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$\phi(\mathbf{F_n}) = \mathbf{F_m}$

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Abstract

We show that 1, 2 and 3 are the only Fibonacci numbers whose Euler functions are also Fibonacci numbers.

1. Introduction

The Fibonacci sequence $(F_n)_{n\ge 0}$ is given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$. For a positive integer m we let $\phi(m)$ be the Euler function of m. We prove the following result:

Theorem 1. *The only positive integers n such that $\phi(F_n) = F_m$ for some positive integer m are $n = 1,\ 2,\ 3$ or 4.*

Recall that if we put $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{for } n = 0,\ 1,\ \ldots.$$

This is sometimes called the Binet formula. We also put $(L_n)_{n\ge 0}$ for the companion Lucas sequence of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \ge 0$. The Binet formula for the Lucas numbers is

$$L_n = \alpha^n + \beta^n \qquad \text{for } n = 0,\ 1,\ \ldots.$$

There are many relations between the Fibonacci and the Lucas numbers, such as

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{1}$$

or $F_{2n} = F_n L_n$, as well as several others which we will mention when they will be needed. We refer the reader to Chapter 5 in [6], or to Ron Knott's web-site on Fibonacci numbers [5] for such formulae.

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2. A Bird's-eye View to the Proof of Theorem 1

We start with a computation showing that there are no other solutions than the obvious ones up to $n \le 256$. Thus, we may assume that $n > 256$. Next we show that any potential solution is very large, at least as large as $3 \cdot 10^{59}$. Let k be the number of distinct prime factors of F_n. Then $2^{k-1} \mid \phi(F_n) = F_m$. Since the power of 2 in a Fibonacci number is small, it follows that k is small. Since F_n does not have too many prime factors, we get that $n - m$ is small. This implies that $\gcd(F_n, F_m)$ is also small. Next we bound iteratively the prime factors of F_n. As a byproduct of this calculation, we get a lower bound for k in terms of n. Since all odd prime factors of F_n are congruent to 1 modulo 4 when n is odd, this lower bound on k compared with the fact that $4^{k-1} \mid F_m$ are sufficient to get a contradiction when n is odd. Hence it suffices to deal with the case when n is even. Writing $n = 2^{\lambda_1} n'$ with n' odd, one proves that $2^{\lambda_1} \mid n - m$, therefore the power of 2 in n is small. Next, we bound $\ell = n - m$. The bound on ℓ together with a recent calculation of McIntosh and Roettger [10] dealing with a conjecture of Ward about the exponent of apparition of a prime in the Fibonacci sequence shows that if one writes $n = UV$, where U and V are coprime, all primes dividing U divide m, and no prime dividing V divides m, then $U \le \ell$. Thus, U is small. Next, we use sieve methods to show that the minimal prime factor p_1 of V is also small. McIntosh and Roettger's calculation together with the Primitive Divisor Theorem now implies that $n' = p_1$, therefore n is a power of 2 times a small prime, and the upper bounds for n are lower than the lower bounds for n obtained previously, which finishes the proof. The entire proof is computer aided and several small calculations are involved at each step.

3. Proof of Theorem 1

We shall assume that $n > 2$ and we shall write

$$F_n = p_1^{\alpha_1} \cdots p_k^{\alpha_k},$$

where $p_1 < \cdots < p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are positive integers. Since $F_n > 1$, it follows that $m < n$.

3.1. The Small Values of n

A Mathematica code confirmed that the only solutions of the equation

$$\phi(F_n) = F_m \tag{2}$$

in positive integers $m \le n \le 256$ have $n \in \{1, 2, 3, 4\}$. From now on, we assume that $n > 256$. We next show that $4 \mid F_m$. Assuming that this is not so, we would get that $4 \nmid \phi(F_n)$. Thus, $F_n \in \{1, 2, 4, p^\gamma, 2p^\gamma\}$ with some prime $p \equiv 3 \pmod 4$ and some positive integer γ. Since $n \geq 257$, it follows that $F_n \in \{p^\gamma, 2p^\gamma\}$. Results from [2] and [3] show that $\gamma > 1$ is impossible in this range for n. Let us now assume that $\gamma = 1$. If $F_n = p$, then

$$F_m = \phi(F_n) = \phi(p) = p - 1 = F_n - 1,$$

which leads to $1 = F_n - F_m \geq F_n - F_{n-1} = F_{n-2} \geq F_{255}$, which is a contradiction. If $F_n = 2p$, then

$$F_m = \phi(F_n) = \phi(2p) = p - 1 = (F_n - 2)/2,$$

therefore $2 = F_n - 2F_m$. If $m = n-1$, we then get $2 = F_n - 2F_{n-1} = F_{n-2} - F_{n-1} = -F_{n-3} < 0$, which is impossible, while if $m \leq n-2$, we then get $2 = F_n - 2F_m \geq F_n - 2F_{n-2} = F_{n-1} - F_{n-2} = F_{n-3} \geq F_{254}$, which is again impossible. Hence, $4 \mid F_m$. In particular, $6 \mid m$. It follows from the results from [7] that $\phi(F_n) \geq F_{\phi(n)}$. Thus

$$m \geq \phi(n) \geq \frac{n}{e^\gamma \log\log n + 2.50637/\log\log n},$$

where the second inequality above is inequality (3.42) on page 72 in [13]. Here, γ is Euler's constant. Since $e^\gamma < 1.782$, and the inequality

$$\frac{n}{1.782\log\log n + 2.50637/\log\log n} > 50$$

holds for all $n \geq 256$, we get that $m \geq 50$. Put $\ell = n - m$. Since m is even, we have that $\beta^m > 0$, therefore

$$\frac{F_n}{F_m} = \frac{\alpha^n - \beta^n}{\alpha^m - \beta^m} > \frac{\alpha^n - 1}{\alpha^m} = \alpha^\ell - \frac{1}{\alpha^m} > \alpha^\ell - 10^{-10}, \tag{3}$$

where we used the fact that $\alpha^{-50} < 3.55319 \times 10^{-11} < 10^{-10}$. We distinguish the following cases.

`Case 1.` $\gcd(n, 6) = 1$.

In this case $\ell \geq 1$, therefore inequality (3) gives

$$\frac{F_n}{F_m} > \alpha - 10^{-10} > 1.61803.$$

For each positive integer s, let $z(s)$ be the smallest positive integer t such that $s \mid F_t$. It is known that this exists and $s \mid F_n$ if and only if $z(s) \mid n$. This is also referred to as the *order of apparition of n in the Fibonacci sequence.* Since n is coprime to 6, it follows that F_n is divisible only by primes p such that $\gcd(z(p), 6) = 1$. Among the first 1000 primes, there are precisely 212 of them with this property. They are

$$\mathcal{P}_1 = \{5,\ 13,\ 37,\ 73,\ \ldots,\ 7873,\ 7901\}.$$

In our case, the following holds:

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)^{-1}=\frac{F_n}{F_m}>1.61803.$$

Writing q_j for the jth prime number in $\mathcal{P}_1$, we checked with Mathematica that the smallest s such that

$$\prod_{j=1}^{s}\left(1-\frac{1}{q_j}\right)^{-1}>1.61803$$

is $s=99$. Thus, $k\geq 99$. Since n is odd and every prime factor p of F_n is also odd, reducing relation (1) modulo p, we get $L_n^2\equiv -4 \pmod p$ for all $p=p_i$ and $i=1,\ldots,k$. Thus, $p_i\equiv 1 \pmod 4$ for all $i=1,\ldots,k$. Hence, $4^k \mid \prod_{i=1}^{k}(p_i-1) \mid \phi(F_n)=F_m$, therefore $2^{2k}\mid F_m$. So, $z(2^{2k})\mid m$. Since $z(2^s)=3\cdot 2^{s-2}$ for all $s\geq 3$, we get that $3\cdot 2^{2k-2}\mid m$. In particular,

$$n\geq 3\cdot 2^{2k-2}\geq 3\cdot 2^{196}>3\cdot 10^{59}. \tag{4}$$

`Case 2.` $2\|n$ and $\gcd(n,3)=1$.

In this case, since m is also even, we have that $\ell=n-m$ is even. Hence, $\ell\geq 2$, and

$$\frac{F_n}{F_m}>\alpha^2-10^{-10}>2.61803.$$

If p is any prime factor of F_n, then, as in Case 1 above, we get that $z(p)$ is coprime to 3 and is not a multiple of 4. There are 1235 primes p among the first 3000 of them with this property. They are

$$\mathcal{P}_2=\{5,\ 11,\ 13,\ 29,\ \ldots,\ 27397,\ 27431\},$$

and

$$\prod_{q\in\mathcal{P}_2}\left(1-\frac{1}{q}\right)^{-1}=2.3756\ldots<2.61803<\frac{F_n}{F_m}.$$

This shows that $k>1235$. Since p_i is odd for all $i=1\ldots,k$, we get that $2^k \mid \phi(F_n)=F_m$, therefore $z(2^k)\mid m$. Thus,

$$n>m\geq 3\cdot 2^{k-2}\geq 3\cdot 2^{1234}>8\cdot 10^{371}. \tag{5}$$

`Case 3.` $3\mid n$ and $\gcd(n,2)=1$.

In this case, since $3\mid m$, we get that $\ell\geq 3$, therefore

$$\frac{F_n}{F_m}>\alpha^3-10^{-10}>4.23606.$$

All prime factors p of F_n have $z(p)$ odd. There are 1005 primes among the first 3000 of them with this property. They are

$$\mathcal{P}_3 = \{2,\ 5,\ 13,\ 17,\ \ldots,\ 27397,\ 27437\}.$$

Since

$$\prod_{q\in\mathcal{P}_3}\left(1-\frac{1}{q}\right)^{-1} < 4.12239 < 4.23606 < \frac{F_n}{F_m},$$

we get that $k \geq 1006$. Since p_i is odd for all $i = 2, \ldots, k$, we get that $2^{k-1} \mid \phi(F_n) \mid F_m$, therefore $z(2^{k-1}) \mid m$. Thus,

$$n > m \geq 3 \cdot 2^{k-3} \geq 3 \cdot 2^{1003} > 2 \cdot 10^{302}. \tag{6}$$

`Case 4.` $4 \mid n$ and $\gcd(n, 3) = 1$.

Write $n = 4n_0$. Since $n > 256$, it follows that $n_0 > 64$. Note that

$$F_{4n_0} = F_{2n_0}L_{2n_0} = F_{n_0}L_{n_0}L_{2n_0}.$$

Since $L_{n_0}^2 - 5F_{n_0}^2 = \pm 4$, and $L_{2n_0} = L_{n_0}^2 \pm 2$, it follows that the three numbers F_{n_0}, L_{n_0}, and L_{2n_0} have disjoint sets of odd prime factors. The sequence $(L_s)_{s\geq 0}$ is periodic modulo 8 with period 12. Listing its first twelve members, one sees that L_s is never a multiple of 8. Thus, there exist two distinct odd primes $q_1 \mid L_{n_0}$ and $q_2 \mid L_{2n_0}$. A result of McDaniel [9] says that if $s > 48$, then F_s has a prime factor $p \equiv 1 \pmod 4$. Let us give a quick proof of this fact. If s has a prime factor $r \geq 5$, then $F_r \mid F_s$ and every prime factor p of F_r is odd (because F_r is even only when $3 \mid r$). Reducing equation (1) with $n = r$ modulo p, we get $L_r^2 \equiv -4 \pmod p$, so $p \equiv 1 \pmod 4$. Thus, it remains to deal with the case when $s = 2^a \cdot 3^b$ for some nonnegative integers a and b. Since $4481 \mid F_{64}$, $769 \mid F_{96}$, $17 \mid F_9$, and 4481, 769, and 17 are all primes congruent to 1 modulo 4, it follows easily that the largest s such that F_s has no prime factor $p \equiv 1 \pmod 4$ is

$$F_{48} = 2^6 \cdot 3^2 \cdot 7 \cdot 23 \cdot 47 \cdot 1103.$$

Since $n_0 > 64 > 48$, it follows that F_{n_0} has a prime factor $q_3 \equiv 1 \pmod 4$. Now $q_1q_2q_3 \mid F_n$, therefore $16 \mid (q_1 - 1)(q_2 - 1)(q_3 - 1) \mid \phi(F_n) \mid F_m$, showing that $z(16) \mid m$. Thus, $12 \mid m$. Since we now know that both n and m are multiples of 4, we get that $\ell \geq 4$. Hence,

$$\frac{F_n}{F_m} > \alpha^4 - 10^{-10} > 6.8541.$$

The prime factors p of F_n have $z(p)$ coprime to 3. There are 1856 such primes p among the first 3000, and they are

$$\mathcal{P}_4 = \{3,\ 5,\ 7,\ 11,\ \ldots,\ 27431,\ 27449\}.$$

Since

$$\prod_{q\in\mathcal{P}_3}\left(1-\frac{1}{q}\right)^{-1} < 5.30404 < 6.8541 < \frac{F_n}{F_m},$$

we get that $k \ge 1857$. Since $2^k \mid \phi(F_n) = F_m$, we deduce that $z(2^k) \mid m$. Thus,

$$n > m \ge 3\cdot 2^{k-2} \ge 3\cdot 2^{1855} > 7\cdot 10^{558}. \tag{7}$$

`Case 5.` $6 \mid n$.

In this case, $\ell \ge 6$, therefore

$$\frac{F_n}{F_m} > \alpha^6 - 10^{-10} > 17.9442.$$

If q_i stands for the ith prime, then we checked that the smallest s such that

$$\prod_{i=1}^{s}\left(1-\frac{1}{q_i}\right)^{-1} > 17.9442$$

is $s = 2624$. Thus, $k \ge 2624$. We now get that $2^{k-1} \mid \phi(F_n) = F_m$, therefore

$$n > m \ge z(2^{k-1}) = 3\cdot 2^{k-3} \ge 3\cdot 2^{2621} > 2\cdot 10^{789}. \tag{8}$$

To summarize, from inequalities (4), (5), (6), (7) and (8), we have that $n > 3\cdot 10^{59}$.

3.2. Bounding ℓ in Terms of n

We saw in the preceding section that $k \ge 99$. We start by bounding k from above. Since n is large, McDaniel's result shows that F_n has at least one prime factor $p \equiv 1 \pmod 4$. Since at least $k-1$ of the prime factors of F_n are odd, and at least one of them is congruent to 1 modulo 4, we get that $2^k \mid \phi(F_n) = F_m$. Thus, $3\cdot 2^{k-2} \mid m$. We now get that

$$n > m \ge 3\cdot 2^{k-2},$$

therefore

$$k < k(n) := \frac{\log n}{\log 2} + 2 - \frac{\log 3}{\log 2}.$$

Let q_j be the jth prime number. Inequality (3.13) on page 69 in [13] shows that in our range we have

$$q_k < q(n) := k(n)(\log k(n) + \log\log k(n)).$$

Now clearly

$$\frac{F_m}{F_n} = \prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) \ge \prod_{2\le p\le q(n)}\left(1-\frac{1}{p}\right) > \frac{1}{e^{\gamma}\log q(n)\,(1+1/(2(\log q(n))^2))},$$

where the last inequality is inequality (3.29) on page 70 in [13]. That inequality is valid only for $q(n) \geq 286$, which is fulfilled for us since $n \geq 3 \cdot 10^{59}$. Therefore, $k(n) \geq 197$ and $q(n) > 1368 > 286$. We thus get that

$$e^{\gamma} \log q(n) + \frac{e^{\gamma}}{2 \log q(n)} > \frac{F_n}{F_m} = \frac{\alpha^n - \beta^n}{\alpha^m - \beta^m} > \frac{\alpha^n - 1}{\alpha^m}.$$

In the above inequality, we used the fact that m is even, and therefore $\beta^m > 0$. Thus,

$$e^{\gamma}(\log q(n))(1+\delta) > \alpha^{n-m},$$

where

$$\delta := \frac{1}{2(\log q(n))^2} + \frac{e^{-\gamma}}{\alpha^m \log q(n)}.$$

Since $q(n) > 1368$, $m \geq 50$ and $e^{-\gamma} < 0.562$, we get that $\delta < 0.0096$. Thus,

$$n - m < \frac{\log(e^{\gamma}(1+\delta))}{\log \alpha} + \frac{\log \log q(n)}{\log \alpha}.$$

We now take a closer look at $q(n)$. We show that

$$q(n) < (k(n) - 2 + \log 3/\log 2)^{1.4}.$$

For this, it suffices that the inequality

$$k(n)(\log k(n) + \log \log k(n)) < (k(n) - 2 + \log 3/\log 2)^{1.4}$$

holds in our range for n. We checked with Mathematica that the last inequality above is fulfilled whenever $k(n) > 90$, which is true in our range for n. Since $k(n) - 2 + \log 3/\log 2 = \log n/\log 2$, we deduce by taking logarithms above that

$$\log q(n) \leq 1.4 \log(\log n/\log 2),$$

leading to

$$\begin{aligned}
\log \log q(n) &\leq \log 1.4 + \log(\log \log n - \log \log 2) \\
&= \log 1.4 + \log \log \log n + \log\left(1 - \frac{\log \log 2}{\log \log n}\right) \\
&< \log \log \log n + \log 1.4 - \frac{\log \log 2}{\log \log n},
\end{aligned}$$

where in the above chain of inequalities we used the fact that the inequality $\log(1+x) < x$ holds for all real numbers $x > -1, x \neq 0$. We thus get that

$$\begin{aligned}
n - m &< \frac{1}{\log \alpha}\left(\log(e^{\gamma} \cdot 1.0096) + \log 1.4 - \frac{\log \log 2}{\log \log n}\right) + \frac{\log \log \log n}{\log \alpha} \\
&< 2.075 + \frac{\log \log \log n}{\log \alpha},
\end{aligned}$$

where we used the fact that $n > 3 \cdot 10^{59}$. We record this for future use as follows.

Lemma 2. *If $n > 4$, then $n > 3 \cdot 10^{59}$ and*

$$n - m < 2.075 + \frac{\log\log\log n}{\log \alpha}.$$

3.3. Bounding the Primes p_i for $i = 1, \ldots, k$

Here, we follow a similar plan of attack as the proof of Theorem 3 in [12]. Write

$$F_n = p_1 \cdots p_k A, \qquad \text{where } A = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}. \tag{9}$$

Clearly, $A \mid \phi(F_n)$, therefore $A \mid F_m$. Since also $A \mid F_n$, we get that $A \mid \gcd(F_n, F_m)$. Now $\gcd(F_n, F_m) = F_{\gcd(n,m)} \mid F_{n-m}$, because $\gcd(n, m) \mid n - m$. Since the inequality $F_s \le \alpha^{s-1}$ holds for all positive integers s, it follows that

$$A \le F_{n-m} \le \alpha^{n-m-1} < \alpha^{1.075} \log\log n, \tag{10}$$

where the last inequality follows from Lemma 2. We next bound the primes p_i for $i = 1, \ldots, k$. We write

$$\prod_{1=1}^{k} \left(1 - \frac{1}{p_i}\right) = \frac{\phi(F_n)}{F_n} = \frac{F_m}{F_n},$$

therefore

$$1 - \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = 1 - \frac{F_m}{F_n} = \frac{F_n - F_m}{F_n} \ge \frac{F_n - F_{n-1}}{F_n} = \frac{F_{n-2}}{F_n}.$$

Using the inequality

$$1 - (1 - x_1) \cdots (1 - x_s) \le x_1 + \cdots + x_s \quad \text{valid for all } x_i \in [0, 1],\ i = 1, \ldots, s, \tag{11}$$

we get

$$\frac{F_{n-2}}{F_n} \le 1 - \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \le \sum_{i=1}^{k} \frac{1}{p_i} < \frac{k}{p_1},$$

therefore

$$p_1 < k \left(\frac{F_n}{F_{n-2}}\right) < 3k, \tag{12}$$

where we used the fact that $F_n < 3F_{n-2}$. (This last inequality is equivalent to $F_{n-1} + F_{n-2} < 3F_{n-2}$, or $F_{n-1} < 2F_{n-2}$, or $F_{n-2} + F_{n-3} < 2F_{n-2}$, or $F_{n-3} < F_{n-2}$, which is certainly true in our range for n.) We now show by induction on the index $i \in \{1, \ldots, k\}$, that if we put

$$u_i := \prod_{j=1}^{i} p_j,$$

then

$$u_i < (2\alpha^{3.075}(\log\log n)k)^{(3^i-1)/2}. \tag{13}$$

For $i = 1$, this becomes

$$p_1 < 2\alpha^{3.075}(\log\log n)k$$

which is implied by estimate (12) and the fact that for $n > 3 \cdot 10^{59}$ we have the estimate $2\alpha^{3.075}\log\log n > 43 > 3$. We now assume that $i \in \{1, \ldots, k-1\}$ and that the estimate (13) is fulfilled, and we shall prove estimate (13) for i replaced by $i+1$. We have

$$\prod_{j=i+1}^{k}\left(1 - \frac{1}{p_j}\right) = \frac{p_1 \cdots p_i}{(p_1 - 1)\cdots(p_i - 1)} \cdot \frac{F_m}{F_n} = \frac{p_1 \cdots p_i}{(p_1 - 1)\cdots(p_i - 1)} \cdot \frac{\alpha^m - \beta^m}{\alpha^n - \beta^n},$$

which we rewrite as

$$\begin{aligned} 1 - \prod_{j=i+1}^{k}\left(1 - \frac{1}{p_j}\right) &= 1 - \frac{p_1 \cdots p_i}{(p_1 - 1)\cdots(p_i - 1)} \cdot \frac{\alpha^m - \beta^m}{\alpha^n - \beta^n} \\ &= \frac{\alpha^m((p_1 - 1)\cdots(p_i - 1)\alpha^{n-m} - p_1 \cdots p_i)}{(p_1 - 1)\cdots(p_i - 1)(\alpha^n - \beta^n)} \\ &+ \frac{\beta^m(p_1 \cdots p_i - \beta^{n-m}(p_1 - 1)\cdots(p_i - 1))}{(p_1 - 1)\cdots(p_i - 1)(\alpha^n - \beta^n)} \\ &=: X + Y, \end{aligned}$$

where

$$\begin{aligned} X &:= \frac{\alpha^m((p_1 - 1)\cdots(p_i - 1)\alpha^{n-m} - p_1 \cdots p_i)}{(p_1 - 1)\cdots(p_i - 1)(\alpha^n - \beta^n)}; \\ Y &:= \frac{\beta^m(p_1 \cdots p_i - \beta^{n-m}(p_1 - 1)\cdots(p_i - 1))}{(p_1 - 1)\cdots(p_i - 1)(\alpha^n - \beta^n)}. \end{aligned}$$

Since m is even and $|\beta| < 1$, we see easily that $Y \geq 0$. Furthermore, since $n-m > 0$, $\beta = -\alpha^{-1}$, and no power of α with positive integer exponent is a rational number, it follows that $XY \neq 0$. Thus, $Y > 0$. Let us suppose first that $X < 0$. Then

$$\begin{aligned} 1 - \prod_{j=i+1}^{k}\left(1 - \frac{1}{p_i}\right) &< Y < \frac{2p_1 \cdots p_i}{\alpha^m(p_1 - 1)\cdots(p_i - 1)(\alpha^n - \beta^n)} \\ &< \frac{2F_n}{\phi(F_n)(\alpha^m - \beta^m)(\alpha^n - \beta^n)} = \frac{2}{5F_m^2}. \end{aligned}$$

Since the left hand side of the above inequality is a positive rational number whose denominator divides $p_{i+1}\cdots p_k \mid F_n$, it follows that this number is at least as large as $1/F_n$. Hence,

$$\frac{1}{F_n} < \frac{2}{5F_m^2},$$

giving

$$F_m^2 < \frac{2}{5}F_n.$$

Since the inequalities $\alpha^{s-2} \le F_s \le \alpha^{s-1}$ hold for all $s \ge 2$, we get

$$\alpha^{2m-4} \le F_m^2 < \frac{2}{5}F_n \le \frac{2}{5}\alpha^{n-1},$$

therefore

$$2m < 3 + \frac{\log(2/5)}{\log\alpha} + n.$$

Using Lemma 2, we have

$$m > n - 2.075 - \frac{\log\log\log n}{\log\alpha}.$$

Combining these inequalities, we get

$$n < 7.15 + \frac{\log(2/5)}{\log\alpha} + \frac{2\log\log\log n}{\log\alpha} < 5.25 + \frac{2\log\log\log n}{\log\alpha},$$

which is impossible in our range for n. Hence, the only chance is that $X > 0$. Since also $Y > 0$, we get that

$$1 - \prod_{j=i+1}^{k}\left(1 - \frac{1}{p_j}\right) > X.$$

Now note that

$$((p_1-1)\cdots(p_i-1)\alpha^{n-m} - p_1\cdots p_i)((p_1-1)\cdots(p_i-1)\beta^{n-m} - p_1\cdots p_i)$$

is a nonzero integer (by Galois theory since β is the conjugate of α), therefore its absolute value is ≥ 1. Since the absolute value of the second factor is certainly $< 2p_1\cdots p_i$ and the first factor is positive (because $X > 0$), we get that

$$(p_1-1)\cdots(p_i-1)\alpha^{n-m} - p_1\cdots p_i > \frac{1}{2p_1\cdots p_i}.$$

Hence,

$$1 - \prod_{j=i+1}^{k}\left(1 - \frac{1}{p_j}\right) > X > \frac{\alpha^m}{2(p_1\cdots p_i)^2(\alpha^n - \beta^n)} > \frac{\alpha^m - \beta^m}{2u_i^2(\alpha^n - \beta^n)} = \frac{F_m}{2u_i^2 F_n},$$

which combined with inequality (11) leads to

$$\frac{F_m}{2u_i^2 F_n} < 1 - \prod_{j=i+1}^{k} \left(1 - \frac{1}{p_j}\right) < \sum_{j=i+1}^{k} \frac{1}{p_j} < \frac{k}{p_{i+1}}.$$

Thus,

$$p_{i+1} < 2ku_i^2 \left(\frac{F_n}{F_m}\right).$$

However,

$$\frac{F_n}{F_m} < \alpha^{n-m+1} < \alpha^{3.075} \log\log n,$$

by Lemma 2. Hence,

$$p_{i+1} < (2\alpha^{3.075} k \log\log n) u_i^2,$$

and multiplying both sides of the above inequality by u_i we get

$$u_{i+1} < (2\alpha^{3.075} k \log\log n) u_i^3.$$

Using the induction hypothesis (13), we get

$$u_{i+1} < (2\alpha^{3.075} k \log\log n)^{1+3(3^i-1)/2} = (2\alpha^{3.075} k \log\log n)^{(3^{i+1}-1)/2},$$

which is precisely inequality (13) with i replaced by $i+1$. This finishes the induction proof and shows that estimate (13) holds indeed for all $i = 1, \ldots, k$. In particular,

$$p_1 \cdots p_k = u_k < (2\alpha^{3.075} k \log\log n)^{(3^k-1)/2},$$

which together with formula (9) and estimate (10) gives

$$F_n = p_1 \cdots p_k A < (2\alpha^{3.075} k \log\log n)^{1+(3^k-1)/2} = (2\alpha^{3.075} k \log\log n)^{(3^k+1)/2}.$$

Since $F_n > \alpha^{n-2}$, we get

$$(n-2)\log\alpha < \frac{(3^k+1)}{2} \log(2\alpha^{3.075} k \log\log n).$$

Assume first that $k \le 2\alpha^{3.075} \log\log n$. We then get that

$$(n-2)\log\alpha < (3^{2\alpha^{3.075} \log\log n} + 1) \log(2\alpha^{3.075} \log\log n),$$

which implies that $n < 10^{16}$. This is false because $n > 3 \cdot 10^{59}$. Thus, $k > 2\alpha^{3.075} \log\log n$, therefore we get

$$(n-2)\log\alpha < (3^k+1)\log k.$$

We also have that

$$k \le k(n) \le \frac{\log n}{\log 2} + 2 - \frac{\log 3}{\log 2} < \frac{\log n}{\log 2} + 0.42.$$

Hence

$$3^k + 1 > \frac{(n-2)\log\alpha}{\log(\log n/\log 2 + 0.42)},$$

so that

$$k > K(n) := \frac{1}{\log 3}\log\left(\frac{(n-2)\log\alpha}{\log(\log n/\log 2 + 0.42)} - 1\right).$$

3.4. The Case When n is Odd

Assume that n is odd. Then every odd prime factor p_i of F_n is congruent to 1 modulo 4. Thus, $4^{k-1} \mid \phi(F_n) = F_m$, therefore $z(2^{2k-2}) \mid m$. So

$$n > m \ge z(2^{2k-2}) = 3 \cdot 2^{2k-4},$$

leading to

$$k \le L(n) := 2 + \frac{\log(n/3)}{2\log 2}.$$

Since also $k > K(n)$, we get that

$$\frac{1}{\log 3}\log\left(\frac{(n-2)\log\alpha}{\log(\log n/\log 2 + 0.42)} - 1\right) < 2 + \frac{\log(n/3)}{2\log 2}.$$

This inequality gives $n < 5 \cdot 10^6$, which is impossible since $n > 3 \cdot 10^{59}$. This shows that the case $n > 4$ and odd is impossible, therefore n has to be even. Returning now to estimates (5), (7), and (8), we also get that $n > 8 \cdot 10^{371}$.

3.5. Bounding ℓ

We write $n = 2^{\lambda_1} n'$, where n' is odd and $\lambda_1 \ge 1$. We start by bounding λ_1. Clearly, $\lambda_1 \ge 1$. If $\lambda_1 \ge 2$, then

$$F_{2^{\lambda_1}} = L_2 \cdots L_{2^{\lambda_1 - 1}}.$$

The numbers L_{2^j} are all odd for $j = 1, \ldots, \lambda_1 - 1$, and since $L_{2^i} = L_{2^{i-1}}^2 \pm 2$ holds for all $i \ge 2$, it follows easily that $L_{2^i} \equiv \pm 2 \pmod{L_{2^j}}$ for all $1 \le j < i$. This shows that $\gcd(L_{2^i}, L_{2^j}) = 1$ for all $1 \le j < i$. In particular, $F_{2^{\lambda_1}}$ is divisible by at least $\lambda_1 - 1$ distinct primes which are all odd. So, $2^{\lambda_1 - 1} \mid \phi(F_n) = F_m$. Thus, assuming that $\lambda_1 \ge 3$, we get that $3 \cdot 2^{\lambda_1 - 3} \mid m$. Hence, $2^{\lambda_1 - 3}$ divides both m and n, so it also divides $n - m$. This argument combined with Lemma 2 shows that,

$$2^{\lambda_1} \le 8(n-m) < 16.6 + \frac{8\log\log\log n}{\log\alpha},$$

and the last inequality above is true for $\lambda_1 < 3$ as well. In particular, if $n' = 1$, we then get that

$$n = 2^{\lambda_1} \le 16.6 + \frac{8\log\log\log n}{\log\alpha},$$

leading to $n < 18$, which is false. Thus $n' > 1$, therefore n has odd prime factors. We deduce more. Write $m = 2^{\mu_1} m'$, where m' is odd. We have already seen that $\mu_1 \geq k - 2 \geq K(n) - 2$. We now show that $\mu_1 > \lambda_1$. Assume that this is not so. Then $\mu_1 \leq \lambda_1$, therefore $2^{\mu_1} \mid n - m$. Hence,

$$\mu_1 \leq \frac{\log(n-m)}{\log 2} < \frac{\log(2.075 + \log\log\log n / \log\alpha)}{\log 2},$$

where the last inequality follows from Lemma 2. We therefore get the inequality

$$K(n) - 2 < \frac{\log(2.075 + \log\log\log n / \log\alpha)}{\log 2},$$

leading to $n < 258$, which is impossible. Thus, $\mu_1 > \lambda_1$. We next rework a bit the relation $\phi(F_n) = F_m$ to deduce a certain inequality relating ℓ to the prime factors of F_n. Write

$$\frac{F_n}{F_m} = \frac{F_n}{\phi(F_n)} = \prod_{p \mid F_n} \left(1 + \frac{1}{p-1}\right).$$

Note that

$$\frac{F_n}{F_m} = \frac{\alpha^n - \beta^n}{\alpha^m - \beta^m} > \frac{\alpha^n - 1}{\alpha^m} = \alpha^\ell \left(1 - \frac{1}{\alpha^n}\right).$$

Thus,

$$\begin{aligned} \ell \log\alpha + \log\left(1 - \frac{1}{\alpha^n}\right) &< \log\left(\frac{F_n}{F_m}\right) = \sum_{p \mid F_n} \log\left(1 + \frac{1}{p-1}\right) \\ &< \sum_{p \mid F_n} \frac{1}{p-1}, \end{aligned} \tag{14}$$

where in the last inequality above we used the fact that $\log(1 + x) < x$ holds for $x > 0$. Next, we note that since the inequality $\log(1 - x) > -2x$ holds for all $x \in (0, 1/2)$, we have that

$$\log\left(1 - \frac{1}{\alpha^n}\right) > -\frac{2}{\alpha^{100}} > -10^{-10}.$$

Thus,

$$\ell \log\alpha - 10^{-10} < \sum_{p \mid F_n} \frac{1}{p+1} + S(n),$$

where we put

$$S(n) := \sum_{p \mid F_n} \left(\frac{1}{p-1} - \frac{1}{p+1} \right).$$

We next bound $S(n)$. Clearly,

$$\begin{aligned} S(n) &< \sum_{\substack{p \mid F_n \\ p<100}} \left(\frac{1}{p-1} - \frac{1}{p+1} \right) + 2 \sum_{p \geq 101} \frac{1}{p(p-1)} \\ &< \sum_{\substack{p \mid F_n \\ p<100}} \left(\frac{1}{p-1} - \frac{1}{p+1} \right) + 0.05. \end{aligned}$$

We distinguish three cases.

`Case 1.` $2 \| n$ and $\gcd(n,3)=1$.

Here, the prime factors of F_n belong to $\mathcal{P}_2$ and the only such below 100 are

$$5,\ 11,\ 13,\ 29,\ 37,\ 59,\ 71,\ 73,\ 89,\ 97.$$

It now follows that

$$S(n) < 0.168.$$

Hence,

$$\ell \log \alpha - 0.168 - 10^{-10} < \sum_{p \mid F_n} \frac{1}{p+1}.$$

Since $\ell \geq 2$, and

$$\frac{\ell \log \alpha - 0.168 - 10^{-10}}{\ell \log \alpha} \geq \frac{2 \log \alpha - 0.168 - 10^{-10}}{2 \log \alpha} > 0.82,$$

we get that

$$0.82 \ell \log \alpha < \sum_{p \mid F_n} \frac{1}{p+1}. \tag{15}$$

`Case 2.` $4 \mid n$ and $\gcd(n,3)=1$.

In this case, if $p \mid F_n$, then $p \in \mathcal{P}_4$. There are 16 primes below 100 in $\mathcal{P}_4$, and using them we get the upper bound

$$S(n) < \sum_{\substack{p \in \mathcal{P}_4 \\ p<100}} \left(\frac{1}{p-1} - \frac{1}{p+1} \right) + 0.05 < 0.463.$$

Since also $4 \mid m$, we get that $\ell \geq 4$. Hence,

$$\ell \log \alpha - 0.463 - 10^{-10} < \sum_{p \mid F_n} \frac{1}{p+1},$$

and since $\ell \geq 4$, and

$$\frac{\ell \log \alpha - 0.463 - 10^{-10}}{\ell \log \alpha} \geq \frac{4 \log \alpha - 0.463 - 10^{-10}}{4 \log \alpha} > 0.75,$$

we get that

$$0.75\ell \log \alpha < \sum_{p|F_n} \frac{1}{p+1}. \tag{16}$$

Case 3. $6 \mid n$.

In this case,

$$S(n) < \sum_{p \geq 2} \left(\frac{1}{p-1} - \frac{1}{p+1} \right) < 1.15,$$

and $\ell \geq 6$. Thus,

$$\ell \log \alpha - 1.15 - 10^{-10} < \sum_{p|F_n} \frac{1}{p+1},$$

and since

$$\frac{\ell \log \alpha - 1.15 - 10^{-10}}{\ell \log \alpha} \geq \frac{6 \log \alpha - 1.15 - 10^{-10}}{6 \log \alpha} > 0.6,$$

we get that

$$0.6\ell \log \alpha < \sum_{p|F_n} \frac{1}{p+1}. \tag{17}$$

From (15), (16) and (17), we get that

$$0.6\ell \log \alpha < \sum_{p|F_n} \frac{1}{p+1}.$$

We now write

$$n = \prod_{i=1}^{u} r_i^{\lambda_i},$$

where $2 = r_1 < \cdots < r_u$ are prime numbers and $\lambda_1, \ldots, \lambda_u$ are positive integers. We organize the prime factors of F_n according to their order of apparition in the Fibonacci sequence. Clearly, for each $p \mid F_n$, we have that $z(p) = d$ for some divisor d of n. Furthermore, $d > 2$, since $F_1 = F_2 = 1$. If p is a prime with $z(p) = d$, then $p \equiv \pm 1 \pmod{d}$, except when $p = d = 5$. Let $\mathcal{Q}_d = \{p : z(p) = d\}$ and let $\ell_d = \#\mathcal{Q}_d$. Then

$$(d-1)^{\ell_d} \leq \prod_{p \in \mathcal{Q}_d} p \leq F_d < \alpha^{d-1},$$

therefore

$$\ell_d < \frac{(d-1)\log\alpha}{\log(d-1)} < \frac{d\log\alpha}{\log d} \tag{18}$$

for all $d \ge 3$. Indeed, the last inequality above follows for $d \ge 4$ because the function $t/\log t$ is increasing for $t \ge 3$, while for $d = 3$ it follows because $\ell_3 = 1 < 3(\log\alpha)/\log 3$. Now note that

$$\sum_{p|F_n} \frac{1}{p+1} = \sum_{\substack{d|n\\d>2}} \sum_{p\in\mathcal{Q}_d} \frac{1}{p+1}.$$

Since all primes $p \in \mathcal{Q}_d$ satisfy $p \equiv \pm 1 \pmod{d}$ for all $d \ne 5$, we get easily that

$$\begin{aligned} Q_d \quad &:= \sum_{p\in\mathcal{Q}_d} \frac{1}{p+1} \le 2 \sum_{\ell\le\lfloor\ell_d/2\rfloor+1} \frac{1}{d\ell} \\ &\le \frac{2}{d}\left(1 + \int_1^{d\log\alpha/(2\log d)+1} \frac{d\ell}{\ell}\right) \\ &\le \frac{2}{d}\log\left(\frac{ed\log\alpha}{2\log d} + e\right), \end{aligned}$$

for $d \ne 5$. Since the inequality

$$\frac{ed\log\alpha}{2\log d} + e < d$$

holds for all $d \ge 5$, we deduce that the inequality

$$Q_d < \frac{2\log d}{d}$$

holds for all $d \ge 6$. The same inequality also holds for $d \in \{3, 4, 5\}$ since

$$Q_3 = \frac{1}{3} < \frac{2\log 3}{3}, \qquad Q_4 = \frac{1}{4} < \frac{2\log 4}{4}, \qquad \text{and} \qquad Q_5 = \frac{1}{6} < \frac{2\log 5}{5}.$$

Hence,

$$\sum_{p|F_n} \frac{1}{p+1} = \sum_{\substack{d|n\\d>2}} Q_d < 2\sum_{d|n} \frac{\log d}{d}.$$

Let us put $\log^* x = \max\{\log x, 1\}$. We next show that the function defined on the set of positive integers and given by $f(a) = 2\log^* a$ for $a > 1$ and $f(1) = 1$ is submultiplicative; i.e.,

$$f(ab) \le f(a)f(b) \qquad \text{holds for all positive integers } a,\ b.$$

The above inequality is clear if one of a and b is 1. If both $a,\ b$ are ≥ 3, then

$$f(ab) = 2\log(ab) = 2\log a + 2\log b < 4\log a\log b = f(a)f(b),$$

because both $2\log a$ and $2\log b$ exceed 2. Finally, assume that one of a and b is 2. Say $a = 2$ and $b \geq 2$. Then the desired inequality is

$$f(ab) = 2\log(2b) = 2\log 2 + 2\log b < 4\log b,$$

which is obviously true. Using the submultiplicativity of the function f, we have

$$0.6\ell\log\alpha < \sum_{d|n}\frac{f(d)}{d} \leq \prod_{r|n}\left(1 + \sum_{\beta\geq 1}\frac{f(r^\beta)}{r^\beta}\right).$$

The contribution of the prime $r = 2$ in the last product above is

$$\begin{aligned} 1 + \frac{2}{2} + \frac{2\log 4}{4} + \frac{2\log 8}{8} + \cdots &= 2 - \log 2 + (\log 2)\left(1 + \frac{2}{2} + \frac{3}{4} + \cdots\right) \\ &= 2 - \log 2 + 4\log 2 = 2 + 3\log 2 < 4.08. \end{aligned}$$

The contribution of an odd prime number r in the above product is

$$1 + \frac{2\log r}{r}\left(1 + \frac{2}{r} + \frac{3}{r^2} + \cdots\right) < 1 + \frac{2r\log r}{(r-1)^2}.$$

Since $0.6/4.08 > 0.14$, we get that

$$0.14\ell\log\alpha < \prod_{\substack{r|n \\ r>2}}\left(1 + \frac{2r\log r}{(r-1)^2}\right). \tag{19}$$

Taking logarithms and using again the fact that $\log(1+x) < x$ holds for all positive real numbers x, we get

$$\log\ell + \log(0.14\log\alpha) < \sum_{\substack{r|n \\ r>2}}\log\left(1 + \frac{2r\log r}{(r-1)^2}\right) < \sum_{\substack{r|n \\ r>2}}\frac{2r\log r}{(r-1)^2}.$$

Separating the prime 3 and using the fact that $r/(r-1)^2 < 1.6/r$ for $r \geq 5$, we get that

$$\log\ell + \log(0.14\log\alpha) < \frac{3\log 3}{2} + 3.2\sum_{\substack{r|n \\ r\geq 5}}\frac{\log r}{r}. \tag{20}$$

We are now finally ready to bound ℓ. Assume that $\ell > 10^8$. Let ω be the number of prime factors of ℓ and let $q_1 < q_2 < \cdots$ be the increasing sequence of all prime numbers. All prime factors $r \geq 5$ of n either divide $\gcd(n,m)$, therefore ℓ, or divide n but not m. Thus,

$$\sum_{\substack{r|n\\ r\geq 5}} \frac{\log r}{r} \leq \sum_{5\leq q\leq q_{\omega+2}} \frac{\log q}{q} + \sum_{\substack{r|n\\ r\nmid m}} \frac{\log r}{r} := S_1 + S_2. \tag{21}$$

In what follows, we bound S_1 and S_2 separately. To bound S_1, note that in order to maximize S_1 as a function of ℓ, we may assume that ℓ is not a multiple of 6. By the Stirling formula, we then have

$$6\ell \geq (\omega+2)! > \left(\frac{\omega+2}{e}\right)^{\omega+2},$$

leading to

$$(\omega+2)(\log(\omega+2)-1) < \log(6\ell).$$

Hence, $2(\omega+2)(\log(\omega+2)-1) < 2\log(6\ell)$. Assume first that

$$2(\omega+2)(\log(\omega+2)-1) < (\omega+2)(\log(\omega+2)+\log\log(\omega+2)).$$

Then

$$\log(\omega+2) < 2 + \log\log(\omega+2),$$

leading to $\omega \leq 21$. In this case,

$$S_1 \leq \sum_{5\leq q\leq 83} \frac{\log q}{q} < 2.56.$$

Assume next that $\omega > 21$. Then

$$2\log(6\ell) > 2(\omega+2)(\log(\omega+2)-1) \geq (\omega+2)(\log(\omega+2)+\log\log(\omega+2)) > q_{\omega+2},$$

where the last inequality is inequality (3.13) on page 69 in [13] (valid for all $\omega \geq 6$, which is our case). Since $\ell > 10^8$, we have that $2\log(6\ell) > 40 > 32$, so formula (3.23) on page 70 in [13] shows that

$$\begin{aligned}
S_1 &< \sum_{5\leq q\leq q_{\omega+2}} \frac{\log q}{q} < \sum_{5\leq q\leq 2\log(6\ell)} \frac{\log r}{r}\\
&< \log(2\log(6\ell)) - \frac{\log 2}{2} - \frac{\log 3}{3} - 1.33 + \frac{1}{\log(2\log(6\ell))}\\
&< \log\log(6\ell) - 1.07 < \log\log(6\ell) - 0.44,
\end{aligned}$$

where the last inequality is valid for $\ell > 10^8$. Since $\log\log(6\ell) - 0.44 > 2.56$ holds for $\ell > 10^8$, it follows that in both cases we have

$$S_1 \leq \log\log(6\ell) - 0.44. \tag{22}$$

We now bound S_2. For this, observe that if $5 \mid n$, then $10 \mid n$. Hence, $11 \mid 55 = F_{10} \mid F_n$. Thus, $10 \mid \phi(F_n) = F_m$, leading to $5 \mid F_m$, so $5 \mid m$. This shows that the smallest prime that can participate in S_2 is ≥ 7 (recall that $6 \mid m$). Let $t \geq 3$, and let $\mathcal{I}_t$ be the set of primes in the interval $[2^t, 2^{t+1}]$ that divide n but not m. Let n_t be the number of elements in $\mathcal{I}_t$. Assume that $n_t \geq 1$ for some t. Let p be a prime in $\mathcal{I}_t$. Then n has at least $2^{n_t - 1}$ squarefree divisors d, such that each one of them is a multiple of p, and such that furthermore each one of them is divisible only by primes $q \in \mathcal{I}_t$. For each one of these divisors d, since $2d \mid n$, we have that $L_d \mid F_{2d} \mid F_n$. Since d is odd and $d > 7$, we get, by the Primitive Divisor Theorem (see [4]), that L_d has a primitive prime factor p_d. Clearly, $p_d \equiv \pm 1 \pmod{d}$, so, in particular, p_d is odd. Reducing relation (1) modulo p_d, we get that $-5F_d^2 \equiv -4 \pmod{p_d}$, therefore $(5/p_d) = 1$. So, $(p_d/5) = 1$ by the Quadratic Reciprocity Law. It now follows that $z(p_d) = d \mid p_d - 1$, showing that $p \mid d \mid p_d - 1 \mid \phi(F_n)$. Since the primitive prime factors p_d are distinct as d runs over the divisors of n composed only of primes $q \in \mathcal{I}_t$, it follows that the exponent of p in $\phi(F_n)$ is at least $2^{n_t - 1}$. On the other hand, since $p \nmid m$, it follows that this exponent is at most the exponent of p in $F_{z(p)}$. Now $z(p) \mid p + \eta$, where $\eta \in \{\pm 1\}$, because $t \geq 3$. Hence, writing a_p for the exponent of p in $F_{z(p)}$, we get that

$$p^{a_p} \mid F_{z(p)} \mid F_{p+\eta} = F_{(p+\eta)/2} L_{(p+\eta)/2}.$$

Relation (1) shows that $\gcd(F_{(p+\eta)/2}, L_{(p+\eta)/2}) \mid 2$. Since p is odd, we get that

$$p^{a_p} \mid F_{(p+\eta)/2}, \qquad \text{or} \qquad p^{a_p} \mid L_{(p+\eta)/2}.$$

In the first case, we have that

$$p^{a_p} \leq F_{(p+1)/2} < \alpha^{(p-1)/2},$$

therefore

$$a_p < \frac{(p-1)\log\alpha}{2\log p} < \frac{(p+1)\log\alpha}{2\log p}. \tag{23}$$

In the second case, we arrive at the same conclusion in the following way. If $\eta = -1$, then since $L_s < \alpha^{s+1}$ for all $s \geq 1$, we have

$$p^{a_p} \leq L_{(p-1)/2} < \alpha^{(p+1)/2},$$

leading again to estimate (23). When $\eta = 1$ and $(p+1)/2$ is odd, then

$$p^{a_p} \leq L_{(p+1)/2} = \alpha^{(p+1)/2} + \beta^{(p+1)/2} < \alpha^{(p+1)/2},$$

leading again to estimate (23). Finally, assume that $\eta = 1$ and $(p+1)/2$ is even. If $L_{(p+1)/2} \neq p^{a_p}$, then

$$p^{a_p} \leq \frac{L_{(p+1)/2}}{2} < \frac{\alpha^{(p+1)/2} + 1}{2} < \alpha^{(p+1)/2},$$

leading again to (23). It remains to deal with the case $L_{(p+1)/2} = p^{a_p}$. Since $p > 7$, it follows easily that $L_{(p+1)/2} > p$. Hence, $a_p > 1$, and therefore $L_{(p+1)/2}$ is a perfect power of exponent > 1, and this is impossible by the main result from [3]. Thus, we have showed that estimate (23) holds for all $p > 7$. We thus get that

$$2^{n_t - 1} \le a_p \le \frac{(p+1)\log\alpha}{2\log p} < \frac{2^{t+1}\log\alpha}{2\log(2^{t+1}-1)}, \tag{24}$$

where for the last inequality we used the fact that $p \le 2^{t+1} - 1$ together with the fact that the function $(s+1)/(2\log s)$ is increasing for $s \ge 7$. We now show that $n_t \le t-2$. Indeed, if not, then $n_t \ge t-1$, which together with inequality (24) leads to

$$2^{t-2} < \frac{2^{t+1}\log\alpha}{2\log(2^{t+1}-1)},$$

therefore

$$\log(2^{t+1}-1) < 4\log\alpha,$$

which is false for $t \ge 3$. Hence, $n_t \le t-2$ holds for all $t \ge 3$. Since the function $\log s/s$ is decreasing for $s \ge 3$, we get that

$$S_2 \le \frac{\log 7}{7} + \sum_{t\ge 3} \frac{(t-2)\log(2^t)}{2^t} < \frac{\log 7}{7} + (\log 2)\sum_{t\ge 3}\frac{t(t-2)}{2^t}.$$

One computes easily that

$$\sum_{t\ge 3}\frac{t(t-2)}{2^t} = 1,$$

therefore

$$S_2 < \frac{\log 7}{7} + \log 2. \tag{25}$$

Estimates (20), (21), (22), and (25) lead to

$$\begin{aligned}\log\ell \quad < \quad & 3.2\log\log(6\ell) \\ & + \left(\frac{3\log 3}{2} - \log(0.14\log\alpha) + 3.2\left(\frac{\log 7}{7} + \log 2 - 0.44\right)\right),\end{aligned}$$

therefore

$$\log\ell < 3.2\log\log(6\ell) + 6.05.$$

The above inequality leads to $\ell < 4 \cdot 10^6$.

3.6. Bounding ℓ Even Better

Now let us write

$$n = U \cdot V, \qquad \text{where} \quad U = \prod_{\substack{1 \le i \le u \\ r_i \mid m}} r_i^{\lambda_i}, \quad \text{and} \quad V = \prod_{\substack{1 \le i \le u \\ r_i \nmid m}} r_i^{\lambda_i}.$$

Let i be such that $r_i \mid U$. Put $r := r_i$ and $\lambda := \lambda_i$. We have already seen that $r^\lambda \mid \ell$ if $i = 1$ because $r_1 = 2$. So, assume that r is odd. Suppose first that $r \ge 5$. Then L_{r^δ} divides F_n for $\delta = 1, 2, \ldots, \lambda$. Each of L_{r^δ} has a primitive prime factor which is congruent to 1 modulo r^δ. Thus $\phi(F_n)$ is divisible by $r^{1+2+\cdots+\lambda} = r^{\lambda(\lambda+1)/2}$. Since $r < 10^{14}$, a calculation of McIntosh and Roettger (see [1] and [10]) shows that $r \| F_{z(r)}$ in this range confirming thus a conjecture of Wall [14]. Thus, $r^{\lambda(\lambda+1)/2-1}$ divides m. If $\lambda \ge 2$, then $\lambda(\lambda+1)/2 - 1 \ge \lambda$, showing that $r^\lambda \mid \gcd(n, m)$. This is also obviously true if $\lambda = 1$ as well. Hence, if $r > 3$, then $r^\lambda \mid \gcd(n, m) \mid \ell$. Assume now that $r = 3$. Then L_{r^δ} divides F_n and has a primitive prime factor congruent to 1 modulo r^δ for all $\delta \ge 2$. It now follows that $3^{\lambda(\lambda+1)/2-1}$ divides $\phi(F_n)$, therefore if $\lambda \ge 2$, then $3^{\lambda(\lambda+1)/2-2}$ divides m. Now $\lambda(\lambda+1)/2-2 \ge \lambda$ holds for all $\lambda \ge 3$. This shows that $3^\lambda \mid \ell$ if $\lambda \ge 3$. This is also true if $\lambda = 1$. If $\lambda = 2$ and there exists another odd prime $q > 3$ dividing n, then also L_{3q} divides F_n and L_{3q} has a primitive prime divisor which is congruent to 1 modulo 3. Since $19 \mid L_9 \mid F_n$, we get that 3^3 divides $\phi(F_n) = F_m$, therefore $9 \mid m$. Thus $3^\lambda \mid \ell$ unless $\lambda = 2$ and $n' = 9$. In this last case we have $n = 2^{\lambda_1} \cdot 9 < 3\ell < 12 \cdot 10^6$, contradicting the fact that $n > 8 \cdot 10^{371}$. Thus, in all cases $U \mid \ell$. Furthermore, since $n > 8 \cdot 10^{371}$ and $\ell < 4 \cdot 10^6$, we get that $V > 1$. We now look at V. Assume that V has w primes in it with $w \ge 1$. Let $p_1 \ge 7$ be the smallest prime factor of V. Then V has 2^{w-1} odd divisors d all divisible by p_1. Since $L_d \mid F_n$ for all such divisors d, and since for each one of these divisors d the number L_d has a primitive divisor $p_d \equiv 1 \pmod{d}$, we get that the power of p_1 in $\phi(F_n)$ is at least 2^{w-1}. Since $p_1 \nmid m$, it follows that $2^{w-1} \le a_{p_1}$, where a_{p_1} is the exponent of p_1 in $F_{z(p_1)}$. It was shown in the preceding section that the inequality $a_{p_1} \le (p_1+1)(\log \alpha)/(2 \log p_1) < (p_1+1)/(4 \log p_1)$ holds for all $p_1 > 7$ because $\log \alpha < 1/2$. This is also true for $p_1 = 7$ because $a_7 = 1 < (7+1)/(4 \log 7)$. We thus get that $2^w < (p_1+1)/(2 \log p_1)$, therefore

$$w < \frac{\log(p_1+1) - \log(2 \log p_1)}{\log 2}.$$

We now return to inequality (19) and use the observation that the function $r \log r/(r-1)^2$ is decreasing for $r \ge 7$, to get that

$$0.14\ell \log \alpha \le \left(\prod_{\substack{r \mid \ell \\ r > 2}} \left(1 + \frac{2r \log r}{(r-1)^2}\right) \right) \left(1 + \frac{2p_1 \log p_1}{(p_1-1)^2}\right)^{(\log(p_1+1) - \log(2 \log p_1))/\log 2}.$$

We can now give a better bound on ℓ. The product of the first 8 primes is $> 9 \cdot 10^6 > \ell$, and the function $(r \log r)/(r-1)^2$ is decreasing for $r \geq 3$. Furthermore, the maximum of the function

$$\left(1 + \frac{2p_1 \log p_1}{(p_1-1)^2}\right)^{(\log(p_1+1)-\log(2\log p_1))/\log 2}$$

as $p_1 \geq 7$ runs over primes is < 1.8. Thus,

$$0.14\ell \log \alpha \leq \prod_{3 \leq q \leq 17} \left(1 + \frac{2r \log r}{(r-1)^2}\right) \cdot 1.8 \leq 51.68,$$

leading to $\ell \leq 766$. The product of the first five primes exceeds 766, so that

$$0.14\ell \log \alpha \leq \prod_{3 \leq q \leq 7} \left(1 + \frac{2r \log r}{(r-1)^2}\right) \cdot 1.8 < 16.82,$$

yielding $\ell \leq 248$. Thus, $U \leq \ell \leq 248$.

We can now see the light at the end of the tunnel. Namely, we shall show that $p_1 < 10^{14}$. Assume that we have proved that. Suppose that n is divisible by $p_1 q$, where q is some other prime factor (which might be p_1 itself). Since $p_1 \geq 7$, it follows that both L_{p_1} and $L_{p_1 q}$ have primitive prime factors which are both congruent to 1 modulo p_1. This shows that $p_1^2 \mid \phi(F_n)$, so $p_1^2 \mid F_m$. By McIntosh's calculation, we get that $p_1 \mid m$, which is impossible. Thus $n' = p_1$, therefore $n = 2^{\lambda_1} p_1 \leq \ell p_1 < 248 \cdot 10^{14}$, contradicting the fact that $n > 8 \cdot 10^{371}$. Thus it remains to bound p_1.

3.7. Bounding p_1

Returning to inequality (14), we have

$$\begin{aligned} \ell \log \alpha - 10^{-10} < \ell \log \alpha + \log\left(1 - \frac{1}{\alpha^n}\right) & < \sum_{p \mid F_n} \frac{1}{p-1} \\ & \leq \sum_{p \mid F_U} \frac{1}{p-1} + \sum_{\substack{p \mid F_n \\ p \nmid F_U}} \frac{1}{p-1}. \end{aligned}$$

Since $U \mid \ell$, a calculation with Mathematica shows that the inequality

$$\ell \log \alpha - 10^{-10} - \sum_{p \mid F_U} \frac{1}{p-1} \geq .3145\ell$$

holds for all even $\ell \le 248$. Thus,

$$.3145\ell \le \sum_{\substack{p \mid F_n \\ p \nmid F_U}} \frac{1}{p-1}.$$

We now assume that $p_1 > 10^{14}$ and we shall get a contradiction. Note that the above sum is

$$\sum_{\substack{p \mid F_n \\ p \nmid F_U}} \frac{1}{p-1} = \sum_{d_1 \mid U} \sum_{\substack{d_2 \mid V \\ d_2 > 1}} Q_{d_1 d_2},$$

where, as in Section 3.5, we have

$$Q_d = \sum_{p \in \mathcal{Q}_d} \frac{1}{p-1}.$$

Since $p \equiv \pm 1 \pmod{d}$, and $d \ge p_1 > 10^{14}$, it follows that $p/(p-1) < .3145/.3144$ for all $p \mid F_n$ but $p \nmid F_U$. Thus we get that

$$.3144\ell \le \sum_{d_1 \mid U} \sum_{\substack{d_2 \mid V \\ d_2 > 1}} \frac{1}{p}. \tag{26}$$

Let $d = d_1 d_2$. We saw that the inequality $\ell_d = \#\mathcal{Q}_d < d \log \alpha / \log d$ holds for all our d (see inequality (18)). Our primes $p \in \mathcal{Q}_d$ have the property that $p \equiv \pm 1 \pmod{d}$. By the large sieve inequality of Montgomery and Vaughan [11], we have that if we write $\pi(t; a, b)$ for the number of primes $p \equiv a \pmod{b}$ which do not exceed t, then the inequality

$$\pi(t; a, b) \le \frac{2t}{\phi(b) \log(t/b)}$$

holds uniformly for $a \le b < t$, with coprime a and b. The calculation from Page 12 in [8], shows that

$$\sum_{\substack{p \in \mathcal{Q}_d \\ 3d < p < d^2}} \frac{1}{p} < \frac{4}{\phi(d) \log d} + \frac{4 \log \log d}{\phi(d)}.$$

For the remaining primes in $\mathcal{Q}_d$ but not in $(3d, d^2)$ we have that

$$\sum_{\substack{p \in \mathcal{Q}_d \\ p \notin (3d, d^2)}} \frac{1}{p} < \frac{1}{d-1} + \frac{1}{d+1} + \frac{1}{2d-1} + \frac{1}{2d+1} + \frac{1}{3d-1} + \frac{\ell_d}{d^2} < \frac{10}{3\phi(d)} + \frac{\log \alpha}{d \log d}.$$

We thus get that

$$\begin{aligned} Q_d &< \frac{4\log\log d}{\phi(d)}\left(1+\frac{1}{(\log d)\log\log d}+\frac{10}{12\log\log d}+\frac{\log\alpha}{(\log d)\log\log d}\right)\\ &< \frac{5.02\log\log d}{\phi(d)}. \end{aligned}$$

Since $d_1 \mid U$, we get that $d_1 \le 248$. Since $d_2 > 1$, we get that $d_2 \ge p_1 > 10^{14}$. Hence, $d_1 d_2 < d_2^{1.2}$ holds uniformly in d_1 and d_2, therefore

$$Q_d < \frac{5.02\log(1.2\log d_2)}{\phi(d_1)\phi(d_2)}.$$

Let $\tau(V)$ be the number of divisors d_2 of V. Of them, $\tau(V/p_1)$ are multiples of p_1, and for each one of these, L_{d_2} has a primitive prime factor p_{d_2} which in particular is congruent to 1 modulo p_1. Hence, the exponent of p_1 in $\phi(F_n)$ is at least $\tau(V/p_1)$. Since $p_1 \nmid m$, we get that

$$\tau(V/p_1) \le a_{p_1} \le \frac{(p_1+1)\log\alpha}{2\log p_1},$$

leading to

$$\tau(V) \le 2\tau(V/p_1) \le \frac{(p_1+1)\log\alpha}{\log p_1}.$$

Now

$$\begin{aligned} \frac{V}{\phi(V)} &\le \prod_{p\mid V}\left(1+\frac{1}{p-1}\right) \le \left(1+\frac{1}{p_1-1}\right)^{\tau(V)}\\ &\le \left(1+\frac{1}{p_1-1}\right)^{(p_1+1)\log\alpha/\log p_1} < 1.02, \end{aligned}$$

where the last inequality holds because $p_1 > 10^{14}$. Thus, the inequality

$$\frac{1}{\phi(d_2)} \le \left(\frac{V}{\phi(V)}\right)\frac{1}{d_2} \le \frac{1.02}{d_2}$$

holds for all divisors d_2 of V. We therefore get that

$$Q_d \le \frac{(5.02\cdot 1.02)\log(1.2\log d_2)}{d_2\phi(d_1)} < \frac{5.13\log(1.2\log d_2)}{d_2\phi(d_1)}.$$

The function $\log(1.2\log s)/s$ is decreasing for $s > 10^{14}$, showing that the inequality

$$Q_d \le \frac{5.13\log(1.2\log p_1)}{p_1}\cdot\frac{1}{\phi(d_1)}$$

holds for all divisors d of n which do not divide U. Thus,

$$\sum_{\substack{p \mid F_n \\ p \nmid F_U}} \frac{1}{p} \le \frac{5.13\tau(V)\log(1.2\log p_1)}{p_1} \sum_{d_1 \mid \ell} \frac{1}{\phi(d_1)} < \frac{5.13(p_1+1)(\log\alpha)\log(1.2\log p_1)}{p_1 \log p_1} h(\ell),$$

where

$$h(\ell) = \sum_{d_1 \mid \ell} \frac{1}{\phi(d_1)} \le \sum_{d_1 \mid \ell} \phi(d_1) = \ell.$$

Thus, comparing the last bound above with inequality (26), we get

$$\frac{p_1 \log p_1}{(p_1+1)\log(1.2\log p_1)} < \frac{5.13 \cdot \log\alpha}{0.3144}.$$

The above inequality implies that $p_1 < 9 \cdot 10^{11} < 10^{14}$, which is the desired contradiction. Theorem 1 is therefore proved.

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