#A22 INTEGERS 9 (2009), 239-259

BETA-EXPANSIONS WITH NEGATIVE BASES

Shunji Ito
Graduate School of Natural Science and Technology, Kanazawa University, Kanazawa, Japan
ito@t.kanazawa-u.ac.jp

Taizo Sadahiro
Prefectural University of Kumamoto, Kumamoto, Japan
sadahiro@pu-kumamoto.ac.jp

Received: 10/10/08, Accepted: 2/17/09

Abstract

This paper investigates representations of real numbers with an arbitrary *negative* base $-\beta < -1$, which we call the $(-\beta)$-expansions. They arise from the orbits of the $(-\beta)$-transformation which is a natural modification of the β-transformation. We show some fundamental properties of $(-\beta)$-expansions, each of which corresponds to a well-known fact of ordinary β-expansions. In particular, we characterize the admissible sequences of $(-\beta)$-expansions, give a necessary and sufficient condition for the $(-\beta)$-*shift* to be sofic, and explicitly determine the invariant measure of the $(-\beta)$-transformations.

1. Introduction

The β-expansions were introduced by Rényi [12] and have been studied extensively. This paper studies representations of real numbers with an arbitrary *negative* base $-\beta < -1$, which we call the $(-\beta)$-expansions, since they are natural modifications of the β-expansions. There exist several studies on expansions with negative bases (see e.g., [7, 5]), which are restricted to the negative integer bases. We show some fundamental properties of $(-\beta)$-expansions which correspond to those of ordinary β-expansions shown by Parry [11] and Bertrand-Mathis [3]: First, we introduce an order on the integer sequences different from that used in the ordinary β-expansions, by which we give a characterization of the digit sequences of $(-\beta)$-expansions. Second, we consider the $(-\beta)$-*shift*, which consists of bi-infinite sequence each of whose finite subword appears in the digit sequence of some $(-\beta)$-expansion. We show the $(-\beta)$-shift is sofic if and only if the $(-\beta)$-expansion of a special point is eventually periodic, just the same as the positive case. We do this by showing an efficient algorithm to construct a graph by which a given $(-\beta)$-shift is presented. Finally, we consider the frequency of the digits. We explicitly determine the absolutely continuous invariant measures of the $(-\beta)$-transformations which generate the $(-\beta)$-expansions. In contrast to the ordinary β-transformations, the invariant measures are not necessarily equivalent to the Lebesgue measure. Our results are formulated in a manner very similar to that of corresponding results for the ordinary β-expansions.

Let $\beta > 1$ be a real number. A $(-\beta)$-*representation* of a real number x is an expression of the form,

$$x = x_{-k}(-\beta)^k + x_{-k+1}(-\beta)^{k-1} + \cdots + x_0 + \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \cdots ,$$

where $k \geq 0$ is a certain integer and $x_i > 0$ for $i \geq -k$. It is denoted by

$$x = (x_{-k}x_{-k+1}\cdots x_0 \,.\, x_1x_2\cdots)_{-\beta}.$$

We denote by I_β the half-open interval $[l_\beta, r_\beta) = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. The $(-\beta)$-transformation T_β on I_β is defined by

$$T_\beta(x) = -\beta x - \lfloor -\beta x - l_\beta \rfloor = \{-\beta x - l_\beta\} + l_\beta,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding a real number x and $\{x\} = x - \lfloor x \rfloor$.

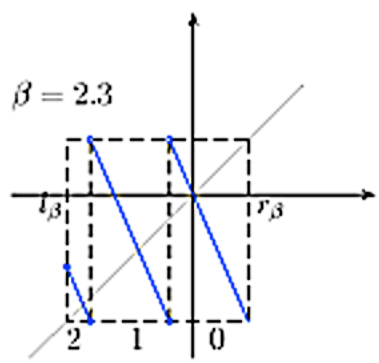

Figure 1: The $(-\beta)$-transformation with $\beta = 2.3$

Then, for each $x \in I_\beta$, we have a particular $(-\beta)$-representation

$$x = (\,.\,x_1x_2\cdots)_{-\beta},$$

where $x_i = \lfloor -\beta T_\beta^{i-1}(x) - l_\beta \rfloor$ for $i \geq 1$. We call this representation the $(-\beta)$-*expansion* of x. For a real number x not contained in I_β, there is an integer d such that $x/(-\beta)^d \in I_\beta$, hence we have the $(-\beta)$-expansion of x:

$$x = (x_{-d+1}x_{-d+2}\cdots x_0 \,.\, x_1x_2\cdots)_{-\beta}$$

where $x_{-d+i} = \lfloor -\beta T_\beta^{i-1}(\frac{x}{(-\beta)^d}) - l_\beta \rfloor$.

If $x \in I_\beta$ has the $(-\beta)$-expansion $x = (\,.\,x_1, x_2\cdots)_{-\beta}$ then we denote

$$d(x, -\beta) = (x_1, x_2, \ldots) \text{ and } d_n(x, -\beta) = x_n.$$

If the $(-\beta)$-representation of a real number x ends up with infinite repetition of 0's, that is, $x = (x_{-l} \cdots x_{-1}x_0 \,.\, x_1x_2 \cdots x_k000\cdots)_{-\beta}$, we occasionally omit writing 0s and denote it as $x = (x_{-l} \cdots x_0 \,.\, x_1x_2 \cdots x_k)_{-\beta}$. We call the $(-\beta)$-expansion of a real number *finite* if it ends up with infinite repetition of 0's. We denote by $(\overline{d_1, d_2, \ldots, d_m})$ the infinite repetition of the word $(d_1, d_2, \ldots, d_m)$, i.e.,

$$(\overline{d_1, d_2, \ldots, d_m}) = (d_1, d_2, \ldots, d_m, d_1, d_2, \ldots, d_m, d_1, d_2, \ldots, d_m, d_1, \ldots).$$

By the definition of $(-\beta)$-expansion, if $\beta \in \mathbb{N}$ then the $(-\beta)$-expansion of l_β is of the form $l_\beta = (\,.\,\beta\beta\beta\cdots)_{-\beta}$, while it has another $(-\beta)$-representation that looks much better:

$$l_\beta = (\,.\,(\beta-1)0(\beta-1)0(\beta-1)0\cdots)_{-\beta}.$$

In Section 2, we will consider this type of representations of l_β in a more general setting, which play the crucial role in our theory.

Example 1. The following are the $(-\beta)$-expansions of some real numbers when $\beta = 2$:

$$2 = (110.)_{-2},\ 3 = (111.)_{-2},\ 4 = (100.)_{-2}, \ldots, 100 = (110100100.)_{-2}, \ldots$$

$$-1 = (11\,.)_{-2},\ \ -2 = (10.)_{-2},\ \ -3 = (1101.)_{-2}, \ldots, -100 = (11101100.)_{-2}$$

$$2/3 = (1.111111\cdots)_{-2},\ \ 1/5 = (.011101110111\cdots)_{-2},$$

$$l_2 = -2/3 = (0.222222\cdots)_{-2}.$$

Example 2. Let $\beta = \frac{3+\sqrt{5}}{2}$. Then Table 1 shows the $(-\beta)$-expansions of several small integers. For this β, we can check that the $(-\beta)$-expansion of every element of $\mathbb{Z}[\beta]$ is finite by a method similar to that for the ordinary β-expansions by Akiyama [1]. In Section 3, we will see that the $(-\beta)$-shift (which is a shift space consisting of the bi-infinite sequences each of whose finite subword appears in the digit sequence of some $(-\beta)$-expansion) is the sofic shift represented by the graph shown in Figure 2.

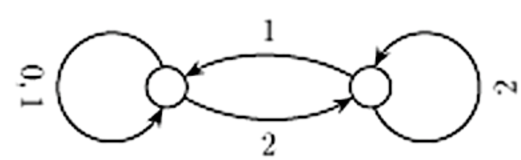

Figure 2: The graph which represents the $(-\beta)$-shift with $\beta = \frac{3+\sqrt{5}}{2}$.

x	$(-\beta)$-expansion of x	x	$(-\beta)$-expansion of x
1	$(1.)_{-\beta}$	-1	$(12.1)_{-\beta}$
2	$(121.21)_{-\beta}$	-2	$(11.1)_{-\beta}$
3	$(122.21)_{-\beta}$	-3	$(10.1)_{-\beta}$
4	$(110.11)_{-\beta}$	-4	$(21.021)_{-\beta}$
5	$(111.11)_{-\beta}$	-5	$(1212.121)_{-\beta}$
6	$(112.11)_{-\beta}$	-6	$(1211.121)_{-\beta}$
7	$(100.01)_{-\beta}$	-7	$(1210.121)_{-\beta}$
8	$(101.01)_{-\beta}$	-8	$(1222.221)_{-\beta}$
9	$(221.1021)_{-\beta}$	-9	$(1221.221)_{-\beta}$

Table 1: $(-\beta)$-expansions of small integers with $\beta = \frac{3+\sqrt{5}}{2}$.

2. Admissible Sequences

We say an integer sequence $(x_1, x_2, \ldots)$ is $(-\beta)$-*admissible*, if there exists a real number $x \in I_\beta$ such that $d(x, -\beta) = (x_1, x_2, \ldots)$. We say a finite word $(x_1, x_2, \ldots, x_n)$ over the alphabet $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor \beta \rfloor\}$ is $(-\beta)$-*admissible* if it appears in a $(-\beta)$-admissible sequence. This section gives a characterization of the $(-\beta)$-admissible sequences.

Proposition 3. *An integer sequence* $(x_1, x_2, \ldots)$ *is* $(-\beta)$*-admissible if and only if*

$$(\,.\,x_i x_{i+1} x_{i+2} \cdots)_{-\beta} \in I_\beta \text{ for all } i \geq 1. \tag{1}$$

Proof. The "only if" part is obvious. So assume (1) and put $x = (\,.\,x_1 x_2 x_3 \cdots)_{-\beta}$. We prove

$$x_i = \lfloor -\beta T_\beta^{i-1}(x) - l_\beta \rfloor, \quad \text{and} \quad T_\beta^i(x) = (\,.\,x_{i+1} x_{i+2} \cdots)_{-\beta} \tag{2}$$

for $i \geq 1$ by induction on i. Since $-\beta x - x_1 = (\,.\,x_2 x_3 \cdots)_{-\beta} \in I_\beta$, (2) holds for $i = 1$. Suppose (2) holds for $i < k$. Then it is easily confirmed that (2) holds for $i = k$. Thus (2) holds for all $i \geq 1$, which means, $x = (\,.\,x_1 x_2 \cdots)_{-\beta}$ is the $(-\beta)$-expansion of x. $\square$

To make Proposition 3 more explicit, we introduce an order $\prec$ on the sequences of integers in the following way. Let $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ be two finite or infinite integer sequences which have the same number of terms. Then we define

$$(x_1, x_2, \ldots) \prec (y_1, y_2, \ldots)$$

if and only if there exists an integer $k \geq 1$ such that $x_i = y_i$ for $i < k$ and $(-1)^k (x_k - y_k) < 0$. We denote $(x_1, x_2, \ldots) \preceq (y_1, y_2, \ldots)$ if $(x_1, x_2, \ldots) \prec (y_1, y_2, \ldots)$ or $(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$.

Let

$$d(l_\beta, -\beta) = (b_1, b_2, \cdots). \tag{3}$$

Then, by putting $b_0 = 0$, we have a $(-\beta)$-*representation* of r_β:

$$r_\beta = (\,.\,b_0 b_1 b_2 \cdots)_{-\beta}.$$

We denote the sequence $(b_0, b_1, b_2, b_3, \ldots)$ by $d(r_\beta, -\beta)$.

Example 4. Let β be a quadratic Pisot number whose minimal polynomial is $X^2 - aX - b$. Frougny and Solomyak [6] showed that the coefficients a and b satisfy

$$a \geq b > 0 \quad \text{or} \quad -a + 1 < b < 0.$$

By using Proposition 3, we have

$$d(l_\beta, -\beta) = \begin{cases} (a, \overline{a - b}), & a \geq b > 0, \\ (\overline{a - 1, -b}), & -a + 1 < b < 0. \end{cases}$$

Numerical experiments suggest that the $(-\beta)$-expansion of every element of $\mathbb{Z}[\beta^{-1}]$ is finite if $-a + 1 < b < 0$.

Proposition 5. *If $(x_1, x_2, \ldots)$ is a $(-\beta)$-admissible sequence, then*

$$d(l_\beta, -\beta) \preceq (x_{n+1}, x_{n+2}, \ldots) \prec d(r_\beta, -\beta) \text{ for all } n \geq 0.$$

In particular,

$$(b_1, b_2, \ldots) \preceq (b_{n+1}, b_{n+2}, \ldots) \prec (b_0, b_1, b_2, \ldots) \text{ for all } n \geq 0,$$

where $(b_1, b_2, \ldots) = d(l_\beta, -\beta)$ and $b_0 = 0$.

Proof. Since $(\,.\,x_1 x_2 \cdots)_{-\beta}$ is the $(-\beta)$-expansion of a real number $x \in I_\beta$, $T_\beta^n(x) = (\,.\,x_{n+1} x_{n+2} \cdots)_{-\beta}$ and hence

$$l_\beta = (\,.\,b_1 b_2 \cdots)_{-\beta} \leq (\,.\,x_{n+1} x_{n+2} \cdots)_{-\beta} < (\,.\,b_0 b_1 b_2 \cdots)_{-\beta} = r_\beta.$$

We first show $d(l_\beta, -\beta) \preceq (x_{n+1}, x_{n+2}, \ldots)$. Suppose that $(b_1, b_2, \ldots) \neq (x_{n+1}, x_{n+2}, \ldots)$, and let k be the integer such that $b_i = x_{n+i}$ for $i < k$ and $b_k \neq x_{n+k}$. Then we have

$$
\begin{aligned}
&(\,.\,x_{n+1}x_{n+2}\cdots)_{-\beta} - (\,.\,b_1b_2\cdots)_{-\beta}\\
&\quad= \left(\frac{x_{n+k}}{(-\beta)^k} + \frac{x_{n+k+1}}{(-\beta)^{k+1}} + \frac{x_{n+k+2}}{(-\beta)^{k+2}} + \cdots\right)\\
&\qquad\qquad - \left(\frac{b_k}{(-\beta)^k} + \frac{b_{k+1}}{(-\beta)^{k+1}} + \frac{b_{k+2}}{(-\beta)^{k+2}} + \cdots\right)\\
&\quad= \frac{1}{(-\beta)^k}\left((x_{n+k} - b_k) + (\,.\,x_{n+k+1}x_{n+k+2}\cdots)_{-\beta} - (\,.\,b_{k+1}b_{k+2}\cdots)_{-\beta}\right)\\
&\quad= \frac{1}{(-\beta)^k}\Big((x_{n+k} - b_k) + T_\beta^{n+k}(x) - T_\beta^k(l_\beta)\Big) < 0.
\end{aligned}
$$

Therefore, since $|T_\beta^{n+k}(x) - T_\beta^k(l_\beta)| < 1$, $x_{n+k} < b_k$ if k is an odd integer, and $x_{n+k} > b_k$ if k is even, that is, $d(l_\beta, -\beta) \prec (x_{n+1}, x_{n+2}, \ldots)$. We can show $(x_{n+1}, x_{n+2}, \ldots) \prec d(r_\beta, -\beta)$ in the same manner. □

The converse of Proposition 5 is not generally true: For example, let β be the real root of $X^3 - 2X^2 + X - 1 = 0$. Then $d(l_\beta, -\beta) = (b_1, b_2, \ldots) = (\overline{1, 0, 1})$. Let $(x_1, x_2, \ldots,) = (\overline{0, 1, 0, 0})$. Then

$$d(l_\beta, -\beta) = (\overline{1, 0, 1}) \prec (x_n, x_{n+1}, \ldots) \prec (0, \overline{1, 0, 1}) = d(r_\beta, -\beta) \text{ for all } n \geq 0.$$

However, $(\,.\,\overline{0100})_{-\beta} = r_\beta \notin I_\beta$ and hence $(\overline{0, 1, 0, 0})$ is not admissible.

We introduce a sequence $d^*(r_\beta, -\beta) = (c_1^*, c_2^*, \ldots)$ as follows:

$$
\begin{aligned}
&d^*(r_\beta, -\beta)\\
&\quad= \begin{cases} (\overline{0, b_1, b_2, \ldots, b_{q-1}, b_q - 1}) & d(l_\beta, -\beta) = (\overline{b_1, b_2, \ldots, b_q}) \text{ for some odd } q,\\ d(r_\beta, -\beta) & \text{otherwise.} \end{cases}
\end{aligned} \tag{4}
$$

Let β again be the real root of of $X^3 - 2X^2 + X - 1 = 0$. Then $d(l_\beta, -\beta) = (b_1, b_2, \ldots) = (\overline{1, 0, 1})$ and hence $d^*(r_\beta, -\beta) = (c_1^*, c_2^*, \ldots) = (\overline{0, 1, 0, 0})$. The following lemmas characterize the sequence $d^*(r_\beta, -\beta)$.

Lemma 6. *Let $d^*(r_\beta, -\beta) = (c_1^*, c_2^*, \ldots)$. Then*

$$d^*(r_\beta, -\beta) = \lim_{x \to r_\beta - 0} d(x, -\beta);$$

that is, for any $n > 0$ there exists an $\varepsilon_n > 0$ such that

$$d_i(x, -\beta) = c_i^* \text{ for } i < n \text{ and } x \in (r_\beta - \varepsilon_n, r_\beta). \tag{5}$$

Proof. We provide the proof by considering the following three cases:

(a) $d(l_\beta, -\beta)$ is not purely periodic.

(b) $d(l_\beta, -\beta)$ is purely periodic with even period q.

(c) $d(l_\beta, -\beta)$ is purely periodic with odd period q.

Here we remark that the case $\beta \in \mathbb{N}$ corresponds to the case (c), where $q = 1$. We use the following interpretation of the $(-\beta)$-expansion: Divide the interval I_β into the following disjoint intervals,

$$I_0 = \left(r_\beta - \tfrac{1}{\beta}, r_\beta\right), \qquad I_1 = \left(r_\beta - \tfrac{2}{\beta}, r_\beta - \tfrac{1}{\beta}\right], \qquad \dots,$$

$$I_{\lfloor\beta\rfloor-1} = \left(r_\beta - \tfrac{\lfloor\beta\rfloor}{\beta}, r_\beta - \tfrac{\lfloor\beta\rfloor-1}{\beta}\right], \qquad I_{\lfloor\beta\rfloor} = \left[l_\beta, r_\beta - \tfrac{\lfloor\beta\rfloor}{\beta}\right].$$

Then $d_i(x, -\beta) = d$ if and only if $T_\beta^{i-1}(x) \in I_d$.

We denote by C_β the set of endpoints of I_i, i.e.,

$$C_\beta = \left\{ l_\beta, r_\beta, r_\beta - \frac{1}{\beta}, \dots, r_\beta - \frac{\lfloor\beta\rfloor}{\beta} \right\}. \tag{6}$$

In case (a), $T_\beta^i(l_\beta)$ is an inner point of $I_{c^*_{i+2}}$ for every $i \geq 1$, at which T_β is continuous. Therefore, (5) holds, if we put

$$\varepsilon_n = \frac{1}{\beta^{n-1}} \min\left(\left\{ \left|T_\beta^i(l_\beta) - c\right| \ : \ i = 1, \dots, n-2,\ c \in C_\beta \right\} \cup \left\{ \frac{\{\beta\}}{\beta} \right\} \right),$$

for $n \geq 1$ Note here that $\frac{\{\beta\}}{\beta}$ is the length of $I_{\lfloor\beta\rfloor}$.

In case (b), we have

$$T_\beta^i(l_\beta) = \begin{cases} \text{left endpoint of } I_{\lfloor\beta\rfloor}, & i \equiv 0 \bmod q, \\ \text{right endpoint of } I_{c^*_{i+2}}, & i \equiv -1 \bmod q, \\ \text{inner point of } I_{c^*_{i+2}}, & \text{otherwise.} \end{cases}$$

Therefore (5) holds if we put

$$\varepsilon_n = \frac{1}{\beta^{n-1}} \min\left(\left\{ \left|T_\beta^i(l_\beta) - c\right| \ : \ i = 1, \dots, q-2,\ c \in C_\beta \right\} \cup \left\{ \frac{\{\beta\}}{\beta} \right\} \right),$$

for $n \geq 1$. In fact, since q is an even integer and $0 < r_\beta - x < \varepsilon_n$, $T_\beta^{i+1}(x) < T_\beta^i(l_\beta)$ for $i \equiv -1 \bmod q$, and $T_\beta^{i+1}(x) > T_\beta^i(l_\beta)$ for $i \equiv 0 \bmod q$.

In case (c), let q be the period length and let a map $\hat{T}_\beta : [l_\beta, r_\beta] \to [l_\beta, r_\beta]$ be defined by

$$\hat{T}_\beta(x) = -\beta x - \rfloor - \beta x - l_\beta \lfloor$$

where $\rfloor x \lfloor$ is the largest integer strictly less than a real number $x > 0$, and $\rfloor 0 \lfloor = 0$. Therefore

$$\hat{T}_\beta(x) = \begin{cases} r_\beta, & x = r_\beta - \frac{k}{\beta} \text{ for some } k \in \{1, \ldots, \lfloor \beta \rfloor\}, \\ l_\beta, & x = r_\beta, \\ T_\beta(x), & \text{otherwise.} \end{cases}$$

From the transformation $\hat{T}_\beta$, we obtain another particular $(-\beta)$-representations of $x \in [l_\beta, r_\beta]$,

$$x = (\,.\,\hat{x}_1 \hat{x}_2 \cdots)_{-\beta},$$

where $\hat{x}_i = \rfloor -\beta \hat{T}_\beta^{i-1}(x) - l_\beta \lfloor$ for $i \geq 1$, which can be explained in the following another way: We divide the interval I_β into the following disjoint intervals,

$$\hat{I}_0 = \left[r_\beta - \tfrac{1}{\beta}, r_\beta \right], \qquad \hat{I}_1 = \left[r_\beta - \tfrac{2}{\beta}, r_\beta - \tfrac{1}{\beta} \right), \qquad \ldots,$$

$$\hat{I}_{\rfloor \beta \lfloor - 1} = \left[r_\beta - \tfrac{\rfloor \beta \lfloor}{\beta}, r_\beta - \tfrac{\rfloor \beta \lfloor - 1}{\beta} \right), \qquad \hat{I}_{\rfloor \beta \lfloor} = \left[l_\beta, r_\beta - \tfrac{\rfloor \beta \lfloor}{\beta} \right).$$

Then $\hat{x}_i = d$ is equivalent to $\hat{T}_\beta^{i-1}(x) \in I_d$ for $i \geq 1$. This representation coincides with the $(-\beta)$-expansion of x if $\hat{T}_\beta^n(x) \neq r_\beta$ for all $n \geq 0$, and $(\hat{x}_1, \hat{x}_2, \ldots) = (c_1^*, c_2^*, \ldots)$ if $x = r_\beta$. Then we have

$$\hat{T}_\beta^i(r_\beta) = \begin{cases} \text{right endpoint of } \hat{I}_0, & i \equiv 0 \bmod (q+1), \\ \text{left endpoint of } \hat{I}_{\rfloor \beta \lfloor}, & i \equiv 1 \bmod (q+1), \\ \text{left endpoint of } \hat{I}_{c_{i+1}^*}, & i \equiv -1 \bmod (q+1), \\ \text{inner point of } \hat{I}_{c_{i+1}^*}, & \text{otherwise.} \end{cases}$$

Therefore (5) holds if we put

$$\varepsilon_n = \frac{1}{\beta^{n-1}} \min \left(\left\{ \left| \hat{T}_\beta^i(r_\beta) - c \right| \ : \ i = 2, \ldots, q-1, \ c \in C_\beta \right\} \cup \left\{ 1 - \frac{\rfloor \beta \lfloor}{\beta} \right\} \right)$$

for $n \geq 1$, where C_β is defined by (6). □

As a corollary of Lemma 6, we immediately obtain the following.

Corollary 7. *We have*

$$r_\beta = (\,.\,c_1^* c_2^* \cdots)_{-\beta}, \quad l_\beta = (\,.\,c_2^* c_3^* \cdots)_{-\beta},$$

where $d^*(r_\beta, -\beta) = (c_1^*, c_2^*, \ldots)$.

Lemma 8. *Let* $d(l_\beta, -\beta) = (\overline{b_1, \dots, b_q})$ *have the odd period* q, *and let* $d^*(r_\beta, -\beta) = (\overline{c_1^*, c_2^*, \dots, c_{q+1}^*})$. *Then,*

$$d(l_\beta, -\beta) \precneqq (c_n^*, c_{n+1}^*, \dots) \precneqq d(r_\beta, -\beta) = (0, b_1, b_2, \dots) \tag{7}$$

for all $n \geq 1$.

Proof. Since the first inequality in (7) for all $n \geq 0$ implies the second one, we prove the first one. It is clear that $(c_n^*, c_{n+1}^*, \dots) \neq d(l_\beta, -\beta)$ and hence it suffices to show

$$d(l_\beta, -\beta) \preceq (c_n^*, c_{n+1}^*, \dots) \text{ for all } n \geq 1. \tag{8}$$

As we have shown in Lemma 6 that $(c_n^*, c_{n+1}^*, \dots, c_{n+m}^*)$ appears in some $(-\beta)$-admissible sequence for any $n \geq 1$ and $m \geq 0$. Therefore, by Proposition 5, we have

$$(b_1, b_2, \dots, b_{m+1}) \preceq (c_n^*, c_{n+1}^*, \dots, c_{n+m}^*) \text{ for all } n \geq 1 \text{ and } m \geq 0,$$

which exactly means that (8) holds. ∎

Lemma 9. *Let* $(x_1, x_2, \dots)$ *be a sequence of* $\mathcal{A}_\beta = \{0, 1, \dots, \lfloor\beta\rfloor\}$ *which satisfies*

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, x_{n+2}, \dots) \prec d^*(r_\beta, -\beta) \quad \textit{for all } n \geq 1. \tag{9}$$

Then

$$(\,.\,x_n x_{n+1} \cdots)_{-\beta} \in I_\beta \quad \textit{for all } n \geq 1.$$

Proof. Let $b_0 = 0$ and $d(l_\beta, -\beta) = (b_1, b_2, \dots)$. We first show that, if $(x_1, x_2, \dots)$ satisfy Condition (9), then,

$$\begin{gathered}(\,.\,x_n x_{n+1} \cdots x_{n+r})_{-\beta} \geq (\,.\,b_m b_{m+1} \cdots)_{-\beta} - \tfrac{1}{\beta^{r+1}} \\ \text{whenever } (x_n, x_{n+1}, \dots, x_{n+r}) \succeq (b_m, b_{m+1}, \dots, b_{m+r})\end{gathered} \tag{10}$$

and

$$\begin{gathered}(\,.\,x_n x_{n+1} \cdots x_{n+r})_{-\beta} \leq (\,.\,b_m b_{m+1} \cdots)_{-\beta} + \tfrac{1}{\beta^{r+1}} \\ \text{whenever } (x_n, x_{n+1}, \dots, x_{n+r}) \preceq (b_m, b_{m+1}, \dots, b_{m+r})\end{gathered} \tag{11}$$

for all $m \geq 0, n \geq 1$ and $r \geq 0$. We prove this by induction on r.

When $r = 0$, if $(x_n) \succeq (b_m)$, i.e., $x_n \leq b_m$, then

$$(\,.\,x_n)_{-\beta} = \frac{x_n}{-\beta} \geq \frac{b_m}{-\beta} = (\,.\,b_m b_{m+1} \cdots)_{-\beta} - \frac{1}{\beta} T^m(l_\beta) \geq (\,.\,b_m b_{m+1} \cdots)_{-\beta} - \frac{1}{\beta}.$$

Thus, (10) holds for $r = 0$. We can prove (11) for $r = 0$ in the same manner as (10).

Now suppose that (10) and (11) hold for all $m \geq 0, n \geq 1$ when $r < k$.

If $(x_n, x_{n+1}, \ldots, x_{n+k}) \succeq (b_m, b_{m+1}, \ldots, b_{m+k})$, then either $x_n = b_m$ and $(x_{n+1}, \ldots, x_{n+k}) \preceq (b_{m+1}, \ldots, b_{m+k})$ or $x_n < b_m$. In the first case, by the assumption on the induction,

$$\begin{aligned}
&(\,.\,x_n \cdots x_{n+k})_{-\beta} - (\,.\,b_m b_{m+1} \cdots)_{-\beta} \\
&\qquad\qquad = \frac{1}{-\beta}\left[(\,.\,x_{n+1} \cdots x_{n+k})_{-\beta} - (\,.\,b_{m+1} b_{m+2} \cdots)_{-\beta}\right] \\
&\qquad\qquad \geq -\frac{1}{\beta^{k+1}}.
\end{aligned}$$

In the latter case, again by the assumption on the induction,

$$\begin{aligned}
&(\,.\,x_n x_{n+1} \cdots x_{n+k})_{-\beta} - (\,.\,b_m b_{m+1} \cdots)_{-\beta} \\
&\quad = \frac{x_n - b_m}{-\beta} + \frac{(\,.\,x_{n+1} \cdots x_{n+k})_{-\beta} - (\,.\,b_{m+1} b_{m+2} \cdots)_{-\beta}}{-\beta} \\
&\quad \geq \frac{1}{\beta}\left\{1 + \left[(\,.\,b_{m+1} b_{m+2} \cdots)_{-\beta} - (\,.\,x_{n+1} \cdots x_{n+k})_{-\beta}\right]\right\} \\
&\quad \geq \frac{1}{\beta}\left\{1 + \left[l_\beta - \left(r_\beta + \frac{1}{\beta^k}\right)\right]\right\} = -\frac{1}{\beta^{k+1}}.
\end{aligned}$$

Thus (10) holds for $r = k$. We can prove (11) for $r = k$ in the same manner as (10). By taking the limit $r \to \infty$ in (10), and respectively (11), we obtain

$$(\,.\,b_m b_{m+1} \cdots)_{-\beta} \leq (\,.\,x_n x_{n+1} \cdots)_{-\beta} \quad \text{whenever} \quad (b_m, b_{m+1}, \ldots) \preceq (x_n, x_{n+1}, \ldots),$$

and, respectively,

$$(\,.\,b_m b_{m+1} \cdots)_{-\beta} \geq (\,.\,x_n x_{n+1} \cdots)_{-\beta} \quad \text{whenever} \quad (b_m, b_{m+1}, \ldots) \succeq (x_n, x_{n+1}, \ldots)$$

for all $m, n \geq 1$. In particular, we have

$$l_\beta = (\,.\,b_1 b_2 \cdots)_{-\beta} \leq (\,.\,x_n x_{n+1} \cdots)_{-\beta} \leq (\,.\,b_0 b_1 b_2 \cdots)_{-\beta} = r_\beta \quad \text{for all } n \geq 1,$$

since $d^*(r_\beta, -\beta) \preceq (b_0, b_1, b_2, \ldots)$.

To complete the proof, we show that $(\,.\,x_n x_{n+1} \cdots)_{-\beta} \neq r_\beta$. Let k be the integer such that $x_{n+i-1} = c_i^*$ for $i < k$ and $(-1)^k(x_{n+k-1} - c_k^*) < 0$. Then, by Lemma 6, there exists a real number $y \in I_\beta$ such that $d(y, -\beta) = (c_1^*, c_2^*, \ldots, c_k^*, y_{k+1}, y_{k+2}, \ldots)$. Therefore we have

$$\begin{aligned}
&(\,.\,x_n x_{n+1} \cdots)_{-\beta} - y \\
&\qquad = \frac{1}{(-\beta)^k}\left((x_{n+k-1} - c_k^*) + (\,.\,x_{n+k} x_{n+k+1} \cdots)_{-\beta} - T_\beta^k(y)\right) \leq 0,
\end{aligned}$$

and $(\,.\,x_n x_{n+1} \cdots)_{-\beta} \leq y < r_\beta$. $\square$

Theorem 10. *An integer sequence* $(x_1, x_2, \ldots)$ *is* $(-\beta)$*-admissible if and only if*

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, x_{n+2}, \ldots) \prec d^*(r_\beta, -\beta) \text{ for all } n \geq 0. \tag{12}$$

Proof. The "if" part is immediate from Proposition 3 and Lemma 9. By using Corollary 7, the "only if" part can be proved in the same manner as Proposition 5. □

3. $(-\beta)$-Shift

We use the terminologies and notations of symbolic dynamical systems following [10]. We define the $(-\beta)$-*shift* $S_{-\beta}$ as the set, endowed with the shift, of all bi-infinite sequences of $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$ for which every finite subword is $(-\beta)$-admissible, i.e., it appears in some $(-\beta)$-admissible sequence.

Theorem 11. *Let* $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots) \in \mathcal{A}_\beta^{\mathbb{Z}}$*. Then* $\mathbf{x} \in S_{-\beta}$ *if and only if*

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, \ldots) \preceq d^*(r_\beta, -\beta) \quad \text{for all } n \in \mathbb{Z} \tag{13}$$

Proof. The "only if" part is clear. So assume (13). Then we have exactly one of the following three cases:

(i) $d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, \ldots) \prec d^*(r_\beta, -\beta)$ for all $n \in \mathbb{Z}$

(ii) There are infinitely many $n \leq 0$ such that

$$(x_n, x_{n+1}, \ldots) = d^*(r_\beta, -\beta).$$

(iii) There exists some $N \in \mathbb{Z}$ such that

$$d(l_\beta, -\beta) \prec (x_n, x_{n+1}, \ldots) \prec d^*(r_\beta, -\beta) \quad \text{for all } n \leq N,$$

and

$$(x_{N+1}, x_{N+2}, \ldots) = d^*(r_\beta, -\beta).$$

In case (i), $\mathbf{x}_n = (x_n, x_{n+1}, \ldots)$ is a $(-\beta)$-admissible sequence for all $n \in \mathbb{Z}$ and clearly every subword of $\mathbf{x}_n$ is $(-\beta)$-admissible. In case (ii), every finite subword of $\mathbf{x}$ becomes some finite subword of $d^*(r_\beta, -\beta)$ which has been shown to be $(-\beta)$-admissible in Lemma 6.

In case (iii), we proceed the proof by showing that, for any s, t with $s \leq t$, the word $\mathbf{x}_{[N+s,N+t]} = (x_{N+s}, x_{N+s+1}, \ldots, x_{N+t})$ is $(-\beta)$-admissible. If $s > 0$, then $\mathbf{x}_{[N+s,N+t]}$ is a subword of $d^*(r_\beta, -\beta)$ which is $(-\beta)$-admissible (Lemma 6). So assume $s \leq 0$ and let m be an integer such that

$$(b_1, \ldots, b_{m-l+1}) \precneqq (x_{N+l}, x_{N+l+1}, \ldots, x_{N+m}) \precneqq (c_1^*, \ldots, c_{m-l+1}^*) \text{ for all } l \leq 0,$$

where $d(l_\beta, -\beta) = (b_1, b_2, \ldots)$ and $d^*(r_\beta, -\beta) = (c_1^*, c_2^*, \ldots)$.

If such an m does not exist, we have some $l < s$ and $m > t$ such that

$$(b_1, \ldots, b_{m-l+1}) = (x_{N+l}, x_{N+l+1}, \ldots, x_{N+m})$$

or

$$(x_{N+l}, x_{N+l+1}, \ldots, x_{N+m}) = (c_1^*, \ldots, c_{m-l+1}^*).$$

$(b_1, \ldots, b_{m-l+1})$, $(c_1^*, \ldots, c_{m-l+1}^*)$ and all their subwords including $\mathbf{x}_{[N+s,N+t]}$ are $(-\beta)$-admissible. If such an m exists, we may assume that $m > t$, and by Lemma 6, there exists a $(-\beta)$-admissible sequence $(y_1, y_2, \ldots)$ such that $y_i = c_i^*$ for $i \le m$. Therefore the concatenation $(x_{N+s}, x_{N+s+1}, \ldots, x_N, y_1, y_2, \ldots)$ is a $(-\beta)$-admissible sequence and hence the word $\mathbf{x}_{[N+s,N+t]} = (x_{N+s}, \ldots, x_N, y_1, y_2, \ldots, y_t)$ is $(-\beta)$-admissible. □

In the rest of this section, our primary concern is in the case where $d(l_\beta, -\beta)$ is eventually periodic. Before proceeding, we recall some basic definitions and results in symbolic dynamics from [10]. Let G be a finite directed graph. $\mathcal{V}(G)$ denotes the vertices of G and $\mathcal{E}(G)$ denotes the edges of G. Let $i(e)$ ($t(e)$, resp.) denote the vertex at which $e \in \mathcal{E}(G)$ starts (ends, resp.). A *labeled graph* G is a finite directed graph whose each edge e carries its *label* $\mathcal{L}(e) \in \mathcal{A}_\beta$. Let $\xi = \ldots, e_{-1}, e_0, e_1, \ldots$ be a bi-infinite path on G, i.e., $e_n \in \mathcal{E}(G)$ and $t(e_n) = i(e_{n+1})$ for all $n \in \mathbb{Z}$. Then the label $\mathcal{L}(\xi)$ is defined by

$$\mathcal{L}(\xi) = (\ldots, \mathcal{L}(e_{-1}), \mathcal{L}(e_0), \mathcal{L}(e_1), \ldots) \in \mathcal{A}_\beta^{\mathbb{Z}}.$$

The set of labels of all bi-infinite paths on G is denoted by

$$X_G = \{\mathcal{L}(\xi) \mid \xi \text{ is a bi-infinite path on } G\},$$

which is known to be a shift space. We say a shift space X is *sofic* if there exists some labeled graph G such that $X = X_G$, and we say a sofic shift X is *presented by G* if $X = X_G$. A labeled graph is called *right resolving* if, for each vertex U, the edges starting at U carry different labels. It is known that every sofic shift can be presented by a right resolving labeled graph (see, e.g., [10]). In the proof of the following theorem, we construct a graph which represents $S_{-\beta}$. Our construction is hinted upon [8], in which Kenyon and Vershik construct graphs which represent sofic covers of hyperbolic toral automorphisms. When we consider applications to hyperbolic toral automorphisms, our algorithm looks much more efficient in general when it is applicable. We implemented their algorithm in [8] and found that it sometimes outputs graphs having a huge number of vertices. A good example of this is the case when β is the minimal Pisot number. Our construction is simple and much more efficient.

Theorem 12. *$S_{-\beta}$ is a sofic shift if and only if $d(l_\beta, -\beta)$ is eventually periodic.*

Proof. We prove the "if" part by showing a concrete algorithm to construct a graph G_β by which $S_{-\beta}$ is presented. We first consider the case when $d(l_\beta, -\beta)$ is not purely periodic with an odd period. In this case, $d^*(r_\beta, -\beta) = d(r_\beta, -\beta) = (0, b_1, b_2, \ldots)$ where $d(l_\beta, -\beta) = (b_1, b_2, \ldots)$. Therefore the condition

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, \ldots) \preceq d^*(r_\beta, -\beta) \quad \text{for all } n \geq 1 \tag{14}$$

is equivalent to

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, \ldots) \quad \text{for all } n \geq 1.$$

Let $d(l_\beta, -\beta) = (b_1, b_2, \ldots, b_p, \overline{b_{p+1}, b_{p+2}, \ldots, b_{p+q}})$ and let

$$l = \begin{cases} q, & q \text{ is an even integer,} \\ 2q, & q \text{ is an odd integer.} \end{cases}$$

Define the map $\underline{\varphi} : \{0, 1, 2, \ldots, p+l\} \times \mathcal{A}_\beta \to \{0, 1, 2, \ldots, p+l\}$ by

$$\underline{\varphi}(i, d) = \begin{cases} i+1, & 1 \leq i < p+l \text{ and } d = b_i, \\ p+1, & i = p+l \text{ and } d = b_{p+l}, \\ 1, & i \neq 0 \text{ and } (-1)^i(b_i - d) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\underline{\varphi}^* : \{0, 1, \ldots, p+l\} \times \mathcal{A}_\beta^* \to \{0, 1, \ldots, p+l\}$ be defined as follows: $\underline{\varphi}^*(i, (d)) = \underline{\varphi}(i, d)$ for all $d \in \mathcal{A}_\beta$ and

$$\underline{\varphi}^*(i, (d_1, d_2, \ldots, d_k)) = \underline{\varphi}(\underline{\varphi}^*(i, (d_1, \ldots, d_{k-1})), d_k).$$

Notice that if $\underline{\varphi}^*(1, (x_1, \ldots, x_k)) = 0$ then there is some subword $(x_l, \ldots, x_k)$ such that $(b_1, \ldots, b_{k-l+1}) \succ (x_l, \ldots, x_k)$, but the converse is not generally true. This is because $\underline{\varphi}^*$ does not check all subsequence of $(x_1, \ldots, x_k)$.

Let G'_β be the graph whose vertices are all subsets of $\{1, 2, \ldots p+l\}$, with one additional vertex F called the *fail state*. Let G'_β have the following edges. From any vertex $U \neq F$, for every $d \in \mathcal{A}_\beta$ there is an edge labeled d to the vertex $\varphi(U, d) \cup \{1\}$ provided this does not contain 0. If this set contains 0, there is instead an edge labeled d from U to the fail state F. Let G_β be the connected component of G'_β which contains the vertex $\{1\}$.

We show G_β to have the desired property. Let X_{G_β} be the shift presented by the graph G_β. It suffices to show $\mathcal{B}(S_{-\beta}) = \mathcal{B}(X_{G_\beta})$, where $\mathcal{B}(X)$ denote the *language* of a shift X. Let $(x_1, x_2, \ldots)$ be a one-sided infinite sequence of $\mathcal{A}_\beta$ which satisfies

$$d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, x_{n+2}, \ldots) \quad \text{for all } n \geq 1. \tag{15}$$

Then it is clear that the bi-infinite sequence

$$(\ldots, 0, 0, 0, x_1, x_2, x_3, \ldots)$$

is contained in $S_{-\beta}$. Thus the language $\mathcal{B}(S_{-\beta})$ consists of the finite prefixes of all one-sided infinite sequences $(x_1, x_2, \ldots)$ which satisfies (15).

We claim that a one-sided sequence $(x_1, x_2, \ldots)$ is the label of an infinite path in G_β starting at the vertex $\{1\}$, if and only if it satisfies the condition (15). In fact, if $(x_1, x_2, \ldots)$ is not the label of any infinite path starting at the vertex $\{1\}$, there is a finite path starting at $\{1\}$ and ending up with the fail state labeled by $(x_1, x_2, \ldots, x_m)$ for some $m > 0$. This means there exists some $n > 0$ and $k > 0$ such that $m = n + k - 1$ and $b_1 = x_n, b_2 = x_{n+1}, \ldots, b_{k-1} = x_{n+k-2}$ and $(-1)^k(b_k - x_{n+k-1}) > 0$ and therefore we have $d(l_\beta, -\beta) \succ (x_n, x_{n+1}, \ldots)$. Since there is an edge from $\{1\}$ to itself labeled 0, the bi-infinite word $(\ldots, 0, 0, x_1, x_2, \ldots)$ is always an element of X_{G_β} if $(x_1, x_2, \ldots)$ is the label of a infinite path starting at $\{1\}$. Since $\{1\} \subset U$ for every vertex U of G_β, $\mathcal{F}(U) \subset \mathcal{F}(\{1\})$, where $\mathcal{F}(U)$ is the *follower set* of U, that is, $\mathcal{F}(U)$ is the set of words

$$\{(\mathcal{L}(e_1), \mathcal{L}(e_2), \ldots, \mathcal{L}(e_k)) \mid \xi = \ldots, e_{-1}, e_0, e_1, \ldots \text{ is some bi-infinite} \\ \text{path on } G_\beta \text{ and } i(e_1) = U\}.$$

Therefore we have $\mathcal{B}(X_{G_\beta}) = \mathcal{F}(\{1\})$, proving $\mathcal{B}(X_{G_\beta}) = \mathcal{B}(S_{-\beta})$.

Then we consider the case when $d(l_\beta, -\beta)$ is purely periodic with an odd period. Let

$$S^-_{-\beta} = \{(x_i)_{i\in\mathbb{Z}} \mid d(l_\beta, -\beta) \preceq (x_n, x_{n+1}, \ldots)\},$$

and

$$S^+_{-\beta} = \{(x_i)_{i\in\mathbb{Z}} \mid (x_n, x_{n+1}, \ldots) \preceq d^*(r_\beta, -\beta)\}.$$

Then clearly

$$S_{-\beta} = S^-_{-\beta} \cap S^+_{-\beta}.$$

We can construct a graph G^-_β which represents $S^-_{-\beta}$ in the same manner as we construct G_β when $d(l_\beta, -\beta)$ is not purely periodic with an odd period. The construction of the graph G^+_β which represents $S^+_{-\beta}$ is similar to that of G^-_β. The only difference from G^-_β is that we use the function $\overline{\varphi}$ instead of $\underline{\varphi}$, which is defined as follows: Let $d^*(r_\beta, -\beta) = (\overline{c^*_1, c^*_2, \ldots, c^*_{q+1}})$. The map $\overline{\varphi} : \{0, 1, 2, \ldots, q+1\} \times \mathcal{A}_\beta \to \{0, 1, 2, \ldots, q+1\}$ is defined by

$$\overline{\varphi}(i, d) = \begin{cases} i+1, & 1 \le i < q+1 \text{ and } d = c^*_i, \\ 1, & i = q+1 \text{ and } d = c^*_{q+1}, \\ 1, & i \ne 0 \text{ and } (-1)^i(c^*_i - d) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now the graph G_β can be constructed by a standard procedure called the *label product* (see e.g. [10, Definition 3.4.8]). The set of vertices of G'_β is $\mathcal{V}(G^-_\beta) \times \mathcal{V}(G^+_\beta)$ where $\mathcal{V}(G)$ denotes the vertices of a graph G. There is an edge from (U, U') to

(V, V') labeled by $d \in \mathcal{A}_\beta$ if and only if there are two edges labeled d, one from U to V in G_β^- and the other from U' to V' in G_β^+. Let G_β be the connected component of G'_β which contains the vertex $(\{1\}, \{1\})$.

Conversely assume $S_{-\beta}$ is a sofic shift presented by a right resolving labeled graph G, and for the sake of contradiction, assume $d(l_\beta, -\beta)$ is not eventually periodic. Then there is a bi-infinite path $\xi = \dots e_{-1}e_0e_1e_2\dots$ in G such that $(\mathcal{L}(e_1), \mathcal{L}(e_2), \dots) = d(l_\beta, -\beta)$. By Theorem 11, we have

$$(-1)^i\mathcal{L}(e_i) = \min\left\{(-1)^i\mathcal{L}(e) \mid e \in \mathcal{E}(G), i(e) = i(e_i)\right\}. \tag{16}$$

Since the number of the vertices of G is finite, there is some vertex U through which the path $\xi_0 = e_1e_2\dots$ passes infinitely many times. So ξ_0 contains some finite path $e_n, e_{n+1}, \cdots, e_{n+l-1}$ with l even starting and ending at the vertex $i(e_n)$. Therefore, by (16) we have $e_i = e_{i+l}$ for all $i \geq n$, which contradicts our assumption that $d(l_\beta, -\beta)$ is not eventually periodic. □

Example 13. Let β be the minimal Pisot number, i.e., the real root of $X^3 - X - 1 = 0$. Let us construct the graph G_β as described in the proof of Theorem 12. We have $d(l_\beta, -\beta) = (1, 0, 0, \overline{1})$ and therefore $p = 3$ and $l = 2$. The following table describes the function $\varphi(i, d)$ for $i \in \{1, 2, \dots, 5\}$ and $d \in \mathcal{A}_\beta = \{0, 1\}$.

$i\backslash d$	0	1
1	1	2
2	3	1
3	4	0
4	0	5
5	1	4
0	0	0

The graph G_β is shown in Figure 3, where the fail state F is omitted.

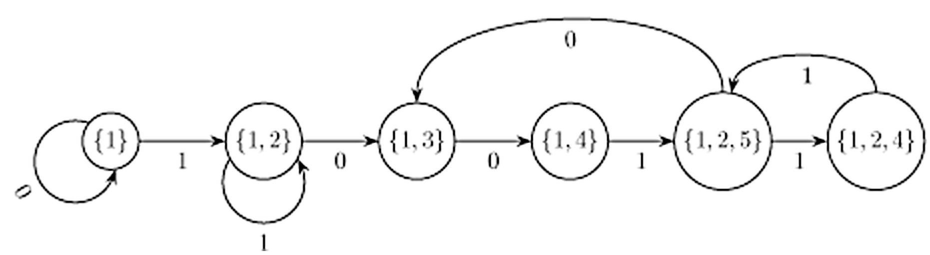

Figure 3: A graph which represents the $(-\beta)$-shift with minimal Pisot β.

Example 14. When $\beta = 2$, $d(l_\beta, -\beta) = (2, 2, 2, \ldots)$ and therefore $d^*(r_\beta, -\beta) = (0, 1, 0, 1, 0, 1, \ldots)$. The values of functions $\underline{\varphi}$ and $\overline{\varphi}$ are shown in Table 2. The graphs G_β^-, G_β^+ and G_β are shown in Figure 4.

$i\backslash d$	0	1	2
1	1	1	2
2	0	0	1

$i\backslash d$	0	1	2
1	2	1	1
2	1	1	0

Table 2: $\underline{\varphi}$ (left) and $\overline{\varphi}$ (right) for $\beta = 2$.

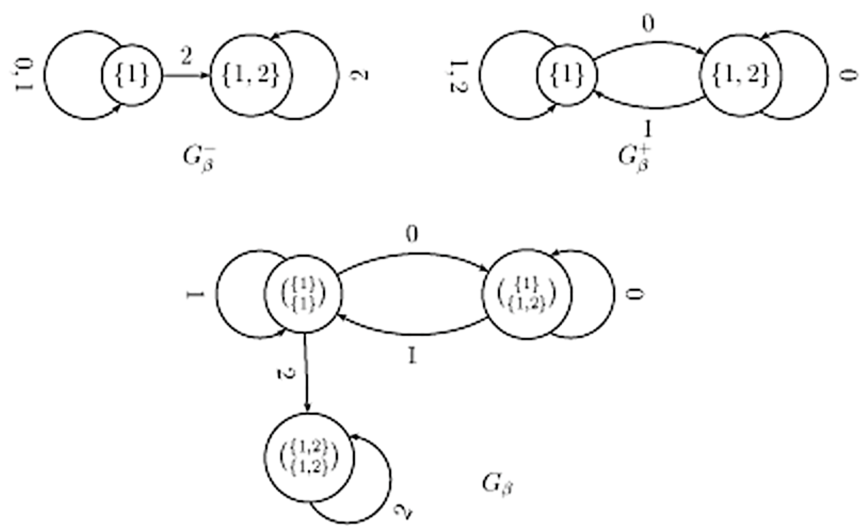

Figure 4: G_β^-, G_β^+ and G_β for $\beta = 2$.

Example 15. Let $\beta > 1$ be a quadratic Pisot number which is a zero of the polynomial of the form

$$X^2 - aX - b \in \mathbb{Z}[X].$$

Then, as we have shown in Example 4, we have

$$d(l_\beta, -\beta) = \begin{cases} (a, a-b, a-b, a-b, \ldots), & a > b > 0, \\ (a-1, -b, a-1, -b, a-1, -b, \ldots), & -a+1 < b < 0. \end{cases}$$

The $(-\beta)$-shift $S_{-\beta}$ is represented by the graph shown in the left side of Figure 5, where the fail state F is omitted.

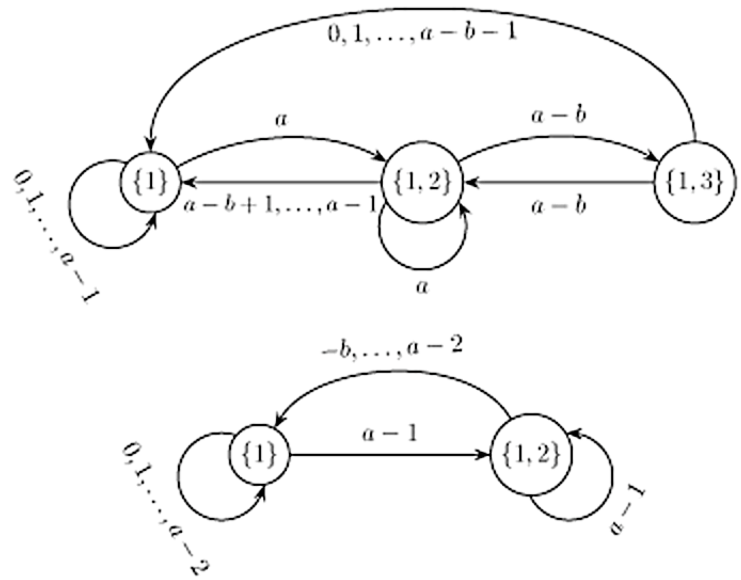

Figure 5: G_β with quadratic Pisot β which satisfies $\beta^2 - a\beta - b = 0$: $a > b > 0$ (top), $-a+1 < b < 0$ (bottom).

4. Invariant Measures

This section considers the frequency of digits in $(-\beta)$-expansions. By applying the theorem of Li and Yorke [9], it can be easily confirmed that T_β has unique invariant measure absolutely continuous with respect to the Lebesgue measure and hence is ergodic.

Theorem 16. *Let $h_{-\beta} : I_\beta \to \mathbb{R}$ be defined by*

$$h_{-\beta}(x) = \sum_{n \geq 0} \frac{d_n(x)}{(-\beta)^n} \quad \textit{where} \quad d_n(x) = \begin{cases} 1, & x \geq T_\beta^n(l_\beta), \\ 0, & \textit{otherwise.} \end{cases}$$

Then the measure $d\mu = h_{-\beta} d\lambda$ is invariant under the transformation T_β, where $d\lambda$ denotes the Lebesgue measure.

Proof. Let $d(l_\beta, -\beta) = (b_1, b_2, \ldots)$. It suffices to show that

$$h_{-\beta}(x) = \frac{1}{\beta} \sum_{y \in T_\beta^{-1}(x)} h_{-\beta}(y)$$

holds for almost every $x \in I_\beta$, that is, $h_{-\beta}$ is invariant under the Perron–Frobenius operator. We have

$$\begin{aligned}
\sum_{y \in T_\beta^{-1}(x)} d_n(y) &= \sum_{i=0}^{\lfloor \frac{\beta^2}{\beta+1} - x \rfloor} d_n\left(\frac{x+i}{-\beta}\right) \\
&= \sharp\left\{ i \in \{0, 1, \ldots, \lfloor \beta \rfloor\} \;\middle|\; T_\beta^n(l_\beta) \le \frac{x+i}{-\beta} \right\} \\
&= \begin{cases} b_{n+1} + 1 & x \le T_\beta^{n+1}(l_\beta) \\ b_{n+1} & \text{otherwise} \end{cases} \\
&= b_{n+1} + 1 - d_{n+1}(x),
\end{aligned}$$

for all $n \ge 0$.

Since $d_0(x) = 1$ for all $x \in I_\beta$

$$\begin{aligned}
\frac{1}{\beta} \sum_{x = T_\beta(y)} h_{-\beta}(y) &= \frac{1}{\beta} \sum_{i=0}^{\lfloor \frac{\beta^2}{\beta+1} - x \rfloor} h_{-\beta}\left(\frac{x+i}{-\beta}\right) = \frac{1}{\beta} \sum_{i=0}^{\lfloor \frac{\beta^2}{\beta+1} - x \rfloor} \sum_{n \ge 0} \frac{d_n\left(\frac{x+i}{-\beta}\right)}{(-\beta)^n} \\
&= \frac{1}{\beta} \sum_{n \ge 0} \frac{\sum_{i=0}^{\lfloor \frac{\beta^2}{\beta+1} - x \rfloor} d_n\left(\frac{x+i}{-\beta}\right)}{(-\beta)^n} \\
&= \frac{1}{\beta} \sum_{n \ge 0} \frac{b_{n+1} + 1 - d_{n+1}(x)}{(-\beta)^n} \\
&= -\sum_{n \ge 1} \frac{b_n}{(-\beta)^n} - \sum_{n \ge 1} \frac{1}{(-\beta)^n} + \sum_{n \ge 1} \frac{d_n(x)}{(-\beta)^n} \\
&= \frac{\beta}{\beta+1} + r_\beta + \left(\sum_{n \ge 0} \frac{d_n(x)}{(-\beta)^n} - d_0(x) \right) \\
&= h_{-\beta}(x).
\end{aligned}$$

$\square$

Example 17. Let β be the golden mean $\frac{1+\sqrt{5}}{2}$. Then $T_\beta^n(l_\beta) = 0$ for $n \ge 1$ and hence

$$h_{-\beta}(x) = \begin{cases} 1, & x < 0, \\ \frac{1}{\beta}, & x \ge 0. \end{cases}$$

Example 18. Let $\beta \approx 1.1347241384\cdots$ be a root of $X^6 - X - 1 = 0$ and put $s_i = T_\beta^i(l_\beta)$ for $i \ge 0$. Then $l_\beta = s_0 < s_5 < s_3 < s_4 < s_6 < s_1 < s_2 < s_7 < r_\beta$, and $s_{i+3} = s_i$ for all $i \ge 5$. The calculation of $h_{-\beta}$ is summarized in the following table. The support of $h_{-\beta}$ consists of three disjoint intervals.

	$s_0 \sim$	$s_5 \sim$	$s_3 \sim$	$s_4 \sim$	$s_6 \sim$	$s_1 \sim$	$s_2 \sim$	$s_7 \sim$
1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta}$						$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^2}$							$\surd$	$\surd$
$-\frac{1}{\beta^3}$			$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^4}$				$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta^5}$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\frac{1}{\beta^6}$					$\surd$	$\surd$	$\surd$	$\surd$
$-\frac{1}{\beta^7}$								$\surd$
$\frac{1}{\beta^8}$		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$h_{-\beta}$	1	$\frac{1}{\beta^3}$	0	$\frac{1}{\beta^4}$	$\frac{1}{\beta}$	0	$\frac{1}{\beta^2}$	$\frac{1}{\beta^5}$

5. Concluding Remarks

We summarize our main results in Table 3 showing the differences and the similarities between $(-\beta)$-expansions and β-expansions, where I is the interval on which the transformation T is defined and h is the density function of the invariant measure of T_β absolutely continuous to the Lebesgue measure. The row "admissible" in Table 3 shows the conditions for an integer sequence $(x_1, x_2, \ldots)$ to be admissible, and $\prec_{\text{lex}}$ stands for the lexicographic order. For the definition of $d^*(1,\beta)$ and $T^n(1)$, see, e.g., [4].

	β-expansion	$(-\beta)$-expansion
$I = [l, r)$	$[0, 1)$	$\left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right)$
T	$\{\beta x\} = \{\beta x - 0\} + 0$	$\{-\beta x - l\} + l$
h	$\displaystyle\sum_{n\geq 0} \frac{d_n(x)}{\beta^n}$	$\displaystyle\sum_{n\geq 0} \frac{d_n(x)}{(-\beta)^n}$
$d_n(x)$	$\begin{cases} 1 & x \leq T^n(1), \\ 0 & \text{otherwise.} \end{cases}$	$\begin{cases} 1 & x \geq T^n(l), \\ 0 & \text{otherwise.} \end{cases}$
admissible	$\forall n{:}(x_n, x_{n+1}, \ldots) \prec_{lex} d^*(1,\beta)$	$\forall n{:}d(l, -\beta) \preceq (x_n, x_{n+1}, \ldots) \prec d^*(r, \beta)$
sofic iff	$d^*(1,\beta)$ is eventually periodic	$d^*(l_\beta, -\beta)$ is eventually periodic

Table 3: Summary

We can consider many problems considered in β-expansions (e.g. [6, 1, 13, 2]), which are not treated in this paper and should be explored in the future.

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