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ANALOGUES OF JACOBI'S TWO-SQUARE THEOREM: AN INFORMAL ACCOUNT

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Abstract

Jacobi's two-square theorem states that the number of representations of a positive integer k as a sum of two squares, counting order and sign, is 4 times the surplus of positive divisors of k congruent to 1 modulo 4 over those congruent to 3 modulo 4. In this paper we give numerous identities, each of which yields an analogue of Jacobi's result. Our identities are drawn from a much larger list, and involve polygonal numbers. The formula for the n^{th} $k-$gonal number is

$$F_k = F_k(n) = n\left((k-2)n - (k-4)\right)/2.$$

1. Introduction

Let f and g be functions from the integers to the non-negative integers, and suppose that

$$\left(\sum_{n=-\infty}^{\infty} q^{f(n)}\right)\left(\sum_{n=-\infty}^{\infty} q^{g(n)}\right) = \sum_{n=0}^{\infty} a_n q^n. \tag{1}$$

Then the number of solutions of the diophantine equation $f(m) + g(n) = k, k \geq 0$, is a_k. Here, as is implied by the limits on the left of (1), m and n can be positive, negative, or zero, and two solutions (m_1, n_1) and (m_2, n_2) are taken to be distinct when $m_1 \neq m_2$ and $n_1 \neq n_2$. In this paper we take $|q| < 1$ and choose f and g so that each sum on the left of (1) converges.

As an example let $f = g = n^2$. Then

$$\begin{aligned}\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right) &= 1 + 4\sum_{n=0}^{\infty}\left(\frac{q^{4n+1}}{1-q^{4n+1}} - \frac{q^{4n+3}}{1-q^{4n+3}}\right) \\ &= 1 + 4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(q^{(4n+1)(m+1)} - q^{(4n+3)(m+1)}\right). \end{aligned} \tag{2}$$

Now let $d_{i,j}(k)$ denote the number of positive divisors of k that are congruent to i mod j. Then (2) yields the following classical result of Jacobi [4, ch 9]:

> *The number of representations of a positive integer k as a sum of two squares, counting order and sign, is* $4\,(d_{1,4}(k) - d_{3,4}(k))$.

In [13] we give a large number of analogues of (2) that involve *polygonal* numbers. The formula for the n^{th} $k-$gonal number is

$$F_k = F_k(n) = \frac{n\,((k-2)n - (k-4))}{2}. \tag{3}$$

Thus, the n^{th} triangular and square numbers are given by $F_3 = n(n+1)/2$, and $F_4 = n^2$, respectively. For information on polygonal numbers see, for example, [12]. In [13] the scope is limited to the polygonal numbers F_k, $3 \le k \le 12$. Furthermore, although in (3) n is usually positive, except for F_3, we maintain the analogy with Jacobi's result by allowing the domain of F_k to be the set of *all* integers.

Let $G_k(q)$ denote the generating function of F_k. Thus,

$$G_3(q) = \sum_{n=0}^{n=\infty} q^{n(n+1)/2}, \quad G_4(q) = \sum_{n=-\infty}^{n=\infty} q^{n^2}$$

denote the generating functions of the triangular numbers and the squares, respectively. In present day usage, these generating functions are usually denoted by $\psi(q) = \sum_{n=0}^{n=\infty} q^{n(n+1)/2}$ and $\phi(q) = \sum_{n=-\infty}^{n=\infty} q^{n^2}$, respectively, which is the notation used by Ramanujan.

There have been many proofs of (2). Three relatively recent papers are worthy of mention. In [2] and [3] the authors give a proof with the use of Ramanujan's ${}_1\psi_1$ formula, while Hirschhorn [9] uses Jacobi's triple product identity.

For detailed information on the broader topic of sums of squares in relation to the older literature, see [7, pp. 231-257], [8, ch. 2], and the forthcoming paper of Cooper [5]. Dr. Cooper's paper investigates the number of representations of a positive integer by the quadratic form

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \lambda_4 y_4^2,$$

where y_1, y_2, y_3, and y_4 are odd, positive integers, for the cases $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) =$ (1, 1, 1, 3), (1, 3, 3, 3), (1, 2, 2, 3), (1, 3, 6, 6), (1, 3, 4, 4), (1, 1, 2, 6), and (1, 3, 12, 12). This paper also contains a comprehensive history of representations by similar forms that dates back to Eisenstein and Liouville.

Even without a formal proof, we can easily convince ourselves of the validity of (2). With the aid of a computer algebra system we simply evaluate, say,

$$\left(\sum_{n=-35}^{35} q^{n^2}\right)\left(\sum_{n=-35}^{35} q^{n^2}\right) - 1 - 4\sum_{n=0}^{250}\sum_{m=0}^{1000}\left(q^{(4n+1)(m+1)} - q^{(4n+3)(m+1)}\right).$$

This verifies that $G_4(q)^2$ and the right side of (2) agree up to q^{1000}. We performed similar checks on all the identities in this paper, as well as those in [13]. In fact, in each case, we checked that both sides, as expansions in powers of q, matched up to powers of q^{4m}, where m is the modulus in question. This was an arbitrary choice that seemed appropriate. For instance, in (9) we checked that both sides were equal up to powers of q^{160}, and in (13) we checked that both sides were equal up to powers of q^{352}. In addition, we checked for symmetry in each of our identities. We checked our formulas meticulously and repeatedly over a period of months, taking this task very seriously, since the presence of errors serves to erode credibility. The strongest statement that we can reasonably make in this regard is that we are as sure as we can be that the formulas are error free.

There are also well-known identities, analogous to (2), for $G_4(q)G_4(q^2)$, $G_4(q)G_4(q^3)$, and $G_4(q)G_4(q^7)$, that are attributed to Dirichlet and Lorenz, Lorenz, and Ramanujan, respectively. Here, and in [13], we list only identities that, to the best of our knowledge, are new and are not consequences of the identities for $G_4(q)G_4(q^k)$, $k = 1, 2, 3, \text{or}, 7$. Without this restriction our lists would have been considerably longer. For instance, identities that we have discovered, but have not stated, are identities for $G_5(q)\,G_5(q)$, $G_5(q)\,G_5\left(q^2\right)$, $G_5(q)\,G_5\left(q^7\right)$, $G_7(q)\,G_7(q)$, $G_8(q)\,G_8(q)$, and $G_{12}(q)\,G_{12}(q)$, to indicate just a few. For further instances of such identities (ie, those that are provable with the use of the four classical identities) see [10] and [11].

Our aim, in the present paper, is to give an abridged version of [13]. Specifically, we present only those identities in [13] that we have called *homogeneous*, meaning that in each such identity the generating functions involve polygonal numbers of the same type. The only homogeneous identities that we have managed to discover (apart from those mentioned in the paragraph above) involve triangular numbers, pentagonal numbers, and heptagonal numbers.

Here, and in [13], our aim has been to put our identities on display in the hope that interested readers may wish to supply proofs. We expect that to prove many of the identities in our lists will call for genuine skill and innovation. To see this, one need only examine the various proofs of (2).

At the time of writing, we see scope for further work, and so we expect to enlarge these lists. Indeed, on many occasions during the process of discovery we felt that the work had reached a plateau. However, on each of these occasions we gained new insights that enabled us to continue. The process is, seemingly, never ending.

Dr. Michael Hirschhorn has taken an interest in this work from the very beginning. At the time of writing he has informed us that he has managed to prove (6) and (7).

In Section 2 we give a worked example to demonstrate how the number-theoretic consequence of one of our identities can be obtained. In Section 3 give some hints relating to our methods of discovery. In Sections 4, 5, and 6 we present our homogeneous identities for the triangular numbers, the pentagonal numbers, and the heptagonal numbers, respectively.

2. A Worked Example

In Sections 4-6 each of (6)-(26) yields a representation result that is analogous to Jacobi's classical result. As an example of the manipulations required in deriving these representation results, we consider identity (6) and denote the right side by $H(q)$. Then

$$\begin{aligned} q^3H(q^4) &= \sum_{n=0}^{\infty}\left(\frac{q^{3(4n+1)}+q^{7(4n+1)}}{1-q^{20(4n+1)}}-\frac{q^{13(4n+3)}+q^{17(4n+3)}}{1-q^{20(4n+3)}}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(q^{(4n+1)(20m+3)}+q^{(4n+1)(20m+7)} \\ &\qquad -q^{(4n+3)(20m+13)}-q^{(4n+3)(20m+17)}\Big). \end{aligned}$$

Each power of q in this double sum is congruent to 3 modulo 4. Denote the coefficient of q^k in $H(q)$ by $C\left[H(q),q^k\right]$. Then

$$\begin{aligned} C\left[H(q),q^k\right] &= C\left[q^3H(q^4),q^{4k+3}\right] \\ &= d_{3,20}(4k+3)+d_{7,20}(4k+3)-d_{13,20}(4k+3)-d_{17,20}(4k+3). \end{aligned}$$

Our conclusion can be stated thus:

> *The number of representations of a positive integer k as a triangular number plus five times a triangular number (in this order) is*
>
> $$d_{3,20}(4k+3)+d_{7,20}(4k+3)-d_{13,20}(4k+3)-d_{17,20}(4k+3).$$

The interested reader can now supply the number-theoretic consequence of any of the remaining identities.

3. Hints Regarding the Methods of Discovery

In this section we give the reader some hints regarding our methods of discovery. Our methods were developed and refined over the four years during which this research took place. Starting very modestly, we developed computer programs to execute the associated algebraic calculations.

To illustrate one of our methods of discovery, we outline how we discovered (6). It is well known (for instance, see [10]) that

$$G_3\left(q\right)G_3\left(q\right) = \sum_{n\geq 0}\left(d_{1,4}(4n+1) - d_{3,4}(4n+1)\right)q^n. \tag{4}$$

Here, the modulus in question is 4, and we require the positive divisors of $4n+1$. If a similar result for $G_3\left(q\right)G_3\left(q^5\right)$ exists, a reasonable guess for the modulus in question is 20. Furthermore, we surmise that we require the positive divisors of $2^r n+s$, in which r could be $1,2,3,\ldots$, and s could be $1,3,5,\ldots$. Of course we settle on upper limits for r and s before our search begins. Then we write

$$G_3\left(q\right)G_3\left(q^5\right) = \sum_{n\geq 0}\left(\sum_{i=1}^{19} c_i d_{i,20}(2^r n+s)\right)q^n. \tag{5}$$

Next, beginning with $(r,s)=(1,1)$, we equate enough coefficients of powers of q on both sides of (5) in order to find the c_i. If the system of linear equations in question is inconsistent, we try $(r,s)=(1,3)$ and proceed similarly. Eventually, for $(r,s)=(2,3)$ we are able to find the required c_i. Once the c_i are known, it is not difficult to construct (6) as we have presented it.

In fact, we discovered the majority of our results by using the method just described. Furthermore, during the early days of the discovery process, after obtaining only a modest number of new results, we profited much by studying the symmetry of such results. This led to an alternative approach for relatively simple types. For instance, the discovery of (7) led us to surmise that an expansion for $G_3\left(q^2\right)G_3\left(q^3\right)$, if such an expansion exists, might look like

$$G_3\left(q^2\right)G_3\left(q^3\right) = \sum_{n=0}^{\infty}\left(\frac{q^{an+b}}{1-q^{24n+3}} + \frac{q^{cn+d}}{1-q^{24n+9}} - \frac{q^{en+f}}{1-q^{24n+15}} - \frac{q^{gn+h}}{1-q^{24n+21}}\right),$$

in which $e-a=g-c=12$, and $d-b$ divides $h-f$. We then checked each such possibility by expanding the right side in powers of q to see if this expansion matched the expansion of the left side. Relatively little trial is required here, but we do require advance knowledge of the form of one identity in a natural pair of identities. As another instance, in [13], this was essentially how we discovered the expansion for $G_3\left(q^2\right)G_5\left(q\right)$ after first finding the expansion for $G_3\left(q\right)G_5\left(q^2\right)$.

4. Triangular Numbers

$$G_3\left(q\right)G_3\left(q^5\right)=\sum_{n=0}^{\infty}\left(\frac{q^{3n}+q^{7n+1}}{1-q^{20n+5}}-\frac{q^{13n+9}+q^{17n+12}}{1-q^{20n+15}}\right). \tag{6}$$

$$G_3\left(q\right)G_3\left(q^6\right)=\sum_{n=0}^{\infty}\left(\frac{q^{7n}}{1-q^{24n+3}}+\frac{q^{5n+1}}{1-q^{24n+9}}-\frac{q^{19n+11}}{1-q^{24n+15}}-\frac{q^{17n+14}}{1-q^{24n+21}}\right). \tag{7}$$

$$G_3\left(q^2\right)G_3\left(q^3\right)=\sum_{n=0}^{\infty}\left(\frac{q^{5n}}{1-q^{24n+3}}+\frac{q^{7n+2}}{1-q^{24n+9}}-\frac{q^{17n+10}}{1-q^{24n+15}}-\frac{q^{19n+16}}{1-q^{24n+21}}\right). \tag{8}$$

$$G_3\left(q\right)G_3\left(q^{10}\right)=\sum_{n=0}^{\infty}\left(\frac{q^{11n}+q^{19n+1}}{1-q^{40n+5}}-\frac{q^{17n+5}+q^{33n+11}}{1-q^{40n+15}}+\frac{q^{7n+3}+q^{23n+13}}{1-q^{40n+25}}-\frac{q^{21n+17}+q^{29n+24}}{1-q^{40n+35}}\right). \tag{9}$$

$$G_3\left(q^2\right)G_3\left(q^5\right)=\sum_{n=0}^{\infty}\left(\frac{q^{7n}+q^{23n+2}}{1-q^{40n+5}}-\frac{q^{21n+7}+q^{29n+10}}{1-q^{40n+15}}+\frac{q^{11n+6}+q^{19n+11}}{1-q^{40n+25}}-\frac{q^{17n+14}+q^{33n+28}}{1-q^{40n+35}}\right). \tag{10}$$

Since our next formula is rather lengthy, it is convenient to define two numerators $N_i=N_i(q,n)$ as follows:

$$\begin{aligned} N_1 &= q^{7n}+q^{11n+1}+q^{15n+2}+q^{19n+3}+q^{31n+6}+q^{47n+10};\\ N_2 &= q^{5n+2}+q^{21n+14}+q^{33n+23}+q^{37n+26}+q^{41n+29}+q^{45n+32}. \end{aligned}$$

Then

$$G_3(q)\,G_3\left(q^{13}\right)=\sum_{n=0}^{\infty}\left(\frac{N_1}{1-q^{52n+13}}-\frac{N_2}{1-q^{52n+39}}\right). \tag{11}$$

Once again it is convenient to define certain numerators. Since there is no danger of confusion, here, and subsequently, we use the same notation used in (11).

Let

$$\begin{aligned}
N_1 &= q^{15n-1}+q^{23n}+q^{31n+1}+q^{47n+3}+q^{71n+6};\\
N_2 &= q^{5n-1}+q^{37n+11}+q^{45n+14}+q^{53n+17}+q^{69n+23};\\
N_3 &= q^{11n+4}+q^{19n+9}+q^{35n+19}+q^{43n+24}+q^{51n+29}+q^{83n+49};\\
N_4 &= q^{17n+12}+q^{33n+26}+q^{41n+33}+q^{57n+47}+q^{65n+54}+q^{73n+61}.
\end{aligned}$$

Then

$$\begin{aligned}G_3(q)\,G_3\left(q^{22}\right)=\sum_{n=0}^{\infty}\Bigg(&\frac{N_1}{1-q^{88n+11}}-\frac{N_2}{1-q^{88n+33}}\\
&+\frac{N_3}{1-q^{88n+55}}-\frac{N_4}{1-q^{88n+77}}\Bigg).\end{aligned} \tag{12}$$

Let

$$\begin{aligned}
N_1 &= q^{5n-1}+q^{37n+3}+q^{45n+4}+q^{53n+5}+q^{69n+7};\\
N_2 &= q^{15n+4}+q^{23n+7}+q^{31n+10}+q^{47n+16}+q^{71n+25};\\
N_3 &= q^{n-1}+q^{9n+4}+q^{25n+14}+q^{33n+19}+q^{49n+29}+q^{81n+49};\\
N_4 &= q^{3n+1}+q^{11n+8}+q^{27n+22}+q^{59n+50}+q^{67n+57}+q^{75n+64}.
\end{aligned}$$

Then

$$\begin{aligned}G_3\left(q^2\right)G_3\left(q^{11}\right)=\sum_{n=0}^{\infty}\Bigg(&-\frac{N_1}{1-q^{88n+11}}+\frac{N_2}{1-q^{88n+33}}\\
&+\frac{N_3}{1-q^{88n+55}}-\frac{N_4}{1-q^{88n+77}}\Bigg).\end{aligned} \tag{13}$$

Put

$$\begin{aligned}
N_1 = {}& q^{15n-1}+q^{19n}+q^{23n+1}+q^{31n+3}+q^{35n+4}+q^{39n+5}\\
&+q^{43n+6}+q^{51n+8}+q^{55n+9}+q^{59n+10}+q^{79n+15}+q^{87n+17}\\
&+q^{91n+18}+q^{103n+21}+q^{119n+25}+q^{131n+28}+q^{135n+29}+q^{143n+31};
\end{aligned}$$

$$\begin{aligned} N_2 &= q^{5n-1}+q^{13n+5}+q^{17n+8}+q^{29n+17}+q^{45n+29}+q^{57n+38} \\ &\quad +q^{61n+41}+q^{69n+47}+q^{89n+62}+q^{93n+65}+q^{97n+68}+q^{105n+74} \\ &\quad +q^{109n+77}+q^{113n+80}+q^{117n+83}+q^{125n+89}+q^{129n+92}+q^{133n+95}. \end{aligned}$$

Then

$$G_3(q)\,G_3\left(q^{37}\right)=\sum_{n=0}^{\infty}\left(\frac{N_1}{1-q^{148n+37}}-\frac{N_2}{1-q^{148n+111}}\right). \tag{14}$$

Set

$$\begin{aligned} N_1 &= q^{3n-7}+q^{11n-6}+q^{19n-5}+q^{27n-4}+q^{43n-2}+q^{75n+2} \\ &\quad +q^{99n+5}+q^{131n+9}+q^{147n+11}+q^{155n+12}+q^{163n+13}+q^{171n+14} \\ &\quad +q^{195n+17}+q^{211n+19}; \\ N_2 &= q^{n-7}+q^{9n-4}+q^{25n+2}+q^{33n+5}+q^{49n+11}+q^{57n+14} \\ &\quad +q^{65n+17}+q^{81n+23}+q^{121n+38}+q^{129n+41}+q^{161n+53}+q^{169n+56} \\ &\quad +q^{209n+71}+q^{225n+77}; \\ N_3 &= q^{15n+2}+q^{31n+12}+q^{39n+17}+q^{47n+22}+q^{55n+27}+q^{79n+42} \\ &\quad +q^{87n+47}+q^{95n+52}+q^{119n+67}+q^{127n+72}+q^{135n+77}+q^{143n+82} \\ &\quad +q^{159n+92}+q^{191n+112}+q^{215n+127}; \\ N_4 &= q^{5n-3}+q^{13n+4}+q^{29n+18}+q^{45n+32}+q^{53n+39}+q^{93n+74} \\ &\quad +q^{109n+88}+q^{117n+95}+q^{125n+102}+q^{141n+116}+q^{149n+123}+q^{165n+137} \\ &\quad +q^{173n+144}+q^{181n+151}+q^{197n+165}. \end{aligned}$$

Then

$$\begin{aligned} G_3(q)\,G_3\left(q^{58}\right)=\sum_{n=0}^{\infty}\Bigg(&-\frac{N_1}{1-q^{232n+29}}+\frac{N_2}{1-q^{232n+87}} \\ &+\frac{N_3}{1-q^{232n+145}}-\frac{N_4}{1-q^{232n+203}}\Bigg). \end{aligned} \tag{15}$$

Define

$$\begin{aligned} N_1 &= q^{15n-2}+q^{31n}+q^{39n+1}+q^{47n+2}+q^{55n+3}+q^{79n+6} \\ &\quad +q^{95n+8}+q^{119n+11}+q^{127n+12}+q^{135n+13}+q^{143n+14}+q^{159n+16} \\ &\quad +q^{191n+20}+q^{215n+23}; \end{aligned}$$

$$\begin{aligned}
N_2 &= q^{5n-2} + q^{13n+1} + q^{45n+13} + q^{53n+16} + q^{93n+31} + q^{109n+37} \\
&\quad + q^{117n+40} + q^{125n+43} + q^{141n+49} + q^{149n+52} + q^{165n+58} + q^{173n+61} \\
&\quad + q^{181n+64} + q^{197n+70}; \\
N_3 &= q^{35n+18} + q^{51n+28} + q^{59n+33} + q^{67n+38} + q^{83n+48} + q^{91n+53} \\
&\quad + q^{107n+63} + q^{115n+68} + q^{123n+73} + q^{139n+83} + q^{179n+108} + q^{187n+113} \\
&\quad + q^{219n+133} + q^{227n+138}; \\
N_4 &= q^{17n+11} + q^{41n+32} + q^{73n+60} + q^{89n+74} + q^{97n+81} + q^{105n+88} \\
&\quad + q^{113n+95} + q^{137n+116} + q^{153n+130} + q^{177n+151} + q^{185n+158} + q^{193n+165} \\
&\quad + q^{201n+172} + q^{217n+186}.
\end{aligned}$$

Then

$$\begin{aligned}
G_3\left(q^2\right) G_3\left(q^{29}\right) = \sum_{n=0}^{\infty} \Bigg(& \frac{N_1}{1-q^{232n+29}} - \frac{N_2}{1-q^{232n+87}} \\
& + \frac{N_3}{1-q^{232n+145}} - \frac{N_4}{1-q^{232n+203}} \Bigg). \qquad (16)
\end{aligned}$$

5. Pentagonal Numbers

Interestingly, the identities we have discovered that involve only pentagonal numbers parallel the cases above for triangular numbers, with two exceptions: we have not been able to find identities for $G_5(q)\, G_5\left(q^6\right)$ and $G_5\left(q^2\right) G_5\left(q^3\right)$.

Let

$$\begin{aligned}
N_1 &= q^n - q^{13n+3} - q^{17n+4} + q^{29n+7} - q^{37n+9} + q^{41n+10} \\
&\quad + q^{49n+12} - q^{53n+13}; \\
N_2 &= q^{7n+5} - q^{11n+8} - q^{19n+14} + q^{23n+17} - q^{31n+23} + q^{43n+32} \\
&\quad + q^{47n+35} - q^{59n+44}.
\end{aligned}$$

Then

$$G_5(q)\, G_5\left(q^5\right) = \sum_{n=0}^{\infty} \left(\frac{N_1}{1-q^{60n+15}} + \frac{N_2}{1-q^{60n+45}} \right). \qquad (17)$$

Put

$$\begin{aligned}
N_1 &= q^{11n} + q^{59n+2}; \\
N_2 &= q^{31n+6} + q^{79n+16};
\end{aligned}$$

$$\begin{aligned} N_3 &= q^{5n+1} + q^{53n+15} + q^{77n+22}; \\ N_4 &= q^{25n+11} + q^{73n+33} + q^{97n+44}; \\ N_5 &= q^{23n+12} + q^{47n+25}; \\ N_6 &= q^{19n+13} + q^{91n+64}; \\ N_7 &= q^{17n+13} + q^{113n+89}; \\ N_8 &= q^{61n+58} + q^{109n+104}. \end{aligned}$$

Then

$$\begin{aligned} G_5(q)\,G_5\left(q^{10}\right) = \sum_{n=0}^{\infty} \Bigg(& \frac{N_1}{1-q^{120n+5}} - \frac{N_2}{1-q^{120n+25}} + \frac{N_3}{1-q^{120n+35}} \\ & - \frac{N_4}{1-q^{120n+55}} + \frac{N_5}{1-q^{120n+65}} + \frac{N_6}{1-q^{120n+85}} \\ & - \frac{N_7}{1-q^{120n+95}} - \frac{N_8}{1-q^{120n+115}} \Bigg). \end{aligned} \tag{18}$$

Set

$$\begin{aligned} N_1 &= q^{31n+1} + q^{79n+3}; \\ N_2 &= q^{11n+2} + q^{59n+12}; \\ N_3 &= q^{n} + q^{25n+7} + q^{49n+14}; \\ N_4 &= q^{5n+2} + q^{29n+13} + q^{101n+46}; \\ N_5 &= q^{19n+10} + q^{91n+49}; \\ N_6 &= q^{23n+16} + q^{47n+33}; \\ N_7 &= q^{61n+48} + q^{109n+86}; \\ N_8 &= q^{17n+16} + q^{113n+108}. \end{aligned}$$

Then

$$\begin{aligned} G_5\left(q^2\right)G_5\left(q^5\right) = \sum_{n=0}^{\infty} \Bigg(& -\frac{N_1}{1-q^{120n+5}} + \frac{N_2}{1-q^{120n+25}} + \frac{N_3}{1-q^{120n+35}} \\ & - \frac{N_4}{1-q^{120n+55}} + \frac{N_5}{1-q^{120n+65}} + \frac{N_6}{1-q^{120n+85}} \\ & - \frac{N_7}{1-q^{120n+95}} - \frac{N_8}{1-q^{120n+115}} \Bigg). \end{aligned} \tag{19}$$

Next, define

$$
\begin{aligned}
N_1 &= q^{7n} + q^{19n+1} + q^{31n+2} + q^{67n+5} + q^{115n+9} + q^{151n+12};\\
N_2 &= q^{11n+4} + q^{47n+19} + q^{59n+24} + q^{71n+29} + q^{83n+34} + q^{119n+49};\\
N_3 &= q^{37n+21} + q^{73n+42} + q^{85n+49} + q^{97n+56} + q^{109n+63} + q^{145n+84};\\
N_4 &= q^{5n+4} + q^{41n+37} + q^{89n+81} + q^{125n+114} + q^{137n+125} + q^{149n+136}.
\end{aligned}
$$

Then

$$
\begin{aligned}
G_5(q)\,G_5\left(q^{13}\right) = \sum_{n=0}^{\infty}\Bigg(&\frac{N_1}{1-q^{156n+13}} + \frac{N_2}{1-q^{156n+65}}\\
&-\frac{N_3}{1-q^{156n+91}} - \frac{N_4}{1-q^{156n+143}}\Bigg).
\end{aligned}
\tag{20}
$$

Next, set

$$
\begin{aligned}
N_1 &= q^{95n+3} + q^{167n+6} + q^{215n+8} + q^{239n+9} + q^{263n+10};\\
N_2 &= q^{67n+13} + q^{91n+18} + q^{115n+23} + q^{163n+33} + q^{235n+48};\\
N_3 &= q^{17n+4} + q^{41n+11} + q^{65n+18} + q^{161n+46} + q^{233n+67};\\
N_4 &= q^{37n+16} + q^{133n+60} + q^{157n+71} + q^{181n+82} + q^{229n+104};\\
N_5 &= q^{35n+18} + q^{83n+44} + q^{107n+57} + q^{131n+70} + q^{227n+122};\\
N_6 &= q^{31n+21} + q^{103n+72} + q^{199n+140} + q^{223n+157} + q^{247n+174};\\
N_7 &= q^{29n+22} + q^{101n+79} + q^{149n+117} + q^{173n+136} + q^{197n+155};\\
N_8 &= q^{n} + q^{25n+23} + q^{49n+46} + q^{97n+92} + q^{169n+161}.
\end{aligned}
$$

Then

$$
\begin{aligned}
G_5(q)\,G_5\left(q^{22}\right) = \sum_{n=0}^{\infty}\Bigg(&-\frac{N_1}{1-q^{264n+11}} - \frac{N_2}{1-q^{264n+55}} - \frac{N_3}{1-q^{264n+77}}\\
&-\frac{N_4}{1-q^{264n+121}} + \frac{N_5}{1-q^{264n+143}} + \frac{N_6}{1-q^{264n+187}}\\
&+\frac{N_7}{1-q^{264n+209}} + \frac{N_8}{1-q^{264n+253}}\Bigg).
\end{aligned}
\tag{21}
$$

Next, let

$$\begin{aligned}
N_1 &= q^{13n}+q^{61n+2}+q^{85n+3}+q^{109n+4}+q^{205n+8};\\
N_2 &= q^{17n+3}+q^{41n+8}+q^{65n+13}+q^{161n+33}+q^{233n+48};\\
N_3 &= q^{67n+19}+q^{91n+26}+q^{115n+33}+q^{163n+47}+q^{235n+68};\\
N_4 &= q^{23n+10}+q^{47n+21}+q^{71n+32}+q^{119n+54}+q^{191n+87};\\
N_5 &= q^{73n+39}+q^{145n+78}+q^{193n+104}+q^{217n+117}+q^{241n+130};\\
N_6 &= q^{29n+20}+q^{101n+71}+q^{149n+105}+q^{173n+122}+q^{197n+139};\\
N_7 &= q^{31n+24}+q^{103n+81}+q^{199n+157}+q^{223n+176}+q^{247n+195};\\
N_8 &= q^{59n+56}+q^{155n+148}+q^{179n+171}+q^{203n+194}+q^{251n+240}.
\end{aligned}$$

Then

$$\begin{aligned}
G_5\left(q^2\right)G_5\left(q^{11}\right) = \sum_{n=0}^{\infty}\Bigg(&\frac{N_1}{1-q^{264n+11}}-\frac{N_2}{1-q^{264n+55}}-\frac{N_3}{1-q^{264n+77}}\\
&+\frac{N_4}{1-q^{264n+121}}-\frac{N_5}{1-q^{264n+143}}+\frac{N_6}{1-q^{264n+187}}\\
&+\frac{N_7}{1-q^{264n+209}}-\frac{N_8}{1-q^{264n+253}}\Bigg). \qquad (22)
\end{aligned}$$

Let

$$\begin{aligned}
N_1 = \ & q^{7n-1}+q^{67n+4}+q^{115n+8}+q^{127n+9}+q^{139n+10}+q^{151n+11}\\
&+q^{175n+13}+q^{211n+16}+q^{223n+17}+q^{247n+19}+q^{271n+21}+q^{295n+23}\\
&+q^{307n+24}+q^{343n+27}+q^{367n+29}+q^{379n+30}+q^{391n+31}+q^{403n+32};\\
N_2 = \ & q^{23n+8}+q^{35n+13}+q^{59n+23}+q^{119n+48}+q^{131n+53}+q^{143n+58}\\
&+q^{167n+68}+q^{179n+73}+q^{191n+78}+q^{203n+83}+q^{227n+93}+q^{239n+98}\\
&+q^{251n+103}+q^{311n+128}+q^{335n+138}+q^{347n+143}+q^{383n+158}\\
&+q^{431n+178};
\end{aligned}$$

$$\begin{aligned}
N_3 &= q^{n-1} + q^{25n+13} + q^{49n+27} + q^{73n+41} + q^{85n+48} + q^{121n+69} \\
&\quad + q^{145n+83} + q^{157n+90} + q^{169n+97} + q^{181n+104} + q^{229n+132} + q^{289n+167} \\
&\quad + q^{337n+195} + q^{349n+202} + q^{361n+209} + q^{373n+216} + q^{397n+230} + q^{433n+251}; \\
N_4 &= q^{5n+3} + q^{17n+14} + q^{29n+25} + q^{89n+80} + q^{113n+102} + q^{125n+113} \\
&\quad + q^{161n+146} + q^{209n+190} + q^{245n+223} + q^{257n+234} + q^{281n+256} + q^{341n+311} \\
&\quad + q^{353n+322} + q^{365n+333} + q^{389n+355} + q^{401n+366} + q^{413n+377} + q^{425n+388}.
\end{aligned}$$

Then

$$G_5(q)\,G_5\left(q^{37}\right) = \sum_{n=0}^{\infty}\left(-\frac{N_1}{1-q^{444n+37}} + \frac{N_2}{1-q^{444n+185}} + \frac{N_3}{1-q^{444n+259}} - \frac{N_4}{1-q^{444n+407}}\right). \tag{23}$$

Put

$$\begin{aligned}
N_1 &= q^{11n-2} + q^{131n+3} + q^{155n+4} + q^{251n+8} + q^{275n+9} + q^{395n+14} \\
&\quad + q^{443n+16} + q^{467n+17} + q^{491n+18} + q^{539n+20} + q^{563n+21} + q^{611n+23} \\
&\quad + q^{635n+24} + q^{659n+25}; \\
N_2 &= q^{7n-1} + q^{103n+19} + q^{151n+29} + q^{175n+34} + q^{199n+39} + q^{223n+44} \\
&\quad + q^{295n+59} + q^{343n+69} + q^{415n+84} + q^{439n+89} + q^{463n+94} + q^{487n+99} \\
&\quad + q^{535n+109} + q^{631n+129}; \\
N_3 &= q^{77n+20} + q^{101n+27} + q^{221n+62} + q^{269n+76} + q^{293n+83} + q^{317n+90} \\
&\quad + q^{365n+104} + q^{389n+111} + q^{437n+125} + q^{461n+132} + q^{485n+139} \\
&\quad + q^{533n+153} + q^{653n+188} + q^{677n+195}; \\
N_4 &= q^{n-2} + q^{25n+9} + q^{49n+20} + q^{121n+53} + q^{169n+75} + q^{241n+108} \\
&\quad + q^{265n+119} + q^{289n+130} + q^{313n+141} + q^{361n+163} + q^{457n+207} \\
&\quad + q^{529n+240} + q^{625n+284} + q^{673n+306};
\end{aligned}$$

$$\begin{aligned} N_5 &= q^{47n+23}+q^{95n+49}+q^{119n+62}+q^{143n+75}+q^{191n+101}+q^{215n+114}\\ &\quad+q^{263n+140}+q^{287n+153}+q^{311n+166}+q^{359n+192}+q^{479n+257}\\ &\quad+q^{503n+270}+q^{599n+322}+q^{623n+335};\end{aligned}$$

$$\begin{aligned} N_6 &= q^{19n+11}+q^{43n+28}+q^{163n+113}+q^{211n+147}+q^{235n+164}+q^{259n+181}\\ &\quad+q^{307n+215}+q^{331n+232}+q^{379n+266}+q^{403n+283}+q^{427n+300}\\ &\quad+q^{475n+334}+q^{595n+419}+q^{619n+436};\end{aligned}$$

$$\begin{aligned} N_7 &= q^{65n+49}+q^{161n+125}+q^{209n+163}+q^{233n+182}+q^{257n+201}+q^{281n+220}\\ &\quad+q^{353n+277}+q^{377n+296}+q^{401n+315}+q^{473n+372}+q^{497n+391}\\ &\quad+q^{521n+410}+q^{545n+429}+q^{593n+467}+q^{689n+543};\end{aligned}$$

$$\begin{aligned} N_8 &= q^{13n+10}+q^{109n+102}+q^{181n+171}+q^{277n+263}+q^{325n+309}+q^{349n+332}\\ &\quad+q^{373n+355}+q^{397n+378}+q^{469n+447}+q^{493n+470}+q^{517n+493}\\ &\quad+q^{589n+562}+q^{613n+585}+q^{637n+608}+q^{661n+631}.\end{aligned}$$

Then

$$\begin{aligned} G_5(q)\,G_5\left(q^{58}\right) = \sum_{n=0}^{\infty}\Bigg(&-\frac{N_1}{1-q^{696n+29}}-\frac{N_2}{1-q^{696n+145}}+\frac{N_3}{1-q^{696n+203}}\\ &+\frac{N_4}{1-q^{696n+319}}+\frac{N_5}{1-q^{696n+377}}-\frac{N_6}{1-q^{696n+493}}\\ &+\frac{N_7}{1-q^{696n+551}}-\frac{N_8}{1-q^{696n+667}}\Bigg). \end{aligned} \tag{24}$$

Set

$$\begin{aligned} N_1 &= q^{7n-1}+q^{103n+3}+q^{151n+5}+q^{175n+6}+q^{199n+7}+q^{223n+8}\\ &\quad+q^{295n+11}+q^{343n+13}+q^{415n+16}+q^{439n+17}+q^{463n+18}+q^{487n+19}\\ &\quad+q^{535n+21}+q^{631n+25};\end{aligned}$$

$$\begin{aligned} N_2 &= q^{11n+1}+q^{131n+26}+q^{155n+31}+q^{251n+51}+q^{275n+56}+q^{395n+81}\\ &\quad+q^{443n+91}+q^{467n+96}+q^{491n+101}+q^{539n+111}+q^{563n+116}\\ &\quad+q^{611n+126}+q^{635n+131}+q^{659n+136};\end{aligned}$$

$$\begin{aligned}
N_3 &= q^{n-1} + q^{25n+6} + q^{49n+13} + q^{121n+34} + q^{169n+48} + q^{241n+69} \\
&\quad + q^{265n+76} + q^{289n+83} + q^{313n+90} + q^{361n+104} + q^{457n+132} + q^{529n+153} \\
&\quad + q^{625n+181} + q^{673n+195}; \\
N_4 &= q^{77n+34} + q^{101n+45} + q^{221n+100} + q^{269n+122} + q^{293n+133} + q^{317n+144} \\
&\quad + q^{365n+166} + q^{389n+177} + q^{437n+199} + q^{461n+210} + q^{485n+221} \\
&\quad + q^{533n+243} + q^{653n+298} + q^{677n+309}; \\
N_5 &= q^{19n+9} + q^{43n+22} + q^{163n+87} + q^{211n+113} + q^{235n+126} + q^{259n+139} \\
&\quad + q^{307n+165} + q^{331n+178} + q^{379n+204} + q^{403n+217} + q^{427n+230} \\
&\quad + q^{475n+256} + q^{595n+321} + q^{619n+334}; \\
N_6 &= q^{47n+32} + q^{95n+66} + q^{119n+83} + q^{143n+100} + q^{191n+134} + q^{215n+151} \\
&\quad + q^{263n+185} + q^{287n+202} + q^{311n+219} + q^{359n+253} + q^{479n+338} \\
&\quad + q^{503n+355} + q^{599n+423} + q^{623n+440}; \\
N_7 &= q^{37n+28} + q^{61n+47} + q^{85n+66} + q^{133n+104} + q^{157n+123} + q^{205n+161} \\
&\quad + q^{229n+180} + q^{253n+199} + q^{301n+237} + q^{421n+332} + q^{445n+351} \\
&\quad + q^{493n+389} + q^{541n+427} + q^{565n+446} + q^{685n+541}; \\
N_8 &= q^{17n+15} + q^{41n+38} + q^{89n+84} + q^{113n+107} + q^{137n+130} + q^{185n+176} \\
&\quad + q^{305n+291} + q^{329n+314} + q^{377n+360} + q^{425n+406} + q^{449n+429} \\
&\quad + q^{569n+544} + q^{617n+590} + q^{641n+613} + q^{665n+636}.
\end{aligned}$$

Then

$$\begin{aligned}
G_5\left(q^2\right) G_5\left(q^{29}\right) = \sum_{n=0}^{\infty} \Bigg(&-\frac{N_1}{1-q^{696n+29}} - \frac{N_2}{1-q^{696n+145}} + \frac{N_3}{1-q^{696n+203}} \\
&+\frac{N_4}{1-q^{696n+319}} - \frac{N_5}{1-q^{696n+377}} + \frac{N_6}{1-q^{696n+493}} \\
&+\frac{N_7}{1-q^{696n+551}} - \frac{N_8}{1-q^{696n+667}} \Bigg). \qquad (25)
\end{aligned}$$

6. Heptagonal Numbers

$$
\begin{aligned}
G_7\left(q\right)G_7\left(q^6\right) = \sum_{n=0}^{\infty}\Bigg(& -\frac{q^{23n-1}}{1-q^{120n+3}} - \frac{q^{61n+3}}{1-q^{120n+9}} + \frac{q^{49n+7}}{1-q^{120n+21}} \\
& -\frac{q^{47n+9}}{1-q^{120n+27}} - \frac{q^{13n+2}}{1-q^{120n+33}} + \frac{q^{11n+2}}{1-q^{120n+39}} \\
& +\frac{q^{39n+15}+q^{79n+32}}{1-q^{120n+51}} + \frac{q^{77n+35}}{1-q^{120n+57}} + \frac{q^{83n+42}}{1-q^{120n+63}} \\
& +\frac{q^{n-1}}{1-q^{120n+69}} + \frac{q^{29n+18}+q^{69n+45}}{1-q^{120n+81}} \\
& -\frac{q^{27n+18}+q^{67n+47}}{1-q^{120n+87}} + \frac{q^{73n+55}}{1-q^{120n+93}} - \frac{q^{71n+57}}{1-q^{120n+99}} \\
& -\frac{q^{19n+16}+q^{99n+90}}{1-q^{120n+111}} - \frac{q^{17n+15}}{1-q^{120n+117}}\Bigg). \qquad (26)
\end{aligned}
$$

We have not been able to find an identity for $G_7\left(q^2\right)G_7\left(q^3\right)$.

7. Final Comments

We would like to acknowledge our gratitude to Dr. Michael Hirschhorn, whose encouraging and highly informed comments have served to significantly streamline this work.

Finally, we would like to express our gratitude to an anonymous referee, who has directed us to several sources that contain material that has relevance to our work. Accordingly, we comment briefly about the relevant material in these sources.

Let a, b, n, x, and y be positive integers. By $t_n(a,b)$, Sun [14] denotes the number of representations of n as $ax(x-1)/2 + by(y-1)/2$. Sun then obtains formulas for $t_n(1,b)$ for fifteen values of b. The referee, in complete detail, has demonstrated that the truth of our conjecture (6) is equivalent to Sun's Theorem 3.3. In a similar manner, the truth of our conjecture (11) is equivalent to Sun's Theorem 3.5, and the truth of our conjecture (14) is equivalent to Sun's Theorem 3.7.

For any integer n, and any integers $k \geq m > 0$, let $r_{(k,m)}(n)$ denote the number of solutions, in integers, of

$$n = kx_1^2 + mx_2^2. \qquad (27)$$

Similarly, let $t_{(k,m)}(n)$ denote the number of solutions, in non-negative integers, of

$$n = k\frac{x_1(x_1+1)}{2} + m\frac{x_2(x_2+1)}{2}. \tag{28}$$

Then Theorem 1 of Adiga, Cooper and Han [1] produces (among many such relations) the following:

$$\begin{aligned}
r_{(3,2)}(8n+5) &= 4t_{(3,2)}(n),\\
r_{(5,1)}(8n+6) &= 4t_{(5,1)}(n),\\
r_{(6,1)}(8n+7) &= 4t_{(6,1)}(n),\\
r_{(5,2)}(8n+7) &= 4t_{(5,2)}(n).
\end{aligned} \tag{29}$$

In (29) we have listed only those relations that are relevant to the present work. For any integer $n \geq 1$, write

$$8n+6 = 2(4n+3) = 2 \times 5^s \times n_0, \tag{30}$$

for an integer $s \geq 0$, and an integer $n_0 \geq 1$ with $(n_0, 20) = 1$. Then by a result in Dickson [6], page 84,

$$\begin{aligned}
r_{(5,1)}(8n+6) = \left(1 - \left(\frac{n_0}{5}\right)\right) &(d_{1,20}(n_0) + d_{3,20}(n_0) + d_{7,20}(n_0) + d_{9,20}(n_0)\\
&-d_{11,20}(n_0) - d_{13,20}(n_0) - d_{17,20}(n_0) - d_{19,20}(n_0)).
\end{aligned} \tag{31}$$

There are four possibilities for n_0 modulo 20, namely

$$n_0 = 20x+3,\ n_0 = 20x+7,\ n_0 = 20x+11,\ \text{or } n_0 = 20x+19,$$

for x a nonnegative integer. From here, using the same path set forth by the referee (see the comments above, on the work of Sun), we used the second entry in in (29) to prove our conjecture (6). Again, in Dickson [6], pages 84-86, there are results for $r_{(3,2)}(n)$, $r_{(6,1)}(n)$, and $r_{(5,2)}(n)$. These results, when coupled with the appropriate entries in (29), yield proofs of our conjectures (7), (8), and (10).

Interestingly, Dickson [6] gives $r_{(k,m)}(n)$ for several other instances of (k, m). For each of these instances, however, Theorem 1 of Adiga, Cooper, and Han [1] is not broad enough in scope to enable any of our remaining conjectures to be proved.

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