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MIN, A COMBINATORIAL GAME HAVING A CONNECTION WITH PRIME NUMBERS

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Abstract

We introduce a two person game played with a pair of nonnegative integers; a move consists of subtracting from the larger integer, a positive integer no greater than the smaller integer. The player who reduces one of the integers to zero wins. The game is curious in several respects: in particular, its Sprague-Grundy values have an interesting connection with prime numbers.

1. Introduction

We introduce a two person, impartial, combinatorial game, that we call MIN (short for minus). The game is played with an unordered pair of nonnegative integers. From position (a, b) with $a \le b$, a legal move consists of subtracting from b, a positive integer no greater than a. The first player who reduces one of the integers to zero wins. The game is a variation of NIM, and it is related to EUCLID, in which the players move by subtracting a positive integer multiple of the smaller integer from the larger integer; see references [1] to [6]. Actually, there are two versions of EUCLID, according to whether one stops when the entries are equal, or one of them is reduced to 0. The latter case is the original game of Cole and Davie [2], while the former case was introduced by Grossman [4]. In both cases, there is an explicit formula for the Sprague-Grundy values. For Grossman's EUCLID, Nivasch [5] gave the formula $\lfloor |\frac{a}{b} - \frac{b}{a}| \rfloor$. For Cole and Davie's EUCLID, the formula involves the continued fraction expansion of b/a; see [1]. For the game MIN the situation seems rather more complicated, and it is very unlikely that there is a simple formula for its Sprague-Grundy values, as will become apparent below.

Let $\mathcal{G}(a, b)$ denote the Sprague-Grundy value of the position (a, b) in MIN. For convenience, throughout this paper we will refer to the Sprague-Grundy value simply as the *nim-value*. Recall that the nim-value of the terminal positions $\mathcal{G}(0, a)$ is zero, and that $\mathcal{G}(a, b)$ is the smallest nonnegative integer not contained in the set of nim-values of the positions that can be reached from (a, b). The game MIN is curious in three respects:

1. For a fixed and $b \geq a-1$, the nim-values of positions $\mathcal{G}(a,b)$ are periodic in b with period $a+1$.

2. The positions with nim-value 1 can be computed recursively without knowledge of the positions of the other nim-values.

3. There is an interesting connection with prime numbers. Our main result is the following:

Theorem 1. *For all $a \geq 1$, we have $\mathcal{G}(a,a) = 1$ if and only if $a+1$ is prime.*

Calculations reveal a number of interesting patterns, which unfortunately die out for sufficiently large integers. For example, if $\mathcal{G}(a-1,a) = 0$ and $a < 6240$, then $a+1$ is prime. But $\mathcal{G}(a-1,a) = 0$ for $a = 6240$, and here $a+1 = 79^2$, which is not prime. Similarly, there are only 4 values of a with $\mathcal{G}(a-2,a) = 0$ and $a < 17226$; in these cases, the values of $a+1$ are the first 4 Fermat primes: $3, 5, 17, 257$. However, $\mathcal{G}(17224, 17226) = 0$, and 17227 isn't a Fermat prime.

2. Preliminaries

Remark 2. Note that for $a > 0$, we have $\mathcal{G}(a,a) \neq 0$, since $\mathcal{G}(0,a) = 0$ and there is a move from (a,a) to $(a,0)$. Moreover, by induction, $\mathcal{G}(1,a) = 0$ if and only if a is even.

We begin by showing that for fixed $a > 0$, the number $\mathcal{G}(a,b)$ is periodic in b with period $a+1$, provided b is sufficiently large. For integers x, y with $y > 0$, we denote the remainder $R_y(x) = x - y \cdot \lfloor x/y \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part.

Lemma 3. *Suppose that $a \geq 1$.*

(a) *If $b \geq 2a$, then $\mathcal{G}(a,b) = \mathcal{G}(a, a-1+R_{a+1}(b-a+1))$.*

(b) *If $b \geq a-1$, then*

$$\{\mathcal{G}(a,b), \mathcal{G}(a,b+1), \ldots, \mathcal{G}(a,b+a)\} = \{0, 1, \ldots, a\}.$$

Proof. We prove (b) first. Note that if $0 \leq j < k \leq a$, then there is a move from $(a, b+k)$ to $(a, b+j)$ and so the elements

$$\mathcal{G}(a,b), \mathcal{G}(a,b+1), \ldots, \mathcal{G}(a,b+a)$$

are all distinct. Furthermore, if $a \leq b$ and $0 \leq i \leq a$, or $a = b+1$ and $0 < i$, there are exactly a moves from the position $(a, b+i)$, while if $a = b+1$ and $i = 0$, there are $a-1$ moves from $(a, b+i)$. Therefore $G(a, b+i) \leq a$ for all $0 \leq i \leq a$. This gives the required result.

From (b) we have $\{\mathcal{G}(a, b+1), \mathcal{G}(a, b+2), \ldots, \mathcal{G}(a, b+a+1)\} = \{0, 1, \ldots, a\} = \{\mathcal{G}(a, b), \mathcal{G}(a, b+1), \ldots, \mathcal{G}(a, b+a)\}$. Hence $\mathcal{G}(a, b+a+1) = \mathcal{G}(a, b)$, which gives (a). □

In general, it is not true that $\mathcal{G}(a, b+a+1) = \mathcal{G}(a, b)$ for $b < a-1$. However, there are partial results. Let $f_j(a)$ be the smallest nonnegative integer i such that $\mathcal{G}(a, a+i) = j$. Note that by the previous lemma, $0 \leq f_j(a) \leq a$.

Lemma 4. *Suppose that* $a > 0$. *Then*

(a) $\mathcal{G}(a, f_0(a) - 1) = 0$.

(b) *If* $f_1(a) > 0$, *then* $\mathcal{G}(a, f_1(a) - 1) = 1$.

Proof. (a) First note that by Remark 2, we have $f_0(a) > 0$. There are exactly a moves from $\mathcal{G}(a, a + f_0(a))$, which lead respectively to

$$(a, f_0(a)), (a, f_0(a) + 1), \ldots, (a, a + f_0(a) - 1),$$

so none of these positions has nim-value 0. Suppose that $\mathcal{G}(a, f_0(a) - 1) \neq 0$. Then none of the positions

$$(a, f_0(a) - 1), (a, f_0(a)), \ldots, (a, a + f_0(a) - 2)$$

would have nim-value 0, and so we would have $\mathcal{G}(a, a + f_0(a) - 1) = 0$, where $f_0(a) - 1$ is nonnegative, contrary to the definition of $f_0(a)$.

(b) Similarly, there are exactly a moves from $\mathcal{G}(a, a + f_1(a))$, leading respectively to $(a, f_1(a)), (a, f_1(a) + 1), \ldots, (a, a + f_1(a) - 1)$, so none of these positions has nim-value 1. Suppose that $\mathcal{G}(a, f_1(a) - 1) \neq 1$. Then none of the positions

$$(a, f_1(a) - 1), (a, f_1(a)), \ldots, (a, a + f_1(a) - 2)$$

would have nim-value 1, and so we would have $\mathcal{G}(a, a + f_1(a) - 1) \leq 1$. We have already observed that $(a, a + f_1(a) - 1)$ does not have nim-value 1. So $\mathcal{G}(a, a + f_1(a) - 1) = 0$. As $f_1(a) > 0$, we have $a + f_1(a) - 1 \geq a$. In fact, as

$\mathcal{G}(a, a + f_1(a) - 1) = 0$ and $\mathcal{G}(a, a) \neq 0$ by Remark 1, we have $a + f_1(a) - 1 > a$; that is, $f_1(a) \geq 2$. As the nim-values

$$\{\mathcal{G}(a, a), \mathcal{G}(a, a + 1), \ldots, \mathcal{G}(a, 2a)\}$$

are distinct, by Lemma 3(b), $f_0(a) = f_1(a) - 1$. Thus, by part (a) of the current lemma, $\mathcal{G}(a, f_1(a) - 2) = 0$. Consider the positions $(a, f_1(a) - 2), (a, f_1(a) - 1), \ldots, (a, a + f_1(a) - 3)$. None of these positions has nim-value 1, and one of them, $(a, f_1(a) - 2)$, has nim-value 0. Hence $(a, a + f_1(a) - 2)$ has nim-value 1, in contradiction with the definition of $f_1(a)$. □

Remark 5. Note that for all a, one has $\mathcal{G}(a, a+1) \neq 1$. Indeed, if $\mathcal{G}(a, a+1) = 1$, then we would have $f_1(a) = 1$ and so by Lemma 4(b), $\mathcal{G}(a, 0) = 1$, which is false. Thus, if $f_1(a) > 0$, then $f_1(a) \geq 2$.

Lemma 6. *For $j = 0, 1$, if $f_j(a) > 0$, then $f_j(f_j(a) - 1) = R_{f_j(a)}(a + 1)$.*

Proof. We have $f_j(a) - 1 < a$. By Lemma 4, $\mathcal{G}(a, f_j(a) - 1) = j$. Hence, by Lemma 3(a), $a - (f_j(a) - 1) \equiv f_j(f_j(a) - 1) \pmod{f_j(a)}$. Thus $a + 1 \equiv f_j(f_j(a) - 1) \pmod{f_j(a)}$. □

Lemma 7. *Let $a \geq 1$. If $\mathcal{G}(a, a) \neq 1$, then* $\gcd(f_1(a), a + 1) > 1$.

Proof. The proof is by induction on a. The condition holds for $a = 1$, since $\mathcal{G}(1, 1) = 1$. Fix $a > 1$ and suppose the condition holds for all $b < a$. Assume that $\mathcal{G}(a, a) \neq 1$. Then $f_1(a) \geq 2$ by Remark 2. Let $b = f_1(a) - 1$. By Lemma 6, $f_1(b) = R_{f_1(a)}(a+1)$. Note that if $f_1(b) = 0$, then $a + 1$ is a multiple of $f_1(a)$, which gives the required result. So we suppose that $f_1(b) \neq 0$. By the inductive hypothesis, $\gcd(f_1(b), b + 1) > 1$. Let us denote $\lambda = \gcd(f_1(b), b + 1)$. So λ divides $f_1(b)$ and $b + 1 = f_1(a)$. Therefore, λ also divides $a + 1$, since $f_1(b) = R_{f_1(a)}(a + 1)$. Hence λ also divides $\gcd(f_1(a), a + 1)$, which gives us the required result. □

3. Proof of the Theorem

We first establish the following result.

Lemma 8. *Suppose that $1 \leq a \leq b$.*

(a) *If $\mathcal{G}(a, b) = 0$, then* $\gcd(a + 1, b + 1) = 1$.

(b) *If $\mathcal{G}(a, b) = 1$, then* $\gcd(a + 1, b + 1)$ *is a prime number.*

(c) *If $a + 1$ is prime, then $\mathcal{G}(a, b) = 1$ iff $a + 1$ is a factor of $b + 1$.*

Proof. The proof is by induction on a. Note that all three parts of the lemma are true for $a = 1$, by Remark 2.

(a) Suppose $\mathcal{G}(a, b) = 0$. So $a < b$, by Remark 2. We have $b = k\cdot(a+1)+R_{a+1}(b)$ for some integer $k \geq 1$. So by Lemma 3, $\mathcal{G}(a, a+1+R_{a+1}(b)) = \mathcal{G}(a, b) = 0$. So by Lemma 4, $\mathcal{G}(a, R_{a+1}(b)) = 0$. Note that $R_{a+1}(b) \leq a$. In fact, since $\mathcal{G}(a, R_{a+1}(b)) = 0$ and $\mathcal{G}(a, a) \neq 0$ by Remark 2, we have $R_{a+1}(b) < a$. Hence, by the inductive hypothesis, $\gcd(a + 1, R_{a+1}(b) + 1) = 1$. Therefore, $\gcd(a + 1, b + 1) = 1$.

(b) Now suppose that $\mathcal{G}(a, b) = 1$. First consider the case where $a = b$. Then none of the positions (a, i), for $0 \leq i < a$ have nim-value 1. Let p be a prime less than a and consider the position $(a, p-1)$. By part (c) of the inductive hypothesis, since p is prime and $\mathcal{G}(a, p-1) \neq 1$, we have that p is not a factor of $a + 1$. Thus $a + 1$ is not a multiple of any prime less than a. Therefore, $a + 1$ is prime; that is, $\gcd(a + 1, a + 1)$ is prime. Now we treat the case where $a < b$. By Lemma 4, we have $\mathcal{G}(a, R_{a+1}(b)) = 1$. Hence, by the inductive hypothesis, $\gcd(a+1, R_{a+1}(b)+1)$ is prime. Since $\gcd(a + 1, b + 1) = \gcd(a + 1, R_{a+1}(b) + 1)$, this gives the required result.

(c) Assume that $a+1$ is prime. If $\mathcal{G}(a, b) = 1$, then by part (b), $\gcd(a+1, b+1) = a + 1$, so $a + 1$ is a factor of $b + 1$. Conversely, if $a + 1$ is a factor of $b + 1$, then $\mathcal{G}(a, b) = \mathcal{G}(a, a)$ by Lemma 3. So it remains to show that $\mathcal{G}(a, a) = 1$. By Lemma 7, if $\mathcal{G}(a, a) \neq 1$, then $\gcd(f_1(a), a + 1) > 1$, which, as $a + 1$ is prime, would imply that $a+1$ is a factor of $f_1(a)$. But this is impossible as $f_1(a) \leq a$, by definition. □

Finally, we deduce the theorem. First, if $\mathcal{G}(a, a) = 1$, then $a+1 = \gcd(a+1, a+1)$ is a prime number, by part (b) of the previous lemma. Conversely, if $a+1$ is prime, then by part (c) of the previous lemma, $\mathcal{G}(a, a) = 1$.

4. Computational Considerations

Notice that knowledge of the function f_1 enables one to decide whether a given position has nim-value 1. Indeed, given position (a, b), one can use Lemma 3(a) to reduce to the case where $a \leq b \leq 2a$. Since there is only one position in this range with nim-value 1, by Lemma 3(b), it remains to check whether $f_1(a) = b - a$.

The function f_1 can be computed recursively, as we will now explain. If $a + 1$ is prime, then $f_1(a) = 0$, by the theorem. Otherwise, $f_1(a) > 0$ and $\mathcal{G}(a, f_1(a) - 1) = 1$, by Lemma 4(b). There may be several positions of nim-value 1 of the form (a, c) with $0 \leq c < a$. The key point is that $f_1(a) - 1$ is the largest of these

c values; indeed, as $(a, a + f_1(a))$ has nim-value 1, none of the positions

$$(a, f_1(a)), (a, f_1(a) + 1), \ldots, (a, a + f_1(a) - 1)$$

has nim-value 1. So, assuming $f_1(x)$ is known for $x < a$, we compute $f_1(a)$ as follows. Consider the successive positions $(a, a - i)$, for $i = 1, 2, 3, \ldots$. Proceed until a position $(a, a - i)$ is found with nim-value 1; as explained above, this is done by checking whether $f_1(a - i) = R_{a-i+1}(i)$. Then, $f_1(a) = a - i + 1$. In this manner, the positions with nim-value 1 can be computed without knowledge of the positions of the other nim-values. By using Lemma 4(a), the positions with nim-value 0 can be determined separately, in the same manner.

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