3} r_1^n + \frac{r_2^2 + r_2 + 1}{2r_2 + 3} r_2^n + \frac{r_3^2 + r_3 + 1}{2r_3 + 3} r_3^n,$$

which was already known to hold.

If we keep the same recurrence relation but replace the initial conditions by $P_0 = 3, P_1 = 0, P_2 = 2$ we obtain the so-called Perrin sequence, whose general term can be again computed with our formulas to obtain:

$$P_n = r_1^n + r_2^n + r_3^n.$$

Finally, if we keep our original initial conditions, that is, $C_0^{(2,2)} = C_1^{(2,2)} = 1$, and $C_2^{(2,2)} = 2$, then the general term of our Padovan-Perrin like sequence turns out to be:

$$C_n^{(2,2)} = \frac{(r_1+1)^2}{2r_1+3} r_1^n + \frac{(r_2+1)^2}{2r_2+3} r_2^n + \frac{(r_3+1)^2}{2r_3+3} r_3^n.$$

## 4. Appendix

In this appendix we give a short and easy procedure, written with Maple®, which computes any number of terms of $C_n^{(k,h)}$. It goes as follows:

```
dr:=proc(k,h,t)
local i;
for i from 0 by 1 to h-1 do c(i):=1 end do;
for i from h by 1 to k+h-2 do c(i):=c(i-1)+c(i-h); end do;
for i from k+h-1 to t do c(i):=sum(c(n), n=i-k-h+1..i-h); end do;
print(seq(c(n),n=0..t));
end proc:
```

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