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GENERALIZING THE COMBINATORICS OF BINOMIAL COEFFICIENTS VIA ℓ-NOMIALS

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Abstract

An *ℓ-sequence* is defined by $a_n = \ell a_{n-1} - a_{n-2}$, with initial conditions $a_0 = 0$, $a_1 = 1$. These ℓ-sequences play a remarkable role in partition theory, allowing ℓ-generalizations of the Lecture Hall Theorem and Euler's Partition Theorem. These special properties are *not* shared with other sequences, such as the Fibonacci sequence, defined by second-order linear recurrences. The ℓ-sequence gives rise to the *ℓ-nomial coefficient* $\binom{n}{k}^{(\ell)} = \prod_{i=1}^{k}(a_{n+1-i}/a_i)$, which is known to be an integer.In this paper, we use algebraic and combinatorial properties of ℓ-sequences to interpret the ℓ-nomial coefficients in terms of weighted lattice paths, integer partitions, and probablility distributions. We show how to use these interpretations to uncover ℓ-generalizations of familiar hypergeometric identities involving binomial coefficients. This leads naturally to an ℓ-analogue of the q-binomial coefficients (Gaussian polynomials) and a corresponding generalization of the "partitions in a box" interpretation of ordinary q-binomial coefficients.

1. Introduction

1.1. The f-Binomial Coefficients and Lucas Sequences

Binomial coefficients and their generalizations occur frequently in combinatorics, number theory, and discrete mathematics. One way to generalize binomial coefficients is to start with a sequence $f = (f_n : n \geq 1)$ of nonzero elements in some field, and define the *f-factorials*

$$[n]!^{(f)} = f_1 f_2 \cdots f_n, \quad [0]!^{(f)} = 1$$

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and f-*binomial coefficients*

$$\binom{n}{k}^{(f)} = \frac{[n]!^{(f)}}{[k]!^{(f)}[n-k]!^{(f)}} = \frac{f_n f_{n-1}\cdots f_{n-k+1}}{f_k f_{k-1}\cdots f_1} \qquad (0 \le k \le n).$$

In many cases of interest, these f-binomial coefficients satisfy certain integrality properties.

Example 1. (a) If $f_n = n$ for all n, $\binom{n}{k}^{(f)}$ is the usual binomial coefficient, which is always an integer. (b) Let q be a variable, and let $f_n = 1+q+q^2+\cdots+q^{n-1} = [n]_q$ for $n \ge 1$. Then $\binom{n}{k}^{(f)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-*binomial coefficient*, which is known to be a polynomial in q with nonnegative integer coefficients (a Gaussian polynomial). (c) If f is the Fibonacci sequence, the quantities $\binom{n}{k}^{(f)}$ are known to be integers [23], which Hoggatt termed *Fibonomials* in 1967 [15].

The third example generalizes as follows. Any sequence defined by a recurrence $f_n = sf_{n-1} + tf_{n-2}$, for integers s and t, with initial conditions $(f_0, f_1) = (0, 1)$ or $(1, s)$ is known as a *Lucas sequence*. The Fibonacci sequence is one example, as are the ℓ-sequences defined below. Lucas sequences are well-studied and are known to have many striking properties, both number-theoretic and combinatorial (see, e.g., [5, 9, 10, 21, 22, 23]). They are even the basis for a proposed public-key cryptosystem, `LUC` [27].

Dating back at least to the work of Lucas (1878) and Carmichael, it has been known that the f-binomial coefficients are integers when f is a Lucas sequence with initial conditions $(f_0, f_1) = (0, 1)$ [23, p. 203]. Both number-theoretic and algebraic properties of these f-binomial coefficients have been studied, especially in the case when f is the Fibonacci sequence [15]. Related work can be found in [14, 16, 18, 19, 30]. As for combinatorial interpretations, a poset interpretation of the Fibonomials was proposed in [20] and recently a tiling interpretation in [4]. Alexanderson and Klosinski [1] confirmed that the q-analogues of the f-binomial coefficients, for Lucas sequences with initial conditions $(f_0, f_1) = (0, 1)$, are polynomials with nonnegative integer coefficients. Picon [24] has shown that iterated substitutions of Lucas numbers (defined by $f'_n = f_{f_n}$, etc.) preserve certain quotients, such as the one used to define f-binomial coefficients.

1.2. Motivation for Studying ℓ-Sequences and ℓ-nomial Coefficients

The goal of this paper is to give a detailed development of the combinatorial properties of the f-binomial coefficients obtained from ℓ-*sequences*. These are Lucas sequences given by $a_0 = 0$, $a_1 = 1$, and $a_n = \ell a_{n-1} - a_{n-2}$ for all $n \ge 2$, where $\ell \geq 2$ is a fixed integer. We call the a-binomial coefficients associated to ℓ-sequences *ℓ-nomial coefficients* and denote them $\binom{n}{k}^{(\ell)}$.

Why do ℓ-sequences and ℓ-nomial coefficients merit special study? On one hand, the ℓ-sequences stand out among all Lucas sequences as being a *generalization* of the nonnegative integers (the case $\ell = 2$). On the other hand, one of our principal motivations in studying ℓ-sequences is their unexpected appearance in partition theory via the *Generalized Lecture Hall Theorem* of Bousquet-Mélou and Eriksson [7, 8]:

> The number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of N satisfying
>
> $$\frac{\lambda_1}{a_n} \geq \frac{\lambda_2}{a_{n-1}} \geq \frac{\lambda_3}{a_{n-2}} \geq \ldots \geq \frac{\lambda_n}{a_1} \geq 0 \tag{1}$$
>
> is equal to the number of partitions of N into parts from the set
>
> $$\{a_i + a_{i-1} \mid 1 \leq i \leq n\}.$$

In particular, this theorem says that the generating function for those partitions satisfying the constraints (1) is $\prod_{i=1}^{n}(1 - q^{a_i + a_{i-1}})^{-1}$. No analogous partition-theoretic results are known for any other Lucas sequence.

The Generalized Lecture Hall Theorem depends on a polynomial analogue of the fact that for ℓ-sequences, the following ratio is integral for all $n \geq 0$:

$$\frac{(a_n)(a_n + a_{n-1})(a_n + a_{n-1} + a_{n-2}) \cdots (a_n + a_{n-1} + \ldots + a_1)}{a_n a_{n-1} \cdots a_1}. \tag{2}$$

It *appears* that among Lucas sequences, the ℓ-sequences are unique in having this property (this is related to a conjecture in [8]).

The ℓ-sequences also play a surprising role in the following generalization of *Euler's Odd-Distinct Partition Theorem*, which can be viewed as a certain limit of the Lecture Hall Theorem:

> **The ℓ-Euler Theorem** [8, 26]: *Let $u = (\ell + \sqrt{\ell^2 - 4})/2$. The number of partitions of N in which the ratio of consecutive parts exceeds u is equal to the number of partitions of N from the set*
>
> $$\{a_i + a_{i-1} \mid i \geq 1\}.$$

(When $\ell = 2$ this says that the number of partitions of N into distinct parts is equal to the number of partitions of N into odd parts.)

Although we will not deal explicitly with these partition theorems in the present paper, we hope that the combinatorics developed here will aid in the resolution of several open problems concerning the role of ℓ-sequences in partition theory. We shall say more about these problems in Section 5.

1.3. Organization of the Paper and Main Results

Section 2 develops some algebraic and combinatorial properties of (a_n) and more general ℓ-sequences. For our later work, the most important results in this section are Theorem 9, which gives several expressions for a_{r+s} in terms of a_r and a_s, and Theorem 13, which shows that (a_n) and related sequences are enumerators for certain regular languages. The words in these languages are the so-called *ℓ-admissible sequences*, which provide unique representations for nonnegative integers generalizing base-b expansions. Such sequences were used in [26] for the combinatorial proof of the ℓ-Euler theorem, but were discovered earlier by Fraenkel [13] in the context of games.

Our main contributions begin in Section 3. Some general theorems on f-binomial coefficients are used to deduce two-term recurrences, fermionic formulas, and combinatorial interpretations for ℓ-nomial coefficients. One combinatorial model involves weighted lattice paths; other models generalize integer partitions by filling the Ferrers diagram of a partition with certain ℓ-admissible sequences. These fillings reduce to ordinary partitions in the case $\ell = 2$, where only the all-zero sequence is allowed. Section 3.3 explains the relationship between ℓ-nomial coefficients and q-binomial coefficients. Some consequences include another proof of the integrality of $\binom{n}{k}^{(\ell)}$ and an ℓ-generalization of the Chu-Vandermonde identity. We show in Section 3.4 that the ℓ-nomial coefficients satisfy a particularly simple *three-term recurrence*. This leads to ℓ-generalizations of the binomial theorem (Section 3.5) and a probabilistic interpretation of ℓ-nomial coefficients in terms of weighted coins (Section 3.6).

Section 4 investigates a q-analogue of the ℓ-nomial coefficient. We use a general theorem on $(f \circ f')$-binomial coefficients to obtain a recurrence characterizing the q-ℓ-nomial coefficients. This recurrence not only proves the positivity and polynomiality of these coefficients, but also leads to a combinatorial formula involving the filled partitions from Section 3. This formula generalizes the usual partition-theoretic interpretation of the q-binomial coefficients.

We conclude in Section 5 by suggesting future directions for study.

2. ℓ-Sequences

It will be useful in the sequel to work with the following somewhat more general notion of an ℓ-sequence.

Definition 2. Given an integer $\ell \geq 2$, an *ℓ-sequence* is a sequence $(s_n)_{n\in\mathbb{Z}}$ of real numbers such that $s_n = \ell s_{n-1} - s_{n-2}$ for all $n \in \mathbb{Z}$.

An ℓ-sequence (s_n) is uniquely determined by any two consecutive values in the sequence, e.g., s_0 and s_1. A linear combination of ℓ-sequences is also an ℓ-sequence. It follows that the set of all ℓ-sequences (for a fixed ℓ) is a two-dimensional real vector space. If $(s_n)_{n\in\mathbb{Z}}$ is an ℓ-sequence and $k \in \mathbb{Z}$, then $(s_{n+k})_{n\in\mathbb{Z}}$ and $(s_{-n})_{n\in\mathbb{Z}}$ are ℓ-sequences.

Definition 3. Given an integer $\ell \geq 2$, let $a^{(\ell)}$, $d^{(\ell)}$, and $p^{(\ell)}$ be the unique ℓ-sequences with initial conditions

$$a_0^{(\ell)} = 0,\ a_1^{(\ell)} = 1; \quad d_0^{(\ell)} = 1,\ d_1^{(\ell)} = 1; \quad p_0^{(\ell)} = 2,\ p_1^{(\ell)} = \ell.$$

We have

$$d_i^{(\ell)} = a_i^{(\ell)} - a_{i-1}^{(\ell)} \qquad (i \in \mathbb{Z}),$$

since both sides are ℓ-sequences that agree for $i = 0, 1$. Similarly,

$$p_i^{(\ell)} = a_{i+1}^{(\ell)} - a_{i-1}^{(\ell)} \qquad (i \in \mathbb{Z}).$$

Example 4. When $\ell = 2$, $a_i^{(2)} = i$, $d_i^{(2)} = 1$, and $p_i^{(2)} = 2$ for all $i \in \mathbb{Z}$. When $\ell = 3$,

$$\begin{aligned}
(a_i^{(3)} : i \geq 0) &= (0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, \ldots) \\
(d_i^{(3)} : i \geq 0) &= (1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, \ldots) \\
(p_i^{(3)} : i \geq 0) &= (2, 3, 7, 18, 47, 123, 322, 843, 2207, 5778, \ldots).
\end{aligned}$$

One may check that $a_n^{(3)} = F_{2n}$ and $d_n^{(3)} = F_{2n-1}$, where F_i is the i'th Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}$). Similarly, $p_n^{(3)} = L_{2n}$ where L_i is the i'th Lucas number ($L_0 = 2$, $L_1 = 1$, $L_i = L_{i-1} + L_{i-2}$). We see that one can recover ordinary integers from ℓ-sequences (when $\ell = 2$) or Fibonacci and Lucas numbers (when $\ell = 3$).

Henceforth, we will omit ℓ's from the notation when there is no danger of confusion.

Many identities are known for general Lucas sequences, as well as various combinatorial interpretations (see, e.g., [5, 9, 21, 23]). In the rest of this section, we highlight and prove some that are most relevant for our work with ℓ-sequences.

2.1. Algebraic Properties of ℓ-Sequences

Definition 5. Let $u = u_\ell$ and $v = v_\ell$ be the roots of the polynomial $x^2 - \ell x + 1$, namely

$$u_\ell = \frac{\ell + \sqrt{\ell^2 - 4}}{2}, \qquad v_\ell = \frac{\ell - \sqrt{\ell^2 - 4}}{2}.$$

Observe that

$$u_\ell + v_\ell = \ell; \qquad u_\ell v_\ell = 1.$$

Theorem 6. *For $\ell > 2$, the sequences $U = (u_\ell^n : n \in \mathbb{Z})$ and $V = (v_\ell^n : n \in \mathbb{Z})$ form a basis for the vector space of ℓ-sequences. Moreover,*

$$a_n^{(\ell)} = \frac{u_\ell^n - v_\ell^n}{u_\ell - v_\ell}, \qquad p_n^{(\ell)} = u_\ell^n + v_\ell^n.$$

Proof. For all $n \in \mathbb{Z}$, $u^n = u^{n-2}u^2 = u^{n-2}(\ell u - 1) = \ell u^{n-1} - u^{n-2}$, so U is an ℓ-sequence. Similarly V is an ℓ-sequence. Noting that $U_0 = 1 = V_0$ and $U_1 = u \neq v = V_1$ (since $\ell > 2$), we see that $\{U, V\}$ is linearly independent, hence a basis for the two-dimensional space of ℓ-sequences. The stated expansions for $a^{(\ell)}$ and $p^{(\ell)}$ follow since the two sides agree for $n = 0$ and $n = 1$. □

For any $\ell \geq 2$ and any *real number* r, we define $p_r = p_r^{(\ell)}$ by the formula $p_r = u^r + v^r$. Since $uv = 1$, it follows that

$$p_{-r} = p_r, \qquad p_r^2 = p_{2r} + 2 \qquad (r \in \mathbb{R}). \tag{3}$$

Theorem 7. *Let $(s_n)_{n\in\mathbb{Z}}$ be an ℓ-sequence. For $n, k \in \mathbb{Z}$, define*

$$g(n,k) = s_n s_k - s_{n-1}s_{k-1}.$$

(a) For fixed k, $(g(n,k))_{n\in\mathbb{Z}}$ is an ℓ-sequence. (b) For all $n, k \in \mathbb{Z}$, $g(n,k) = g(n-1, k+1)$. (c) If $s_0 = 0$ or $\ell s_0 = 2s_1$, then $g(n,k) = (s_1^2 - s_0^2)a_{n+k-1}$. (d) If $s_0 = s_1$, then $g(n,k) = s_0^2(\ell - 2)a_{n+k-2}$.

Proof. (a) holds because $(g(n,k))_{n\in\mathbb{Z}}$ is a linear combination of ℓ-sequences. For (b), compute

$$\begin{aligned} g(n,k) &= s_n s_k - s_{n-1}s_{k-1} = (\ell s_{n-1} - s_{n-2})s_k - s_{n-1}s_{k-1} \\ &= s_{n-1}(\ell s_k - s_{k-1}) - s_{n-2}s_k = s_{n-1}s_{k+1} - s_{n-2}s_k \\ &= g(n-1, k+1). \end{aligned}$$

(c) Assuming $s_0 = 0$ or $\ell s_0 = 2s_1$, we must show that the two ℓ-sequences $(g(n,k))_{n\in\mathbb{Z}}$ and $((s_1^2 - s_0^2)a_{n+k-1})_{n\in\mathbb{Z}}$ have the same values when $n = k$ and $n = k+1$. Iteration of (b) gives

$$g(k,k) = g(k-1,k+1) = \cdots = g(1,2k-1) = s_1 s_{2k-1} - s_0 s_{2k-2};$$

$$g(k+1,k) = g(k,k+1) = \cdots = g(1,2k) = s_1 s_{2k} - s_0 s_{2k-1}.$$

It now suffices to show that the ℓ-sequences $(s_1 s_t - s_0 s_{t-1})_{t\in\mathbb{Z}}$ and $((s_1^2 - s_0^2)a_t)_{t\in\mathbb{Z}}$ are equal. Again, we do this by comparing the initial values $t = 0$ and $t = 1$. Both sequences evaluate to $s_1^2 - s_0^2$ when $t = 1$. When $t = 0$, $(s_1^2 - s_0^2)a_0 = 0 = s_1 s_0 - s_0 s_{-1}$, since the hypothesis of (c) guarantees that $s_0 = 0$ or $s_{-1} = \ell s_0 - s_1 = s_1$.

(d) Assume $s_0 = s_1$. As in (c), we are reduced to verifying that the ℓ-sequences $(s_1 s_t - s_0 s_{t-1})_{t\in\mathbb{Z}}$ and $(s_0^2(\ell-2)a_{t-1})_{t\in\mathbb{Z}}$ agree for $t = 1$ and $t = 2$. Both sequences are zero when $t = 1$, since $s_0 = s_1$. When $t = 2$,

$$s_1 s_2 - s_0 s_1 = \ell s_1^2 - 2s_0 s_1 = s_0^2(\ell - 2)a_1.$$

□

Corollary 8. *For all $n, k \in \mathbb{Z}$,*

$$\begin{aligned} a_n a_k - a_{n-1}a_{k-1} &= a_{n+k-1};\ d_n d_k - d_{n-1}d_{k-1} \\ &= (\ell - 2)a_{n+k-2};\ p_n p_k - p_{n-1}p_{k-1} \\ &= (\ell^2 - 4)a_{n+k-1}. \end{aligned}$$

Theorem 9. *For all $r, s \in \mathbb{Z}$,*

$$\begin{aligned} a_{r+s} &= d_s a_r + d_{r+1}a_s = d_{s+1}a_r + d_r a_s \\ &= -a_{s-1}a_r + a_{r+1}a_s = a_{s+1}a_r - a_{r-1}a_s \\ &= u^s a_r + v^r a_s = v^s a_r + u^r a_s \\ &= (p_s/2)a_r + (p_r/2)a_s. \end{aligned}$$

Proof. By the first part of the preceding corollary,

$$\begin{aligned} d_s a_r + d_{r+1}a_s &= (a_s - a_{s-1})a_r + (a_{r+1} - a_r)a_s \\ &= a_{r+1}a_s - a_r a_{s-1} = a_{r+s}; \\ d_{s+1}a_r + d_r a_s &= (a_{s+1} - a_s)a_r + (a_r - a_{r-1})a_s \\ &= a_{s+1}a_r - a_s a_{r-1} = a_{r+s}; \\ a_{r+1}a_s - a_r a_{s-1} &= a_{r+s} = a_{s+1}a_r - a_s a_{r-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
u^s a_r + v^r a_s &= (u-v)^{-1}(u^s(u^r - v^r) + v^r(u^s - v^s)) \\
&= (u-v)^{-1}(u^{r+s} - v^{r+s}) = a_{r+s}; \\
v^s a_r + u^r a_s &= (u-v)^{-1}(v^s(u^r - v^r) + u^r(u^s - v^s)) \\
&= (u-v)^{-1}(u^{r+s} - v^{r+s}) = a_{r+s}.
\end{aligned}$$

Adding these equations and dividing by 2 gives $(p_s/2)a_r + (p_r/2)a_s = a_{r+s}$. □

The proof of Theorem 7 can be readily adapted to establish the following theorem.

Theorem 10. *Let $(s_n)_{n\in\mathbb{Z}}$ be an ℓ-sequence. For $n, k \in \mathbb{Z}$, define*

$$f(n,k) = s_{n-1}s_k - s_n s_{k-1}.$$

(a) *For fixed k, $(f(n,k))_{n\in\mathbb{Z}}$ is an ℓ-sequence.*

(b) *For all $n, k \in \mathbb{Z}$, $f(n,k) = f(n-1,k-1)$.*

(c) *For all $n, k \in \mathbb{Z}$, $f(n,k) = (s_0^2 + s_1^2 - \ell s_0 s_1)a_{n-k}$.*

Corollary 11. *For all $n, k \in \mathbb{Z}$, $a_{n-1}a_k - a_n a_{k-1} = a_{n-k}$, $d_{n-1}d_k - d_n d_{k-1} = (2-\ell)a_{n-k}$, and $p_{n-1}p_k - p_n p_{k-1} = (4-\ell^2)a_{n-k}$.*

2.2. Combinatorial Properties of ℓ-Sequences

This section describes some collections of combinatorial objects that are counted by the integers $a_n^{(\ell)}$, $d_n^{(\ell)}$, and $p_n^{(\ell)}$.

Definition 12. For $\ell \geq 2$, an *ℓ-admissible word* is a word $w = w_1 w_2 \cdots w_n$ with $0 \leq w_i < \ell$ for all i, such that w contains no subword matching the pattern $(\ell - 1)(\ell-2)^*(\ell-1)$, where $(\ell-2)^*$ denotes zero or more occurrences of $\ell - 2$. Let $W = W^{(\ell)}$ be the set of all ℓ-admissible words. Let W' (resp. W'') be the subset of W consisting of words that do not begin with 0 (resp. 00). For any set Z of words, let Z_n be the set of words in Z of length n. Finally, let $W_n^\dagger$ be the set of words in W_n that weakly precede the word $(\ell-2)^n$ in lexicographic order.

Theorem 13. *For all $\ell \geq 2$:*

(a) *$a_n = |W_{n-1}|$ for $n \geq 0$;*

(b) *$d_n = |W'_{n-1}| = |W^\dagger_{n-1}|$ for $n \geq 1$;*

(c) *$p_n = |W''_n|$ for $n \geq 1$.*

First Proof of (a). Let $s_n = |W_{n-1}|$ for $n \geq 0$. We have $s_0 = 0 = a_0$ and $s_1 = 1 = a_1$, so it suffices to prove $\ell s_{n-1} = s_n + s_{n-2}$ for $n \geq 2$. We define a bijection $h : W_{n-2} \times \{0, 1, \ldots, \ell - 1\} \to W_{n-1} \cup W_{n-3}$. Fix (w, x) with $w \in W_{n-2}$ and $0 \leq x < \ell$. If wx is admissible, let $h(w, x) = wx \in W_{n-1}$. Otherwise, w ends in $(\ell - 1)(\ell - 2)^*$ and $x = \ell - 1$, and we define $h(w, x) = w_1 \cdots w_{n-3} \in W_{n-3}$. To invert h, map $z = w_1 \cdots w_{n-1} \in W_{n-1}$ to $(w_1 \cdots w_{n-2}, w_{n-1})$, and map $z \in W_{n-3}$ to $(zy, \ell - 1)$, where $y = \ell - 2$ if z ends in $(\ell - 1)(\ell - 2)^*$, and $y = \ell - 1$ otherwise. □

Second Proof of (a). Define a "norm map" $N : W_{n-1}^{(\ell)} \to \{0, 1, \ldots, a_n^{(\ell)} - 1\}$ by $N(w_1 \cdots w_{n-1}) = \sum_{i=1}^{n-1} w_i a_{n-i}^{(\ell)}$. It was shown in [26] that N is a bijection such that $w \leq_{\text{lex}} v$ in W_{n-1} iff $N(w) \leq N(v)$. □

Proof of (b). Now let $s_n = |W'_{n-1}|$ for $n \geq 1$. Note $s_1 = 1 = d_1$ and $s_2 = \ell - 1 = d_2$. For $n \geq 3$, it is routine to check that the bijection h defined in part (a) restricts to a bijection $h' : W'_{n-2} \times \{0, 1, \ldots, \ell - 1\} \to W'_{n-1} \cup W'_{n-3}$, which proves that (s_n) and (d_n) satisfy the same recursion. Thus, $s_n = d_n$ for all $n \geq 1$. Alternatively, since $W'_n = W_n \sim \{0w : w \in W_{n-1}\}$, we can deduce $|W'_{n-1}| = d_n$ from (a) and $d_n = a_n - a_{n-1}$.

Recall that the norm bijection N is compatible with the lexicographic ordering on words. So, to prove that $|W^{\dagger}_{n-1}| = d_n$, it suffices to verify that the norm of $(\ell - 2)^{n-1}$ is $d_n - 1$, or equivalently $(\ell - 2)(a_1 + a_2 + \cdots + a_{n-1}) = a_n - a_{n-1} - 1$ for all $n \geq 1$. This identity is true for $n = 1$ and $n = 2$. Assuming by induction that the identity holds for some $n \geq 2$, then the identity also holds for $n + 1$, since

$$\begin{aligned}
(\ell - 2)(a_1 + \cdots + a_n) &= (\ell - 2)(a_1 + \cdots + a_{n-1}) + (\ell - 2)a_n \\
&= (a_n - a_{n-1} - 1) + \ell a_n - 2a_n \\
&= (\ell a_n - a_{n-1}) - a_n - 1 = a_{n+1} - a_n - 1.
\end{aligned}$$

□

Proof of (c). Now let $s_n = |W''_n|$ for $n \geq 1$. Note $s_1 = \ell = p_1$ and $s_2 = \ell^2 - 2 = p_2$. For $n \geq 3$, it is routine to check that the bijection h defined in part (a) (with n replaced by $n+1$) restricts to a bijection $h'' : W''_{n-1} \times \{0, 1, \ldots, \ell - 1\} \to W''_n \cup W''_{n-2}$, which proves that (s_n) and (p_n) satisfy the same recursion. Thus, $s_n = p_n$ for all $n \geq 1$. Alternatively, (c) can be proved using (a) and $p_n = a_{n+1} - a_{n-1}$. □

Example 14. For $\ell = 3$, the 21 words in W_3 (in lexicographic order) are:

000	001	002	010	011	012	020	021
100	101	102	110	111	112	120	121
200	201	202	210	211.			

W_3' consists of the 13 words in the second and third rows, whereas $W_3^\dagger$ consists of the 13 words up to and including 111. W_3'' consists of the 18 words following 002. We have $|W_3| = 21 = a_4$, $|W_3'| = |W_3^\dagger| = 13 = d_4$, and $|W_3''| = 18 = p_3$.

By using an obvious modification of the bijection h in the first proof of Theorem 13(a), one can also obtain the following result.

Proposition 15. *Suppose $\ell \geq 2$, $n \geq 0$, and i, j are distinct elements of $A = \{0, 1, \ldots, \ell - 1\}$. Then $a_n^{(\ell)}$ is the number of words in A^{n-1} in which i is never immediately followed by j.*

The integers $a_n^{(\ell)}$ also have the following graph-theoretic interpretation. (This generalizes problems 2.2.15 and 2.2.16 in [31].)

Theorem 16. *For $\ell \geq 3$ and $m \geq 1$, let $G_m = G_m^{(\ell)}$ be any graph consisting of a sequence of m ℓ-cycles such that neighboring cycles in the sequence share a common edge (see Figure 1 for an example with $\ell = 6$ and $m = 5$). The number of spanning trees of $G_m^{(\ell)}$ is $a_{m+1}^{(\ell)}$.*

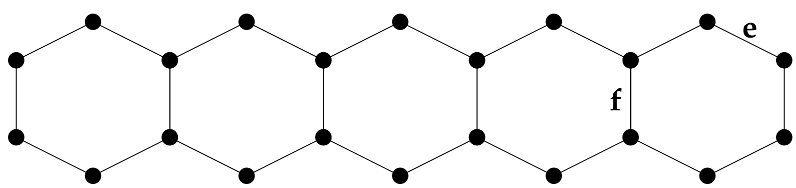

Figure 1: Example of $G_5^{(6)}$.

Proof. Define G_0 to be one of the edges of G_1 not belonging to G_2. For each $m \geq 0$, let T_m be the set of spanning trees of G_m, and let $t_m = |T_m|$. Then $t_0 = 1 = a_1$. It suffices to show that $t_1 = a_2$ and $t_m = \ell t_{m-1} - t_{m-2}$ for all $m \geq 2$. When $m = 1$, $G_m^{(\ell)}$ is a single ℓ-cycle. We obtain a spanning tree by removing any one edge from this cycle, so $t_1 = \ell = a_2$. We now fix $m \geq 2$ and prove that $\ell t_{m-1} = t_m + t_{m-2}$. The expression ℓt_{m-1} counts pairs (e, T) where T is a spanning tree of G_{m-1} and e is one of the edges on the ℓ-cycle C that we add to G_{m-1} to get G_m. It suffices to define a bijection h from the set of such pairs to $T_m \cup T_{m-2}$. Let f be the unique edge that belongs to both C and G_{m-1}. If $e \neq f$, let $h(e, T)$ be the spanning tree of G_m obtained by adding all edges of C except e and f to T. If $e = f$ is an edge of T, let $h(e, T)$ be the spanning tree of G_m obtained by deleting e from T and replacing it with all the other edges in C. If $e = f$ is not an edge of T, let $h(e, T)$ be the spanning tree of G_{m-2}

obtained by erasing all edges in T that are edges of G_{m-1} but not edges of G_{m-2}. It is routine to check that h is a bijection. □

3. The ℓ-nomial Coefficients

Recall that the ℓ-nomial coefficients are defined by

$$\binom{n}{k}^{(\ell)} = \frac{a_n^{(\ell)} a_{n-1}^{(\ell)} \cdots a_{n-k+1}^{(\ell)}}{a_k^{(\ell)} a_{k-1}^{(\ell)} \cdots a_1^{(\ell)}} \qquad (0 \le k \le n).$$

For example,

$$\binom{9}{4}^{(3)} = \frac{2584 \cdot 987 \cdot 377 \cdot 144}{21 \cdot 8 \cdot 3 \cdot 1} = 174,715,376.$$

Figure 2 depicts the beginning of "Pascal's triangle" for 3-nomial coefficients, in which row n from the top contains $\binom{n}{k}^{(3)}$ for $0 \le k \le n$:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 1 & & & & \\
 & & & 1 & & 8 & & 8 & & 1 & & & \\
 & & 1 & & 21 & & 56 & & 21 & & 1 & & \\
 & 1 & & 55 & & 385 & & 385 & & 55 & & 1 & \\
1 & & 144 & & 2640 & & 6930 & & 2640 & & 144 & & 1
\end{array}$$

Figure 2: Pascal's triangle for 3-nomial coefficients.

3.1. Two-Term Recursions

Recall that the usual binomial coefficients satisfy the recursion $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. This can be written more symmetrically using multinomial coefficients as $\binom{r+s}{r,s} = \binom{r+s-1}{r-1,s} + \binom{r+s-1}{r,s-1}$, for $r, s > 0$. We would like to find analogous recursions for ℓ-nomial coefficients. We will deduce these from the following general result for the f-binomial coefficients defined by $\binom{r+s}{r,s}^{(f)} = \frac{[r+s]!^{(f)}}{[r]!^{(f)}[s]!^{(f)}}$.

Theorem 17. *Suppose $f = (f(n) : n \in \mathbb{N}^+)$ is a sequence of nonzero field elements, and g_1, g_2 are functions defined on $\mathbb{N}^+ \times \mathbb{N}^+$ such that*

$$f(r+s) = g_1(r,s)f(r) + g_2(r,s)f(s) \qquad (r, s \in \mathbb{N}^+). \tag{4}$$

Then the f-binomial coefficients satisfy the recursion

$$\binom{r+s}{r,s}^{(f)} = g_1(r,s)\binom{r+s-1}{r-1,s}^{(f)} + g_2(r,s)\binom{r+s-1}{r,s-1}^{(f)} \qquad (r,s>0) \tag{5}$$

and initial conditions $\binom{r}{r,0}^{(f)} = 1 = \binom{s}{0,s}^{(f)}$.

Proof. Dividing both sides of (5) by the nonzero common factor $\frac{[r+s-1]!^{(f)}}{[r-1]!^{(f)}[s-1]!^{(f)}}$, we see that (5) is equivalent to

$$\frac{f(r+s)}{f(r)f(s)} = \frac{g_1(r,s)}{f(s)} + \frac{g_2(r,s)}{f(r)},$$

which is clearly equivalent to (4). □

Corollary 18. *For all $\ell \geq 2$ and all $r, s > 0$, $\binom{r+s}{r,s}^{(\ell)}$ equals each expression shown here:*

$$\begin{aligned}
&d_s\binom{r+s-1}{r-1,s}^{(\ell)} + d_{r+1}\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = d_{s+1}\binom{r+s-1}{r-1,s}^{(\ell)} + d_r\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = -a_{s-1}\binom{r+s-1}{r-1,s}^{(\ell)} + a_{r+1}\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = a_{s+1}\binom{r+s-1}{r-1,s}^{(\ell)} - a_{r-1}\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = u^s\binom{r+s-1}{r-1,s}^{(\ell)} + v^r\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = v^s\binom{r+s-1}{r-1,s}^{(\ell)} + u^r\binom{r+s-1}{r,s-1}^{(\ell)}\\
&\quad = \frac{p_s}{2}\binom{r+s-1}{r-1,s}^{(\ell)} + \frac{p_r}{2}\binom{r+s-1}{r,s-1}^{(\ell)}.
\end{aligned}$$

In particular, each ℓ-nomial coefficient is a positive integer.

Proof. The recursions follow from Theorem 9 and Theorem 17. The integrality assertion follows from the first recursion by induction on $r+s$. □

3.2. Annotated Lattice Paths and Partitions

It is well-known that the ordinary binomial coefficient $\binom{r+s}{r,s}$ counts lattice paths from $(0,0)$ to (r,s), which can be identified with words consisting of r copies of E (east step) and s copies of N (north step). This follows from the recursion $\binom{r+s}{r,s} = \binom{r+s-1}{r-1,s} + \binom{r+s-1}{r,s-1}$, which classifies such paths based on whether they arrive at (r,s) via an east step or a north step. By iterating the recursion (5), we obtain the following analogous result for f-binomial coefficients.

Theorem 19. *Assume $f(r+s) = g_1(r,s)f(r) + g_2(r,s)f(s)$ as in Theorem 17. Let $P(r,s)$ be the set of lattice paths from $(0,0)$ to (r,s). For $\pi \in P(r,s)$, let $E(\pi)$ (resp. $N(\pi)$) be the set of $(x,y) \in \mathbb{N}^+ \times \mathbb{N}^+$ such that π arrives at (x,y) by an east (resp. north) step. Then, for all $r,s \in \mathbb{N}$,*

$$\binom{r+s}{r,s}^{(f)} = \sum_{\pi \in P(r,s)} \prod_{(x,y)\in E(\pi)} g_1(x,y) \prod_{(x,y)\in N(\pi)} g_2(x,y).$$

If g_1 and g_2 take values in $\mathbb{N}$ (or in $\mathbb{N}[q]$), we obtain a combinatorial interpretation for $\binom{r+s}{r,s}^{(f)}$ as counting *annotated lattice paths* ending at (r,s), in which each lattice point (x,y) on the path in the positive quadrant is labeled by an object counted by $g_1(x,y)$ (if (x,y) is reached by an east step) or by $g_2(x,y)$ (if (x,y) is reached by a north step).

In particular, the first two formulas in Corollary 18 give two interpretations for ℓ-nomial coefficients in which points on the lattice path are labeled by words in the language W' or $W^\dagger$ (see Theorem 13). If we allow signed objects, the next two formulas in the corollary give interpretations where the labels come from the language W. Finally, if we multiply through by suitable powers of 2, the last formula in the corollary gives a combinatorial formula for $2^{r+s}\binom{r+s}{r,s}^{(\ell)}$ in which lattice points are labeled by words in the language W''. (One can show that every p_s is even for ℓ even, so in this case, we can avoid the extra powers of 2 by passing to a language in which we retain half the words in each W''_n.)

Next we reformulate Theorem 19 in terms of integer partitions. By considering the array of lattice squares to the left of a lattice path, we can identify lattice paths π ending at (r,s) with partitions λ that fit in an $s \times r$ box, i.e., with integer sequences $\lambda = (\lambda_1, \ldots, \lambda_s)$ such that

$$r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0.$$

Let λ^* be the complementary partition located below the lattice path (where the parts of λ^* give the column heights from right to left), so

$$s \geq \lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_r^* \geq 0.$$

Then a lattice point (x, y) is in $N(\pi)$ if and only if $\lambda_{s+1-y} = x > 0$, whereas a lattice point (x, y) is in $E(\pi)$ if and only if $\lambda^*_{r+1-x} = y > 0$. We can now restate Theorem 19 as follows.

Theorem 20. *Assume $f(r+s) = g_1(r,s)f(r) + g_2(r,s)f(s)$ as in Theorem 17. Let $P(r,s)$ be the set of integer partitions that fit in an $s \times r$ box. Then, for all $r, s \in \mathbb{N}$,*

$$\binom{r+s}{r,s}^{(f)} = \sum_{\lambda \in P(r,s)} \prod_{j:\lambda_j^*>0} g_1(r+1-j, \lambda_j^*) \prod_{i:\lambda_i>0} g_2(\lambda_i, s+1-i).$$

Example 21. Combining Corollary 18 and Theorem 20, we can write down various "fermionic formulas" expressing ℓ-nomial coefficients as sums of products of a's, d's, or p's. For example,

$$\begin{aligned}
\binom{r+s}{r,s}^{(\ell)} &= \sum_{\lambda \in P(r,s)} d_{\lambda_1+1} d_{\lambda_2+1} \cdots d_{\lambda_1^*} d_{\lambda_2^*} \cdots \\
&= \sum_{\lambda \in P(r,s)} (-1)^{\mathrm{len}(\lambda^*)} a_{\lambda_1+1} a_{\lambda_2+1} \cdots a_{\lambda_1^*-1} a_{\lambda_2^*-1} \cdots \\
&= 2^{-(r+s)} \sum_{\lambda \in P(r,s)} p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_s} p_{\lambda_1^*} p_{\lambda_2^*} \cdots p_{\lambda_r^*},
\end{aligned}$$

where subscripts in the first two sums range over the nonzero λ_i and λ_j^*. We can interpret these formulas combinatorially by filling the rows of λ and columns of λ^* with suitable words counted by the a's, d's, or p's (see Theorem 13). For example, the first equation above tells us that $\binom{r+s}{r,s}^{(\ell)}$ counts objects constructed as follows: start with a partition $\lambda \in P(r,s)$; fill each nonzero row of λ (from left to right) with a word in $W^\dagger_{\lambda_i}$; fill each nonzero column of λ^* (from bottom to top, say) with a word in $0W^\dagger_{\lambda_j^*-1}$. (The initial zero is added so that every box in the $s \times r$ rectangle is filled. One could also use the languages W'_n instead of $W^\dagger_n$.) For example, Figure 3 illustrates one object counted by $\binom{9}{5,4}^{(3)}$.

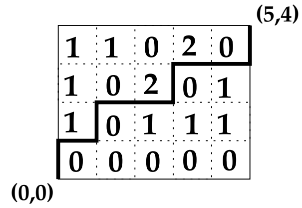

Figure 3: An object counted by $\binom{9}{5,4}^{(3)}$.

As in Example 21, we obtain a simple "semi-combinatorial" interpretation of the ℓ-nomial coefficient by taking $g_1(r,s) = u^s$ and $g_2(r,s) = v^r$:

$$\binom{r+s}{r,s}^{(\ell)} = \sum_{\lambda \in P(r,s)} u^{|\lambda|} v^{|\lambda^*|}. \tag{6}$$

This formula expresses the ℓ-binomial coefficient as a sum over *weighted* partitions in an $s \times r$ box, where the real-valued weight is obtained by filling each box of λ with $u = u_\ell$, filling each box of λ^* with $v = v_\ell$, and multiplying together the numbers in all the boxes. Here is one application of this formula.

Theorem 22. *For all $n \geq 0$ and $\ell \geq 2$,*

$$\sum_{k=0}^{\infty} \binom{n+k}{n,k}^{(\ell)} z^k = \prod_{i=0}^{n} \frac{1}{(1 - zu^i v^{n-i})}.$$

Proof. The left side counts partitions $\lambda \in \bigcup_{k \geq 0} P(n,k)$, weighted by $z^k u^{|\lambda|} v^{|\lambda^*|}$. The right side builds all such partitions by choosing, for each $i \leq n$, the number of parts of λ equal to i. Each such part contributes $zu^i v^{n-i}$ to the weight, and the geometric series factor $(1 - zu^i v^{n-i})^{-1}$ allows any number of such parts to be chosen. □

3.3. Comparison to q-Binomial Coefficients

There is a close connection between ℓ-nomial coefficients and q-binomial coefficients (see Example 1(b)). To describe this, we introduce homogenized versions of the q-binomial coefficients. Let x and y be variables, and define

$$\begin{bmatrix} r+s \\ r,s \end{bmatrix}_{x,y} = \binom{r+s}{r,s}^{(f)}$$

where

$$f(n) = x^{n-1} + x^{n-2}y^1 + x^{n-3}y^2 + \cdots + y^{n-1} = \frac{x^n - y^n}{x - y} \ (n \geq 1).$$

Note that $\left[{r+s \atop r,s}\right]_q = \left[{r+s \atop r,s}\right]_{q,1} = \left[{r+s \atop r,s}\right]_{1,q}$. We have

$$f(r+s) = y^s f(r) + x^r f(s) = x^s f(r) + y^r f(s) \qquad (r, s > 0).$$

Using Theorem 17 and Theorem 20, we deduce

$$\begin{aligned}
\begin{bmatrix} r+s \\ r,s \end{bmatrix}_{x,y} &= y^s \begin{bmatrix} r+s-1 \\ r-1,s \end{bmatrix}_{x,y} + x^r \begin{bmatrix} r+s-1 \\ r,s-1 \end{bmatrix}_{x,y} \\
&= x^s \begin{bmatrix} r+s-1 \\ r-1,s \end{bmatrix}_{x,y} + y^r \begin{bmatrix} r+s-1 \\ r,s-1 \end{bmatrix}_{x,y} \\
&= \sum_{\lambda \in P(r,s)} x^{|\lambda|} y^{|\lambda^*|} \in \mathbb{N}[x,y].
\end{aligned}$$

Comparing these formulas to those in the last subsection, we see that

$$\binom{r+s}{r,s}^{(\ell)} = \left.\begin{bmatrix} r+s \\ r,s \end{bmatrix}_{x,y}\right|_{x=u,y=v}. \tag{7}$$

Thus, the ℓ-nomial coefficients are just a particular specialization of the (homogenized) q-binomial coefficients. We can therefore use known q-binomial identities to deduce identities (involving u and v) for ℓ-nomial coefficients. The proof of Theorem 23 below (which is an ℓ-analogue of the Chu-Vandermonde identity) illustrates how this technique can be used to derive *integer* identities.

We should point out that many combinatorial properties of the ℓ-nomial coefficients are not always evident from the description in terms of q-binomial coefficients, since u and v are real numbers. For instance, the integrality of the ℓ-nomial coefficients is not immediately clear from (7). Nevertheless, we can use that equation to give another proof of this integrality, as follows. We see from the definitions that $\left[{r+s \atop r,s}\right]_{x,y}$ is a *symmetric* polynomial in the variables x and y with integer coefficients. Therefore, by the fundamental theorem of symmetric polynomials [29], there exists a two-variable polynomial g with integer coefficients such that $\left[{r+s \atop r,s}\right]_{x,y} = g(e_1(x,y), e_2(x,y))$, where $e_1(x,y) = x + y$ and $e_2(x,y) = xy$ are the elementary symmetric polynomials in x and y. Specializing $x = u_\ell$ and $y = v_\ell$, we know that $e_1(x,y) = \ell$ and $e_2(x,y) = 1$ (Definition 5). Thus, $\binom{r+s}{r,s}^{(\ell)} = g(\ell, 1)$ is certainly an integer.

Theorem 23. *For all $C, D, E \in \mathbb{N}$,*

$$\binom{C+D+E+1}{C+D+1,E}^{(\ell)} = \sum_{i=0}^{E} \binom{D+i}{D,i}^{(\ell)} \binom{C+E-i}{C,E-i}^{(\ell)} \frac{p_{i(C+1)-(E-i)(D+1)}}{2}.$$

Proof. Classify lattice paths from $(0,0)$ to $(C+D+1,E)$ based on the height i of the east step going from $x=D$ to $x=D+1$. Weighting area cells above the path by x and area cells below by y, we get

$$\begin{bmatrix} C+D+E+1 \\ C+D+1,E \end{bmatrix}_{x,y} = \sum_{i=0}^{E} \begin{bmatrix} D+i \\ D,i \end{bmatrix}_{x,y} \begin{bmatrix} C+E-i \\ C,E-i \end{bmatrix}_{x,y} y^{(C+1)i} x^{(D+1)(E-i)}.$$

Since the left side is symmetric in x and y, we also have

$$\begin{bmatrix} C+D+E+1 \\ C+D+1,E \end{bmatrix}_{x,y} = \sum_{i=0}^{E} \begin{bmatrix} D+i \\ D,i \end{bmatrix}_{x,y} \begin{bmatrix} C+E-i \\ C,E-i \end{bmatrix}_{x,y} x^{(C+1)i} y^{(D+1)(E-i)}.$$

Add these two identities, divide by 2, and set $x=u$, $y=v$. The proof is completed by observing that $u^n v^m + u^m v^n = (uv)^n (v^{m-n} + u^{m-n}) = p_{m-n} = p_{n-m}$. □

3.4. Three-Term Recursion

Theorem 24. *For all $\ell \geq 2$ and all $r, s \in \mathbb{N}$ with $r+s \geq 2$,*

$$\binom{r+s}{r,s}^{(\ell)} = \binom{r+s-2}{r-2,s}^{(\ell)} + p_{r+s-1} \binom{r+s-2}{r-1,s-1}^{(\ell)} + \binom{r+s-2}{r,s-2}^{(\ell)}.$$

The initial conditions are $\binom{0}{0,0}^{(\ell)} = \binom{1}{1,0}^{(\ell)} = \binom{1}{0,1}^{(\ell)} = 1$ and $\binom{r+s}{r,s}^{(\ell)} = 0$ if $r<0$ or $s<0$.

Proof. If $r \geq 2$ and $s \geq 2$, we can use the recursions in Corollary 18 to compute

$$\begin{aligned} \binom{r+s}{r,s}^{(\ell)} &= u^s \binom{r+s-1}{r-1,s}^{(\ell)} + v^r \binom{r+s-1}{r,s-1}^{(\ell)} \\ &= u^s \left[v^s \binom{r+s-2}{r-2,s}^{(\ell)} + u^{r-1} \binom{r+s-2}{r-1,s-1}^{(\ell)} \right] \\ &\quad + v^r \left[v^{s-1} \binom{r+s-2}{r-1,s-1}^{(\ell)} + u^r \binom{r+s-2}{r,s-2}^{(\ell)} \right]. \end{aligned}$$

Using $u^s v^s = v^r u^r = 1$ and $u^{s+r-1} + v^{r+s-1} = p_{r+s-1}$ completes the proof in this case. Now, if $r = 1$ and $s \geq 1$, the desired recursion becomes $a^{(\ell)}_{s+1} = 0 + p^{(\ell)}_s + a^{(\ell)}_{s-1}$, which is true. Similarly, the result holds in the cases: $s = 1$ and $r \geq 1$; $s = 0$ and $r \geq 2$; $r = 0$ and $s \geq 2$. $\square$

Theorem 25. *For all $\ell \geq 2$ and $r, s \in \mathbb{N}$, $\binom{r+s}{r,s}^{(\ell)}$ counts annotated lattice paths satisfying the following conditions:*

(a) *the path starts at $(0,0)$ or $(1,0)$ or $(0,1)$ and ends at (r,s);*

(b) *each step in the path is a long east step from $(x-2, y)$ to (x, y), or a long north step from $(x, y-2)$ to (x, y), or a diagonal step from $(x-1, y-1)$ to (x, y);*

(c) *each diagonal step, say ending at (x, y), is labeled with a word in W''_{x+y-1}.*

Proof. The labeled paths just described evidently satisfy the three-term recursion and initial conditions in Theorem 24, as we see by removing the last step in any such path ending at (r, s). $\square$

Another way to phrase condition (a) is to say that all the paths start at the origin, but if $r + s$ is odd, the first step of the path must be a "short" (length 1) north or east step.

Example 26. Recall (Definition 12) that W''_{x+y-1} consists of ℓ-admissible words of length $x + y - 1$ that do not begin with two zeroes. Given a path with a diagonal step ending at (x, y), there are $x + y - 1$ lattice squares in the "hook" consisting of the square occupied by the diagonal step and all squares due west or due south of this square. Hence, we can enter the letters of the word labeling this diagonal step in these squares, starting at the y-axis and moving east and south to the x-axis. For example, Figure 4 illustrates an object counted by $\binom{15}{9,6}^{(3)}$ according to this model.

When $\ell = 2$, we have the following bijection from the objects in Theorem 25 to ordinary lattice paths from $(0,0)$ to (r,s). As $p^{(2)}_n = 2$ for all n, there are two possible labels for each diagonal step (no matter where it ends). The label chosen for a given diagonal step tells us whether to convert this step to NE or EN (where these north and east steps have length 1).

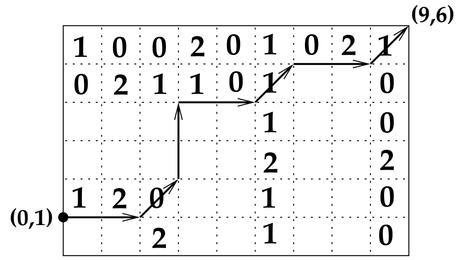

Figure 4: An object counted by $\binom{15}{9,6}^{(3)}$.

3.5. Analogues of the Binomial Theorem

The usual binomial theorem states that $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$. We now derive ℓ-analogues of this formula.

Theorem 27. *For all $\ell \geq 2$ and $n \geq 1$,*

$$\sum_{r=0}^{n} \binom{n}{r}^{(\ell)} z^r = \begin{cases} (1+p_1 z+z^2)(1+p_3 z+z^2)\cdots(1+p_{n-1}z+z^2) \\ \hfill \textit{for even } n; \\ (1+z)(1+p_2 z+z^2)(1+p_4 z+z^2)\cdots(1+p_{n-1}z+z^2) \\ \hfill \textit{for odd } n. \end{cases}$$

Proof. For $r, s \geq 0$, let $f(r,s)$ be the coefficient of z^r when we expand the products on the right side of the theorem taking $n = r+s$. Using the distributive law on the rightmost factor, one sees immediately that the quantities $f(r,s)$ satisfy the recursion and initial conditions in Theorem 24. Hence $f(r,s) = \binom{r+s}{r,s}^{(\ell)}$. Alternatively, the formulas here can be deduced easily from Theorem 25 (weight paths ending at (r,s) by z^r). $\square$

We can now deduce ℓ-analogues of the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Corollary 28.

$$\sum_{r=0}^{n} \binom{n}{r}^{(\ell)} = \left\{ \begin{array}{ll} (p_1+2)(p_3+2)(p_5+2)\cdots(p_{n-1}+2) & \textit{for even } n \\ 2(p_2+2)(p_4+2)(p_6+2)\cdots(p_{n-1}+2) & \textit{for odd } n \end{array} \right\}$$

$$= \prod_{i=0}^{n-1} p_{i-(n-1)/2}.$$

Proof. The first equality follows from the theorem by letting $z = 1$. The second equality then follows from (3). $\square$

Example 29. When $\ell = 3$ and $n = 5$, the corollary says

$$1 + 55 + 385 + 385 + 55 + 1 = 882 = 2(7 + 2)(47 + 2) = 7 \cdot 3 \cdot 2 \cdot 3 \cdot 7.$$

For $\ell = 3$ and $n = 6$, we get

$$\begin{aligned} &1 + 144 + 2640 + 6930 + 2640 + 144 + 1 = 12,500 \\ &\qquad = (3 + 2)(18 + 2)(123 + 2) = \sqrt{125}\sqrt{20}\sqrt{5}\sqrt{5}\sqrt{20}\sqrt{125}. \end{aligned}$$

We can also write the ℓ-nomial theorem in terms of u and v.

Corollary 30. *For all $\ell \geq 2$ and $n \geq 1$,*

$$\sum_{r=0}^{n} \binom{n}{r}^{(\ell)} z^r = \prod_{i=0}^{n-1} (u^{i-(n-1)/2} + v^{i-(n-1)/2} z).$$

Proof. Setting $j = i - (n-1)/2$, we can rewrite the right side as $\prod_{j=-(n-1)/2}^{(n-1)/2} (u^j + v^j z)$. For $0 < j \leq (n-1)/2$, the product of the factor indexed by j and the factor indexed by $-j$ is

$$(u^j + v^j z)(u^{-j} + v^{-j} z) = 1 + ((v/u)^j + (u/v)^j)z + z^2 = 1 + p_{2j} z + z^2.$$

Also, if n is odd, the factor indexed by $j = 0$ is $1 + z$. The result therefore follows from Theorem 27. $\square$

3.6. Probabilistic Interpretation of ℓ-nomial Coefficients

Next we give a probabilistic interpretation of ℓ-nomial coefficients. The starting point is the following formula, which expresses $\binom{n}{r}^{(\ell)}$ as a sum of weighted r-element subsets of an n-element set.

Theorem 31. *We have*

$$\binom{n}{r}^{(\ell)} = \sum_{\substack{S \subseteq \{0,1,\ldots,n-1\} \\ |S| = r}} \prod_{i \in S} v_\ell^{i-(n-1)/2} \prod_{i \in \{0,1,\ldots,n-1\} - S} u_\ell^{i-(n-1)/2}.$$

Proof. In Corollary 30, expand the product on the right side using the distributive law, and then extract the coefficient of z^r on both sides. $\square$

Definition 32. For any integer $\ell \geq 2$ and real number r, let $C_r^{(\ell)}$ denote a *weighted coin* that comes up tails with probability u^r/p_r and heads with probability v^r/p_r. (Note that these probabilities reduce to 1/2 for all r when $\ell = 2$.)

Theorem 33. *For $n \geq r \geq 0$, the probability of getting exactly r heads when tossing the n coins*

$$C^{(\ell)}_{-(n-1)/2}, \quad C^{(\ell)}_{1-(n-1)/2}, \quad C^{(\ell)}_{2-(n-1)/2}, \quad \ldots, \quad C^{(\ell)}_{(n-1)/2}$$

is $\binom{n}{r}^{(\ell)}/D$, where $D = \prod_{i=0}^{n-1} p^{(\ell)}_{i-(n-1)/2}$.

Proof. In n tosses, the probability of getting r heads in positions specified by an r-element subset $S \subseteq \{0, 1, \ldots, n-1\}$ is

$$D^{-1} \prod_{i \in S} v^{i-(n-1)/2} \prod_{i \in \{0,1,\ldots,n-1\}-S} u^{i-(n-1)/2} .$$

Summing over all possible S and using Theorem 31 gives the result. $\square$

Example 34. Take $n = \ell = 3$. We are tossing three coins, which come up heads with probability $(3+\sqrt{5})/6$, $1/2$, and $(3-\sqrt{5})/6$, respectively. The denominator in the theorem is $D = p_{-1}p_0p_1 = 3 \cdot 2 \cdot 3 = 18 = \sum_{r=0}^{3} \binom{3}{r}^{(3)}$. We have $P(0 \text{ heads}) = P(3 \text{ heads}) = 1/18$ and $P(1 \text{ head}) = P(2 \text{ heads}) = 8/18 = 4/9$.

4. q-Analogues of ℓ-nomial Coefficients

Recall that

$$[\,n\,]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1} \in \mathbb{N}[q].$$

Definition 35. For $n \geq k \geq 0$, the *q-ℓ-nomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(\ell)}_q = \frac{[\,a_n\,]_q[\,a_{n-1}\,]_q \cdots [\,a_{n-k+1}\,]_q}{[\,a_k\,]_q[\,a_{k-1}\,]_q \cdots [\,a_1\,]_q}.$$

Observe that $\begin{bmatrix} n \\ k \end{bmatrix}^{(\ell)}_q$ is an $(f \circ f')$-binomial coefficient, where $f'(n) = a_n^{(\ell)}$ and $f(m) = [m]_q$. We can therefore apply the following general result to obtain properties of the q-ℓ-nomial coefficients.

Theorem 36. *Assume f, f', g_1, g_2, g_1', g_2' and h are such that*

$$\begin{aligned} f(r+s) &= g_1(r,s)f(r) + g_2(r,s)f(s); \\ f'(r+s) &= g_1'(r,s)f'(r) + g_2'(r,s)f'(s); \\ f(rs) &= h(r,s)f(s) \end{aligned}$$

for all r, s. Setting $f^* = f \circ f'$,

$$g_1^*(r,s) = g_1(g_1'(r,s)f'(r), g_2'(r,s)f'(s))h(g_1'(r,s), f'(r)),$$

$$g_2^*(r,s) = g_2(g_1'(r,s)f'(r), g_2'(r,s)f'(s))h(g_2'(r,s), f'(s)),$$

we then have

$$f^*(r+s) = g_1^*(r,s)f^*(r) + g_2^*(r,s)f^*(s).$$

Proof. We compute

$$\begin{aligned} f^*(r+s) &= f(f'(r+s)) = f(g_1'(r,s)f'(r) + g_2'(r,s)f'(s)) \\ &= g_1(g_1'(r,s)f'(r), g_2'(r,s)f'(s))f(g_1'(r,s)f'(r)) \\ &\quad + g_2(g_1'(r,s)f'(r), g_2'(r,s)f'(s))f(g_2'(r,s)f'(s)) \\ &= g_1(g_1'(r,s)f'(r), g_2'(r,s)f'(s))h(g_1'(r,s), f'(r))f(f'(r)) \\ &\quad + g_2(g_1'(r,s)f'(r), g_2'(r,s)f'(s))h(g_2'(r,s), f'(s))f(f'(s)) \\ &= g_1^*(r,s)f^*(r) + g_2^*(r,s)f^*(s). \end{aligned}$$

□

This theorem immediately yields a profusion of recurrences satisfied by the q-ℓ-nomial coefficients. We state one of these below, which is a q-analogue of the first recurrence in Corollary 18.

Corollary 37. *For all $\ell \geq 2$ and all $r, s > 0$,*

$$\left[\begin{array}{c} r+s \\ r,s \end{array}\right]_q^{(\ell)} = [d_s]_{q^{a_r}} \left[\begin{array}{c} r+s-1 \\ r-1,s \end{array}\right]_q^{(\ell)} + q^{d_s a_r}[d_{r+1}]_{q^{a_s}} \left[\begin{array}{c} r+s-1 \\ r,s-1 \end{array}\right]_q^{(\ell)}.$$

The initial conditions are $\left[\begin{array}{c} r \\ r,0 \end{array}\right]_q^{(\ell)} = \left[\begin{array}{c} s \\ 0,s \end{array}\right]_q^{(\ell)} = 1$. In particular, each q-ℓ-nomial coefficient is a polynomial with nonnegative integer coefficients.

Proof. Note that $[r+s]_q = [r]_q + q^r[s]_q$ and $[rs]_q = [r]_{q^s}[s]_q$ for $r, s \in \mathbb{N}^+$. So, Theorem 36 is applicable to the functions given by $f(m) = [m]_q$, $f'(n) = a_n^{(\ell)}$,

$g_1(r,s) = 1$, $g_2(r,s) = q^r$, $g'_1(r,s) = d_s$, $g'_2(r,s) = d_{r+1}$, and $h(r,s) = [r]_{q^s}$. The functions in the conclusion of Theorem 36 are given by $f^*(n) = [a_n]_q$, $g^*_1(r,s) = [d_s]_{q^{a_r}}$, and $g^*_2(r,s) = q^{d_s a_r}[d_{r+1}]_{q^{a_s}}$. The recurrence in the corollary now follows from Theorem 17. Polynomiality of the q-ℓ-nomial coefficients follows from the recurrence by induction on $r+s$. □

The recurrence in the corollary reduces to the usual recurrence for the q-binomial coefficients when $\ell = 2$, since in that case, $a_j = j$ and $d_j = 1$ for all $j > 0$. Other recurrences arise by using $g_1(r,s) = q^s$ and $g_2(r,s) = 1$, or by using other choices of g'_1 and g'_2. For example, taking $f(r) = (1-q^r)/(1-q)$ for *real* r, $g_1(r,s) = 1$, $g_2(r,s) = q^r$, $h(r,s) = (1-q^{rs})/(1-q^s)$, $f'(n) = a^{(\ell)}_n$, $g'_1(r,s) = u^s$, and $g'_2(r,s) = v^r$, we obtain (for suitable real values of q)

$$\left[\begin{matrix} r+s \\ r,s \end{matrix}\right]^{(\ell)}_q = \frac{1-q^{u^s a_r}}{1-q^{a_r}}\left[\begin{matrix} r+s-1 \\ r-1,s \end{matrix}\right]^{(\ell)}_q + q^{u^s a_r}\frac{1-q^{v^r a_s}}{1-q^{a_s}}\left[\begin{matrix} r+s-1 \\ r,s-1 \end{matrix}\right]^{(\ell)}_q. \tag{8}$$

Just as before, Theorem 20 lets us convert recurrences for q-ℓ-nomial coefficients into fermionic formulas. For instance, here is one q-analogue of the first formula from Example 21.

Theorem 38. *For all $r,s \in \mathbb{N}^+$,*

$$\left[\begin{matrix} r+s \\ r,s \end{matrix}\right]^{(\ell)}_q = \sum_{\lambda \in P(r,s)} \prod_{i:\lambda_i>0} [d_{\lambda_i+1}]_{q^{a_{s+1-i}}}\, q^{d_{s+1-i}a_{\lambda_i}} \prod_{j:\lambda^*_j>0} [d_{\lambda^*_j}]_{q^{a_{r+1-j}}}.$$

When $\ell = 2$, this again reduces to the usual interpretation of the q-binomial coefficients as the sum over all partitions $\lambda \in P(r,s)$ of $q^{|\lambda|}$, since $a_j = j$ and $d_j = 1$ for all $j > 0$.

We can obtain a combinatorial interpretation for q-ℓ-nomial coefficients by assigning a suitable q-weight to each of the filled partitions described in Example 21. Recall that a filled partition consists of a partition $\lambda \in P(r,s)$, together with a word $w_i \in W^\dagger_{\lambda_i}$ filling each nonzero part λ_i, and a word $w^*_j \in 0W^\dagger_{\lambda^*_j-1}$ filling each nonzero part λ^*_j. Call the resulting filling $F = (\{w_i\},\{w^*_j\})$ of the $s \times r$ box λ-*admissible*.

We attach q-weights to λ and F as follows. The *shape weight* of λ is

$$\sigma(\lambda) = \sum_{i=1}^{s} d_{s+1-i}a_{\lambda_i}.$$

The *fill weight* of a λ-admissible filling F is

$$\rho(\lambda, F) = \sum_{i=1}^{s} a_{s+1-i} N(w_i) + \sum_{j=1}^{r} a_{r+1-j} N(w_j^*).$$

Recall from the proof of Theorem 13 that the map $N : W_{n-1}^{\dagger} \to \{0, 1, \ldots, d_n - 1\}$ given by $N(w_1 \cdots w_{n-1}) = \sum_{i=0}^{n-1} w_i a_{n-i}$ is a bijection. It follows that

$$\sum_{w \in W_{n-1}^{\dagger}} q^{N(w)} = [d_n]_q.$$

Using this remark and the preceding definitions, we deduce the following combinatorial version of Theorem 38.

Theorem 39. *For all $r, s \in \mathbb{N}^+$ and $\ell \geq 2$,*

$$\left[\begin{array}{c} r+s \\ r,\ s \end{array} \right]_q^{(\ell)} = \sum_{(\lambda, F)} q^{\sigma(\lambda) + \rho(\lambda, F)},$$

where $\lambda \in P(r, s)$ and F is a λ-admissible filling of the cells of the $s \times r$ box using letters from the alphabet $\{0, 1, \ldots, \ell - 1\}$.

We can get variations on this theorem by starting with other two-term recurrences for the ℓ-nomial coefficient.

Example 40. Let $\ell = 3$, $r = 5$, and $s = 4$. The object (λ, F) shown in Figure 3 has shape weight

$$\sigma(5, 3, 1, 0) = d_4 a_5 + d_3 a_3 + d_2 a_1 + d_1 a_0 = 757$$

and fill weight

$$\begin{aligned} \rho(\lambda, F) \quad = \quad & a_4 N(11020) + a_3 N(102) + a_2 N(1) \\ & + a_5 N(011) + a_4 N(010) + a_3 N(01) + a_2 N(00) + a_1 N(0) = 2096. \end{aligned}$$

So this object contributes the term q^{2853} to $\left[\begin{array}{c} 9 \\ 5, 4 \end{array} \right]_q^{(\ell)}$.

5. Directions for Future Work

This section lists some questions and research topics that are related to ℓ-nomial coefficients. We hope that the present work may help shed light on some of these problems by providing a combinatorial framework for manipulating ℓ-nomial coefficients.

- *Hypergeometric identities.* We have given several ℓ-analogues of well-known identities such as the binomial theorem and the Chu-Vandermonde formula. Are there natural ℓ-analogues, or q-ℓ-analogues, for other hypergeometric identities? Can a WZ-type method [32] be developed in this setting?

- *Lattice point enumeration.* It was shown in [11, 12] that binomial coefficients occur naturally in the enumeration of lattice points $(\lambda_1, \ldots, \lambda_k)$ satisfying

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n+1-k} \geq 0.$$

(These are truncated version of the lecture hall partitions of [7, 8]). It is known that ℓ-nomial coefficients arise in a similar way when enumerating lattice points $(\lambda_1, \ldots, \lambda_k)$ satisfying

$$\frac{\lambda_1}{a_n} \geq \frac{\lambda_2}{a_{n-1}} \geq \ldots \geq \frac{\lambda_k}{a_{n+1-k}} \geq 0,$$

but the combinatorial or geometric significance of this is not yet understood.

- *Partitions and q-series.* In [26], the second author and Yee gave a bijective proof of the ℓ-Euler theorem discovered by Bousquet-Mélou and Eriksson [8], which relied heavily on the combinatorics of ℓ-sequences. There are several q-series identities related to Euler's theorem, such as Lebesgue's identity [2, 6], the Rogers-Fine identity [3, 33], and Cauchy's identity [17, 33]. Are there ℓ-analogues or q-ℓ-analogues of any of these? Do any of the classical partition identities extend to the filled partitions that are enumerated by ℓ-nomial coefficients? In particular, it was shown in [26] that the partitions involved in the ℓ-Euler theorem have a representation as a filling of the shape $(n, n-1, \ldots, 1)$ with ℓ-admissible words such that the rows form a lexicographically decreasing sequence. We would like a better understanding of how this result is related to our combinatorial interpretations of ℓ-nomial coefficients.

- *Extension to real ℓ.* We can view $a_n = a_n(\ell)$ as a polynomial in ℓ and therefore consider all real values of ℓ as arguments. In fact, $a_n(\ell) = U_n(\ell/2)$, where $U_n(x)$ is the n'th Chebyshev polynomial of the second kind, defined recursively by $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$, with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. We have

$$\lim_{\ell \to 2^+} \binom{n}{k}^{(\ell)} = \binom{n}{k}.$$

This suggests using ℓ-nomial coefficients and continuity arguments to obtain information about the binomial coefficients.

Motivated by work in [25], Stanton [28] considers partitions whose parts are polynomials. We can connect this to our work here if we realize a polynomial part $f_n(x)$ by a word of length n from a language counted by $f_n(x)$. It should be worthwhile to investigate interpretations of the (q, t) binomial coefficients in [25] in terms of filled partitions.

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