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ON THE β-EXPANSION OF AN ALGEBRAIC NUMBER IN AN ALGEBRAIC BASE β

Yann Bugeaud
Université Louis Pasteur, U. F. R. de mathématiques,Strasbourg, France
bugeaud@math.u-strasbg.fr

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Abstract

Let α in $(0,1]$ and $\beta > 1$ be algebraic numbers. We study the asymptotic behaviour of the function that counts the number of digit changes in the β-expansion of α.

1. Introduction

Let $\beta > 1$ be a real number. The β-transformation T_β is defined on $[0,1]$ by $T_\beta : x \longmapsto \beta x \bmod 1$. In 1957, Rényi [12] introduced the β-expansion of a real x in $[0,1]$, denoted by $d_\beta(x)$ and defined by

$$d_\beta(x) = 0.x_1 x_2 \dots x_k \dots,$$

where $x_k = \lfloor \beta T_\beta^{k-1}(x) \rfloor$ for $k \geq 1$, except when β is an integer and $x = 1$, in which case $d_\beta(1) := 0.(\beta - 1) \dots (\beta - 1) \dots$ Here and throughout the present paper, $\lfloor \cdot \rfloor$ denotes the integer part function. Clearly, we have

$$x = \sum_{k \geq 1} \frac{x_k}{\beta^k}.$$

For $x < 1$, this expansion coincides with the representation of x computed by the 'greedy algorithm'. If β is an integer b, then the digits x_i of x lie in the set $\{0, 1, \dots, b-1\}$ and, if $x < 1$, then $d_b(x)$ corresponds to the b-ary expansion of x. If β is not an integer, then the digits x_i lie in the set $\{0, 1, \dots, \lfloor \beta \rfloor\}$. We direct the reader to [2] and to the references quoted therein for more on β-expansions. Throughout this note, we say that $d_\beta(x)$ is finite (resp. infinite) if there are only finitely many (resp. there are infinitely many) non-zero digits in the β-expansion of x.

We stress that the β-expansion of 1 has been extensively studied, for it yields a lot of information on the β-shift. In particular, Blanchard [5] proposed a classification of the β-shifts according to the properties of the (finite or infinite) word given by $d_\beta(1)$, see Section 4 of [2]. The occurrences of consecutive 0's in $d_\beta(1)$ play a crucial role in Blanchard's classification of the β-shifts. This motivates the following problem first investigated in [17].

Let $\beta > 1$ be a real number such that $d_\beta(1)$ is infinite and let $(a_k)_{k\geq 1}$ be the β-expansion of 1. Assume that there exist a sequence of positive integers $(r_n)_{n\geq 1}$ and an increasing sequence of positive integers $(s_n)_{n\geq 1}$ such that

$$a_{s_n+1} = a_{s_n+2} = \cdots = a_{s_n+r_n} = 0, \quad a_{s_n+r_n+1} \neq 0,$$

and $s_{n+1} > s_n + r_n$ for every positive integer n. The problem is then to estimate the gaps between two consecutive non-zero digits in $d_\beta(1)$, that is, to estimate the asymptotic behaviour of the ratio r_n/s_n.

The main result of [17], quoted as Theorem VG below, mainly shows that $d_\beta(1)$ cannot be 'too lacunary' when β is an algebraic number. Recall that the Mahler measure of a real algebraic number θ, denoted by $M(\theta)$, is, by definition, equal to the product

$$M(\theta) := a \prod_{i=1}^{d} \max\{1, |\theta_i|\},$$

where $\theta = \theta_1, \theta_2, \ldots, \theta_d$ are the complex conjugates of θ and a is the leading coefficient of the minimal defining polynomial of θ over the integers.

Theorem VG. *Let $\beta > 1$ be a real algebraic number. Then, with the above notation, we have*

$$\limsup_{n\to\infty} \frac{r_n}{s_n} \leq \frac{\log M(\beta)}{\log \beta} - 1.$$

Theorem VG was extended in [2], where, roughly speaking, repetitions of arbitrary (finite) blocks in the β-expansion of an algebraic number (where $\beta > 1$ is algebraic) are studied, see Theorem 2 from [2] for a precise statement.

The purpose of the present note is to study the β-expansion of an algebraic number α from another point of view, introduced in [8]. We aim at estimating the asymptotic behaviour of the number of digit changes in $d_\beta(\alpha)$. For α in $(0, 1]$, write

$$d_\beta(\alpha) = 0.a_1a_2\ldots,$$

and define the function nbdc_β, 'number of digit changes in the β-expansion', by

$$\mathrm{nbdc}_\beta(n, \alpha) = \mathrm{Card}\{1 \leq k \leq n : a_k \neq a_{k+1}\},$$

for any positive integer n. This function was first studied in [8] when β is an integer, see also [9] for an improvement of the main result of [8]. The short Section 6 of [8] is devoted to the study of nbdc_β for β algebraic, but it contains some little mistakes (see below) and its main result can be strengthened (see Theorem 2 below).

The present note is organized as follows. Our results on the behaviour of the function nbdc_β when β is an algebraic number are stated in Section 2 and proved in Section 4. New results on values of lacunary series at algebraic points are discussed in Section 3.

2. Results

We begin by stating a consequence of Theorem 2 from [2], that can also be obtained with the tools used in [17].

Theorem 1. *Let $\beta > 1$ be a real algebraic number. Let α be an algebraic number in $(0,1]$. If $d_\beta(\alpha)$ is infinite, then*

$$\liminf_{n\to+\infty} \frac{\mathrm{nbdc}_\beta(n,\alpha)}{\log n} \geq \left(\log\left(\frac{\log M(\beta)}{\log\beta}\right)\right)^{-1}. \tag{1}$$

For the sake of completeness, Theorem 1 is established in Section 4 along with the proof of Theorem 2.

A Pisot (resp. Salem) number is an algebraic integer greater than 1 whose conjugates are of modulus less than 1 (resp. less than or equal to 1, with at least one conjugate on the unit circle). In particular, an algebraic number $\beta > 1$ is a Pisot or a Salem number if, and only if, $M(\beta) = \beta$. In that case, Theorem 1 implies that

$$\frac{\mathrm{nbdc}_\beta(n,\alpha)}{\log n} \underset{n\to+\infty}{\longrightarrow} +\infty. \tag{2}$$

The main purpose of the present note is to show how the use of a suitable version of the Quantitative Subspace Theorem allows us to strengthen (2.2).

Theorem 2. *Let β be a Pisot or a Salem number. Let α be an algebraic number in $(0,1]$ such that $d_\beta(\alpha)$ is infinite and write*

$$d_\beta(\alpha) = 0.a_1a_2\ldots a_k\ldots$$

Then, there exists an effectively computable constant $c(\alpha,\beta)$, depending only on α and β, such that

$$\mathrm{nbdc}_\beta(n,\alpha) \geq c(\alpha,\beta)\,(\log n)^{3/2}\cdot(\log\log n)^{-1/2}, \tag{3}$$

for every positive integer n.

We stress that the exponent of $(\log n)$ in (3) is independent of β, unlike in Theorem 3 of [8]. This is a consequence of the use of the Parametric Subspace Theorem, exactly as in Theorem 3.1 of [9]. Note that Theorem 3 of [8] is not correctly stated: indeed, it claims a result valid for all expansions, whereas in the proof we are led to construct good algebraic approximations to α and to use one property of the β-expansion (see (4.14) below) to ensure that, roughly speaking, all these approximations are different.

We display two immediate corollaries of Theorem 2. A first one is concerned with the number of non-zero digits in the β-expansion of an algebraic number for β being a Pisot or a Salem number.

Corollary 3. *Let ε be a positive real number. Let β be a Pisot or a Salem number. Let α be an algebraic number in $(0,1]$ whose β-expansion is infinite. Then, for n large enough, there are at least*

$$(\log n)^{3/2} \cdot (\log \log n)^{-1/2-\varepsilon}$$

non-zero digits among the first n digits of the β-ary expansion of α.

For $\beta = 2$, Corollary 3 gives a much weaker result than the one obtained by Bailey, Borwein, Crandall, and Pomerance [3], who proved that, among the first n digits of the binary expansion of a real irrational algebraic number ξ of degree d, there are at least $c(\xi)n^{1/d}$ occurrences of the digit 1, where $c(\xi)$ is a suitable positive constant (see also Rivoal [14]).

Recall that β is called a *Parry number* if $d_\beta(1)$ is finite or eventually periodic. Every Pisot number is a Parry number [15, 4] and K. Schmidt [15] conjectured that all Salem numbers are Parry numbers. This was proved for all Salem numbers of degree 4 by Boyd [6], who gave in [7] a heuristic suggesting the existence of Salem numbers of degree 8 that are not Parry numbers.

We highlight the special case of the β-expansion of 1 in a base β that is a Salem number.

Corollary 4. *Let ε be a positive real number. Let β be a Salem number. Assume that $d_\beta(1)$ is infinite and write*

$$d_\beta(1) = 0.a_1 a_2 \ldots$$

For any sufficiently large integer n, we have

$$a_1 + \ldots + a_n > (\log n)^{3/2} \cdot (\log \log n)^{-1/2-\varepsilon},$$

and there are at least $(\log n)^{3/2} (\log \log n)^{-1/2-\varepsilon}$ indices j with $1 \leq j \leq n$ and $a_j \neq 0$.

In view of Theorem 2, our Corollaries 3 and 4 can be (very) slightly improved.

3. On Values of Lacunary Series at Algebraic Points

The following problem was posed in Section 7 of [8].

Problem 5. Let $\mathbf{n} = (n_j)_{j \geq 1}$ be a strictly increasing sequence of positive integers and set

$$f_{\mathbf{n}}(z) = \sum_{j \geq 1} z^{n_j}. \qquad (4)$$

If the sequence $\mathbf{n}$ increases sufficiently rapidly, then the function $f_{\mathbf{n}}$ takes transcendental values at every non-zero algebraic point in the open unit disc.

By a clever use of the Schmidt Subspace Theorem, Corvaja and Zannier [10] proved that the conclusion of Problem 5 holds for $f_{\mathbf{n}}$ given by (3.1) when the strictly increasing sequence $\mathbf{n}$ is lacunary, that is, satisfies

$$\liminf_{j \to +\infty} \frac{n_{j+1}}{n_j} > 1.$$

Under the weaker assumption that

$$\limsup_{j \to +\infty} \frac{n_{j+1}}{n_j} > 1,$$

it follows from the Ridout Theorem that the function $f_{\mathbf{n}}$ given by (3.1) takes transcendental values at every point $1/b$, where $b \geq 2$ is an integer (see, e.g., Satz 7 from Schneider's monograph [16]), and even at every point $1/\beta$, where β is a Pisot or a Salem number [1] (see also Theorem 3 of [10]).

The latter result can be improved with the methods of the present paper. Namely, we extend Corollary 4 of [8] and Corollary 3.2 of [9] as follows.

Corollary 6. *Let β be a Pisot or a Salem number. For any real number $\eta > 2/3$, the sum of the series*

$$\sum_{j \geq 1} \beta^{-n_j}, \quad \text{where } n_j = 2^{\lfloor j^\eta \rfloor} \text{ for } j \geq 1, \tag{5}$$

is transcendental.

The growth of the sequence $(n_j)_{j \geq 1}$ defined in (5) shows that our Corollary 6 is not a consequence of the results of [10].

To establish Corollary 6, it is enough to check that, for any positive integer N, the number of positive integers j such that $2^{\lfloor j^\eta \rfloor} \leq N$ is less than some absolute constant times $(\log N)^{1/\eta}$, and to apply Theorem 2.

To be precise, to establish Corollary 3, we prove that any real number α having an expansion in base β given by (5) is transcendental. We do not need to assume (or to prove) that (5) is the β-expansion of α. Namely, this assumption is used in the proof to guarantee that the approximants α_j constructed in the proof of Theorem 2 are (essentially) all different. Under the assumption of Corollary 3, this condition is automatically satisfied.

4. Proofs

The proof of Theorem 2 follows the same lines as that of Theorem 1 of [8]. For convenience, we first explain the case where β is an integer $b \geq 2$. Then, we point out which changes have to be made to treat the case of a real algebraic number $\beta > 1$.

The key point for our argument is the following result of Ridout [13].

For a prime number ℓ and a non-zero rational number x, we set $|x|_\ell := \ell^{-u}$, where u is the exponent of ℓ in the prime decomposition of x. Furthermore, we set $|0|_\ell = 0$. With this notation, the main result of [13] reads as follows.

Theorem (Ridout, 1957) *Let S_1 and S_2 be disjoint finite sets of prime numbers. Let θ be a real algebraic number. Let ε be a positive real number. Then there are only finitely many rational numbers p/q with $q \geq 1$ such that*

$$0 < \left|\theta - \frac{p}{q}\right| \cdot \prod_{\ell \in S_1} |p|_\ell \cdot \prod_{\ell \in S_2} |q|_\ell < \frac{1}{q^{2+\varepsilon}}. \tag{6}$$

More precisely, we need a quantitative version of Ridout's theorem, namely an explicit upper bound for the number of solutions to (6). In this direction, Locher [11] proved that, if $\varepsilon < 1/4$, the degree of θ is at most d and its Mahler measure at most H, then (6) has at most

$$\mathcal{N}_1(\varepsilon) := c_1(d)\, e^{7s} \varepsilon^{-s-4} \log(\varepsilon^{-1}) \tag{7}$$

solutions p/q with $q \geq \max\{H, 4^{4/\varepsilon}\}$, where s denotes the cardinality of the set $S_1 \cup S_2$, and $c_1(d)$ depends only on d.

Actually, as will be apparent below, in the present application of the quantitative Ridout's theorem, S_1 is the empty set and we have actually to estimate the total number of solutions to the system of inequalities

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{c}{q^{1+\varepsilon}}, \quad \prod_{\ell \in S_2} |q|_\ell < \frac{c}{q}, \tag{8}$$

where c is a positive integer. Every solution to (8) with q large is a solution to (6), with ε replaced by 2ε, but the converse does not hold. Furthermore, the best known upper bound for the total number of large solutions to (8) does not depend on the set S_2. Namely, if $\varepsilon < 1/4$, then there exists an explicit number $c_2(d)$, depending only on the degree d of θ, such that (8) has at most

$$\mathcal{N}_2(\varepsilon) := c_2(d)\varepsilon^{-3} \log(\varepsilon^{-1}) \tag{9}$$

solutions p/q with $q \geq \max\{2H, 4^{4/\varepsilon}\}$; see Corollary 5.2 of [9]. Since there is no dependence on s in (9), unlike in (4.2), this explains the improvement obtained in [9] on the result from [8].

After these preliminary remarks, let us explain the method of the proof. Let α be an irrational (otherwise, the result is clearly true) real number in $(0, 1)$ and write

$$\alpha = \sum_{k \geq 1} \frac{a_k}{b^k} = 0.a_1 a_2 \ldots$$

Define the increasing sequence of positive integers $(n_j)_{j\geq 1}$ by $a_1 = \ldots = a_{n_1}$, $a_{n_1} \neq a_{n_1+1}$ and $a_{n_j+1} = \ldots = a_{n_{j+1}}$, $a_{n_{j+1}} \neq a_{n_{j+1}+1}$ for every $j \geq 1$. Observe that

$$\mathrm{nbdc}_b(n, \alpha) = \max\{j : n_j \leq n\} \tag{10}$$

for $n \geq n_1$, and that $n_j \geq j$ for $j \geq 1$. To construct good rational approximations to α, we simply truncate its b-ary expansion at rank a_{n_j+1} and then complete with repeating the digit a_{n_j+1}. Precisely, for $j \geq 1$, we define the rational number

$$\alpha_j = \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \sum_{k=n_j+1}^{+\infty} \frac{a_{n_j+1}}{b^k} = \sum_{k=1}^{n_j} \frac{a_k}{b^k} + \frac{a_{n_j+1}}{b^{n_j}(b-1)} =: \frac{p_j}{b^{n_j}(b-1)}.$$

Set $q_j := b^{n_j}(b-1)$ and take for S_2 the set of prime divisors of b. Observe that

$$0 < |\alpha - \alpha_j| < \frac{1}{b^{n_{j+1}}}, \quad \prod_{\ell \in S_2} |q_j|_\ell = \frac{b-1}{q_j}\cdot \tag{11}$$

On the other hand, the Liouville inequality as stated by Waldschmidt [18], p. 84, asserts that there exists a positive constant c, depending only on α, such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^d}, \quad \text{for all positive integers } p,\ q,$$

where d is the degree of α. Consequently, we have

$$n_{j+1} \leq 2dn_j, \tag{12}$$

for every sufficiently large integer j, say for $j \geq j_0$.

It then follows from (11) and (12) that

$$0 < \left|\alpha - \frac{p_j}{q_j}\right| < \frac{(b-1)^{2d}}{q_j^{n_{j+1}/n_j}}, \quad \prod_{\ell \in S_2} |q_j|_\ell = \frac{b-1}{q_j}\cdot \tag{13}$$

Note that all the p_j/q_j's are different. We are in position to apply the quantitative form of the Ridout Theorem to (4.8). Let ε be a real number with $0 < \varepsilon < 1/4$. Let $J > j_0$ be a large positive integer. It follows from Corollary 5.2 of [9] that there exist at most $\mathcal{N}_2(\varepsilon)$ positive integers $j > J$ such that $n_{j+1} \geq (1+\varepsilon)n_j$. Consequently, we infer from (9) and (12) that

$$\frac{n_J}{n_{j_0}} = \frac{n_J}{n_{J-1}} \times \ldots \times \frac{n_{j_0+1}}{n_{j_0}} \leq (1+\varepsilon)^J \, (2d)^{\mathcal{N}_2(\varepsilon)},$$

and

$$\log n_J \ll J\varepsilon + \varepsilon^{-4},$$

where the numerical constant implied in $\ll$ depends only on α. Selecting $\varepsilon = J^{-1/5}$, we get that

$$\log n_J \ll J^{4/5}. \tag{14}$$

By (10), this implies a lower bound for $\mathrm{nbdc}_b(n, \alpha)$. Here, to get (14), we have used a rather crude upper bound for $\mathcal{N}_2(\varepsilon)$. A further refinement can be obtained by means of the trick that allowed us to prove Theorem 3.1 of [9], which is similar to Theorem 2 for $\beta = b$.

Replacing b by an algebraic number $\beta > 1$, everything goes along the same lines, except that we have to apply a suitable extension of Ridout's theorem, and several technical difficulties arise.

We proceed exactly as above, keep the same notation, and set

$$\alpha_j = \sum_{k=1}^{n_j} \frac{a_k}{\beta^k} + \sum_{k=n_j+1}^{+\infty} \frac{a_{n_j+1}}{\beta^k} = \sum_{k=1}^{n_j} \frac{a_k}{\beta^k} + \frac{a_{n_j+1}}{\beta^{n_j}(\beta-1)} =: \frac{p_j}{\beta^{n_j}(\beta-1)}. \tag{15}$$

Here, p_j is an element of the number field generated by β. We have to prove that α_j is distinct from α: unlike when β is an integer, this is not straightforward.

Recall that $a_{n_j+1} = \ldots = a_{n_{j+1}}$ and $a_{n_{j+1}} \neq a_{n_{j+1}+1}$. Assume first that

$$a_{n_j+1} > a_{n_{j+1}+1}. \tag{16}$$

Then, using (15), we have

$$\alpha_j - \sum_{k=1}^{n_{j+1}} \frac{a_k}{\beta^k} \geq \frac{a_{n_j+1}}{\beta^{n_{j+1}+1}} + \frac{a_{n_j+1}}{\beta^{n_{j+1}+2}}, \tag{17}$$

while

$$\alpha - \sum_{k=1}^{n_{j+1}} \frac{a_k}{\beta^k} \leq \frac{a_{n_{j+1}+1}}{\beta^{n_{j+1}+1}} + \frac{1}{\beta^{n_{j+1}+1}} \leq \frac{a_{n_j+1}}{\beta^{n_{j+1}+1}}, \tag{18}$$

since, by the property of the β-expansion,

$$\sum_{k \geq r+1} \frac{a_k}{\beta^k} \leq \frac{1}{\beta^r}, \quad \text{for every } r \geq 0. \tag{19}$$

Note that $a_{n_j+1} \geq 1$, by (16). Combining this with (17) and (18), we get that

$$\alpha_j - \alpha \geq \frac{1}{\beta^{n_{j+1}+2}}. \tag{20}$$

Assume now that

$$a_{n_j+1} < a_{n_{j+1}+1}. \tag{21}$$

Then, we have

$$\alpha_j - \sum_{k=1}^{n_{j+1}} \frac{a_k}{\beta^k} = \frac{a_{n_j+1}}{\beta^{n_{j+1}}(\beta-1)}, \tag{22}$$

while

$$\alpha - \sum_{k=1}^{n_{j+1}} \frac{a_k}{\beta^k} > \frac{a_{n_{j+1}+1}}{\beta^{n_{j+1}+1}} \geq \frac{a_{n_j+1}+1}{\beta^{n_{j+1}+1}}, \tag{23}$$

by (21). Since $a_{n_j+1} < \beta - 1$, we infer from (22) that

$$\alpha_j - \sum_{k=1}^{n_{j+1}} \frac{a_k}{\beta^k} < \frac{1}{\beta^{n_{j+1}+1}},$$

and then from (23) that

$$\alpha - \alpha_j > 0.$$

Note also that, by (23), at least one of the following statements holds:

$$n_{j+2} = n_{j+1} + 1 \quad \text{and} \quad 0 = a_{n_{j+2}+1} < a_{n_{j+1}+1} \tag{24}$$

or

$$\alpha - \alpha_j > \frac{1}{\beta^{n_{j+1}+2}}. \tag{25}$$

On the other hand, we check that

$$|\alpha - \alpha_j| \ll \frac{1}{\beta^{n_{j+1}}}, \quad \text{for } j \geq 1. \tag{26}$$

Here, and throughout the end of the paper, the constants implied by $\ll$ depend only on α and β. Disregarding the indices j for which we are in case (24) (and this concerns at most one index in every pair $(j, j+1)$), we infer from (20), (25), and (26) that the number of occurrences of a given element in the sequence $(\alpha_j)_{j\geq 1}$ is bounded by an absolute constant.

Now, we apply the extension to number fields of the aforementioned results of Ridout and Locher. We keep the notation from Section 6 of [2], noticing that the r_n (resp. s_n) in that paper corresponds to our n_j (resp. to 1). In particular, the height function H is defined as in [2].

Let $\mathbf{K}$ be the number field generated by α and β and denote by D its degree. We consider the following linear forms, in two variables and with algebraic coefficients. For the place v_0 corresponding to the embedding of $\mathbf{K}$ defined by $\beta \hookrightarrow \beta$, set $L_{1,v_0}(x,y) = x$ and $L_{2,v_0}(x,y) = \alpha(\beta-1)x + y$. It follows from (15) and (26) that

$$|L_{2,v_0}(\beta^{n_j}, -p_j)|_{v_0} \ll \frac{1}{\beta^{(n_{j+1}-n_j)/D}},$$

where we have chosen the continuation of $|\cdot|_{v_0}$ to $\overline{\mathbf{Q}}$ defined by $|x|_{v_0} = |x|^{1/D}$.

Denote by S'_∞ the set of all other infinite places on $\mathbf{K}$ and by S_0 the set of all finite places v on $\mathbf{K}$ for which $|\beta|_v \neq 1$. For any v in $S_0 \cup S'_\infty$, set $L_{1,v}(x,y) = x$ and $L_{2,v}(x,y) = y$. Denote by S the union of S_0 and the infinite places on $\mathbf{K}$. Clearly, for any v in S, the linear forms $L_{1,v}$ and $L_{2,v}$ are linearly independent.

To simplify the exposition, set

$$\mathbf{x}_j = (\beta^{n_j}, -p_j).$$

We wish to estimate the product

$$\Pi_j := \prod_{v \in S} \prod_{i=1}^{2} \frac{|L_{i,v}(\mathbf{x}_j)|_v}{|\mathbf{x}_j|_v}$$

from above. Arguing exactly as in [2], we get that

$$\begin{aligned}\Pi_j &\ll n_j^D\, \beta^{-n_{j+1}/D}\, M(\beta)^{n_j/D} \prod_{v \in S} |\mathbf{x}_j|_v^{-2} \\ &\ll n_j^D\, \beta^{-n_{j+1}/D}\, M(\beta)^{n_j/D}\, H(\mathbf{x}_j)^{-2}, \end{aligned} \tag{27}$$

since $|\mathbf{x}_j|_v = 1$ if v does not belong to S.

Note that

$$\beta^{n_j} \ll H(\mathbf{x}_j) \ll n_j^D\, M(\beta)^{n_j}. \tag{28}$$

Let ρ denote a positive real number that is strictly smaller than the right-hand side of (2.1). Assume that there are arbitrarily large integers n such that

$$\mathrm{nbdc}_\beta(n, \alpha) \leq \rho \log n.$$

Consequently, there must be infinitely many indices j with

$$n_{j+1} \geq \exp\{\rho^{-1}\} n_j.$$

It then follows from (4.22) and (4.23) that there are a positive real number ε and arbitrarily large integers j such that

$$\Pi_j \ll H(\mathbf{x}_j)^{-2-\varepsilon}.$$

We then get infinitely many indices j such that p_j/β^{n_j} takes the same value. This contradicts the fact that the number of occurrences of a given element in the sequence $(\alpha_j)_{j \geq 1}$ is bounded by an absolute constant, and proves Theorem 1.

From now on, we assume that $M(\beta) = \beta$. We infer from (4.22) and (4.23) that

$$\Pi_j \ll H(\mathbf{x}_j)^{-2-(n_{j+1}/n_j - 1)/(2D)}$$

as soon as j is sufficiently large.

We need a suitable extension of Corollary 5.2 from [9] to conclude (unfortunately, the notations used in [9] differ from ours). Exactly as in the case when β is an integer, we do not have to consider a product of linear forms, but rather a system

$$
\begin{aligned}
&|L_{1,v_0}(\mathbf{x}_j)|_{v_0} \le \kappa H(\mathbf{x}_j), && |L_{2,v_0}(\mathbf{x}_j)|_{v_0} \le H(\mathbf{x}_j)^{-\delta},\\
&|L_{1,v}(\mathbf{x}_j)|_{v} \le \kappa H(\mathbf{x}_j)^{-c_v}, && |L_{2,v}(\mathbf{x}_j)|_{v} \le H(\mathbf{x}_j)^{\eta} \quad (v \in S'_\infty),\\
&|L_{1,v}(\mathbf{x}_j)|_{v} \le \kappa H(\mathbf{x}_j)^{-c_v}, && |L_{2,v}(\mathbf{x}_j)|_{v} \le 1 \quad (v \in S_0).
\end{aligned}
$$

Here, κ is a positive real number, the c_v are defined by $|\beta|_v = \beta^{-c_v/D}$ and $\delta = (s'+1)\eta$, where s' is the cardinality of S'_∞. Observe that $\sum_{v\in S'_\infty \cup S_0} c_v = 1$. We do not work out the technical details. Everything goes along the same lines as in [9]. It remains to note that, by the general form of the Liouville inequality (as in [18], p. 83), we get that

$$|\alpha - \alpha_j| \gg \beta^{-Dn_j}.$$

This provides us with the needed extension of (12) and completes the sketch of the proof of Theorem 2.

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