THE (EXPONENTIAL) BIPARTITIONAL POLYNOMIALS AND POLYNOMIAL SEQUENCES OF TRINOMIAL TYPE: PART I

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Abstract

The aim of this paper is to investigate and present the general properties of the (exponential) bipartitional polynomials. After establishing relations between bipartitional polynomials and polynomial sequences of binomial and trinomial type, a number of identities are deduced.

1. Introduction

For $(m, n) \in \mathbb{N}^2$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$, the complete bipartitional polynomial

$$A_{m,n} \equiv A_{m,n}\left(x_{0,1}, x_{1,0}, x_{0,2}, x_{1,1}, x_{2,0}, \ldots, x_{m,n}\right)$$

with variables $x_{0,1}, x_{1,0}, \ldots, x_{m,n}$ is defined by the sum

$$A_{m,n} := \sum \frac{m!n!}{k_{0,1}!k_{1,0}!\cdots k_{m,n}!}\left(\frac{x_{0,1}}{0!1!}\right)^{k_{0,1}}\left(\frac{x_{1,0}}{1!0!}\right)^{k_{1,0}}\cdots\left(\frac{x_{m,n}}{m!n!}\right)^{k_{m,n}}, \qquad (1)$$

where the summation is carried out over all partitions of the bipartite number (m, n), that is, over all nonnegative integers $k_{0,1}, k_{1,0}, \ldots, k_{m,n}$ which are solution to the equations

$$\sum_{i=1}^{m} i \sum_{j=0}^{n} k_{i,j} = m, \quad \sum_{j=1}^{n} j \sum_{i=0}^{m} k_{i,j} = n. \qquad (2)$$

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The partial bipartitional polynomial $A_{m,n,k} \equiv A_{m,n,k}(x_{0,1}, \ldots, x_{m,n})$ with variables $x_{0,1}, x_{1,0}, \ldots, x_{m,n}$, of degree $k \in \mathbb{N}$, is defined by the sum

$$A_{m,n,k} := \sum \frac{m!n!}{k_{0,1}!k_{1,0}! \cdots k_{m,n}!} \left(\frac{x_{0,1}}{0!1!}\right)^{k_{0,1}} \left(\frac{x_{1,0}}{1!0!}\right)^{k_{1,0}} \cdots \left(\frac{x_{m,n}}{m!n!}\right)^{k_{m,n}}, \quad (3)$$

where the summation is carried out over all partitions of the bipartite number (m, n) into k parts, that is, over all nonnegative integers $k_{0,1}, k_{1,0}, \ldots, k_{m,n}$ which are solution to the equations

$$\sum_{i=1}^{m} i \sum_{j=0}^{n} k_{i,j} = m, \quad \sum_{j=1}^{n} j \sum_{i=0}^{m} k_{i,j} = n, \quad \sum_{i=0}^{m} \sum_{j=0}^{n} k_{i,j} = k, \quad \text{with the convention } k_{0,0} = 0. \quad (4)$$

These polynomials were introduced in [1, pp. 454] with properties such as those given in (5), (6), (7), (8) below. Indeed, for all real numbers α, β, γ, we have

$$A_{m,n,k}\left(\beta\gamma x_{0,1}, \alpha\gamma x_{1,0}, \beta^2\gamma x_{0,2}, \ldots, \alpha^m \beta^n \gamma x_{m,n}\right) = \beta^m \alpha^n \gamma^k A_{m,n,k}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right), \quad (5)$$

and

$$A_{m,n}\left(\alpha x_{0,1}, \beta x_{1,0}, \ldots, \alpha^m \beta^n x_{m,n}\right) = \alpha^m \beta^n A_{m,n}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right). \quad (6)$$

Moreover, we can check that the exponential generating functions for $A_{m,n}$ and $A_{m,n,k}$ are respectively provided by

$$1 + \sum_{m+n \geq 1} A_{m,n}\left(x_{0,1}, \ldots\right) \frac{t^m}{m!} \frac{u^n}{n!} = \exp\left(\sum_{i+j \geq 1} x_{i,j} \frac{t^i}{i!} \frac{u^j}{j!}\right), \quad (7)$$

and

$$\sum_{m+n \geq k} A_{m,n,k}\left(x_{0,1}, \ldots\right) \frac{t^m}{m!} \frac{u^n}{n!} = \frac{1}{k!}\left(\sum_{i+j \geq 1} x_{i,j} \frac{t^i}{i!} \frac{u^j}{j!}\right)^k. \quad (8)$$

From the above definitions and properties, we attempt to present more properties and identities for the partial and complete bipartitional polynomials. We also consider the connection between these polynomials and polynomials $(f_{m,n}(x))$ of trinomial type, defined by

$$\left(\sum_{i,j \geq 0} f_{i,j}(1) \frac{t^i}{i!} \frac{u^j}{j!}\right)^x = \sum_{m,n \geq 0} f_{m,n}(x) \frac{t^m}{m!} \frac{u^n}{n!}, \quad \text{with } f_{0,0}(x) := 1. \quad (9)$$

For the investigation of polynomials of trinomial and multinomial type, we can refer to [8].

We use the notations $A_{m,n,k}(x_{i,j})$ or $A_{m,n,k}(x_{I,J})$ for $A_{m,n,k}(x_{0,1}, x_{1,0}, \ldots)$, and $A_{m,n}(x_{i,j})$ or $A_{m,n}(x_{I,J})$ for $A_{m,n}(x_{0,1}, \ldots)$. Moreover, we represent respectively by $B_{n,k}(x_j)$ and $B_n(x_j)$, the partial and complete Bell polynomials $B_{n,k}(x_1, x_2, \ldots)$ and $B_n(x_1, x_2, \ldots)$. We recall that these polynomials are defined by their generating functions

$$\sum_{n=k}^{\infty} B_{n,k}(x_j) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k,$$

$$1 + \sum_{n=1}^{\infty} B_n(x_j) \frac{t^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right),$$

which admit the following explicit expressions:

$$B_{n,k}(x_j) = \sum_{k_1+k_2+\cdots=k,\ \ k_1+2k_2+\cdots=n} \frac{n!}{k_1! k_2! \cdots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots,$$

$$B_n(x_j) = \sum_{k_1+2k_2+3k_3+\cdots=n} \frac{n!}{k_1! k_2! \cdots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots.$$

For $m < 0$ or $n < 0$, we set $f_{m,n}(x) = 0$ and $A_{m,n}(x_{i,j}) = 0$, and for $m < 0$ or $n < 0$ or $k < 0$, we set $A_{m,n,k}(x_{i,j}) = 0$. For $x \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, we set

$$\binom{x}{k} := \frac{1}{k!} x(x-1)\cdots(x-k+1) \text{ for } k = 1, 2, \ldots,$$

$$\binom{x}{0} = 1 \text{ and } \binom{x}{k} = 0 \text{ otherwise},$$

which gives $\binom{n}{k} = 0$ for $k > n$, $n \in \mathbb{N}$.

2. Properties of Bipartitional Polynomials

In this section, we give some recurrence relations for the bipartitional polynomials from which we deduce connections with Bell polynomials. In the following, we consider a sequence $(x_{n,m})$ of real numbers with $x_{0,0} = 0$.

Theorem 1. *We start with*

$$\sum_{i+j\leq m+n-k+1}\binom{m}{i}\binom{n}{j}x_{i,j}A_{m-i,n-j,k-1}\left(x_{I,J}\right)=kA_{m,n,k}\left(x_{I,J}\right), \tag{10}$$

$$\sum_{i+j\leq m+n-k+1}\binom{m}{i}\binom{n}{j}ix_{i,j}A_{m-i,n-j,k-1}\left(x_{I,J}\right)=mA_{m,n,k}\left(x_{I,J}\right), \tag{11}$$

$$\sum_{i+j\leq m+n-k+1}\binom{m}{i}\binom{n}{j}jx_{i,j}A_{m-i,n-j,k-1}\left(x_{I,J}\right)=nA_{m,n,k}\left(x_{I,J}\right), \tag{12}$$

and

$$\sum_{i,j}\binom{m}{i}\binom{n}{j}ix_{i,j}A_{m-i,n-j}\left(x_{I,J}\right)=mA_{m,n}\left(x_{I,J}\right), \tag{13}$$

$$\sum_{i,j}\binom{m}{i}\binom{n}{j}jx_{i,j}A_{m-i,n-j}\left(x_{I,J}\right)=nA_{m,n}\left(x_{I,J}\right). \tag{14}$$

Proof. From (8), we obtain

$$\frac{d}{dx_{r,s}}\frac{1}{k!}\left(\sum_{i+j\geq 1}x_{i,j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^k=\frac{1}{(k-1)!}\left(\sum_{i+j\geq 1}x_{i,j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^{k-1}\frac{t^r}{r!}\frac{u^s}{s!}$$

$$\begin{aligned}
&= \sum_{m+n\geq k-1}\binom{m+r}{r}\binom{n+s}{s}A_{m,n,k-1}\left(x_{I,J}\right)\frac{t^{m+r}}{(m+r)!}\frac{u^{n+s}}{(n+s)!}\\
&= \sum_{m+n\geq r+s+k-1}\binom{m}{r}\binom{n}{s}A_{m-r,n-s,k-1}\left(x_{I,J}\right)\frac{t^m}{m!}\frac{u^n}{n!},
\end{aligned}$$

we also have

$$\frac{d}{dx_{r,s}}\frac{1}{k!}\left(\sum_{i+j\geq 1}x_{i,j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^k=\sum_{m+n\geq k}\frac{d}{dx_{r,s}}A_{m,n,k}\left(x_{I,J}\right)\frac{t^m}{m!}\frac{u^n}{n!}.$$

From these two expressions, we deduce that

$$\frac{d}{dx_{r,s}}A_{m,n,k}\left(x_{I,J}\right)=\binom{m}{r}\binom{n}{s}A_{m-r,n-s,k-1}\left(x_{I,J}\right). \tag{15}$$

For $\alpha=\beta=1$, we take the derivatives on both sides of (5) with respect to γ and, by means of (15), we obtain (10). The same goes for identities (11) and (12). When we sum over all possible values of k for both sides of each of the identities (11) and (12), we obtain identities (13) and (14) respectively. □

The next theorem provides the link between the bipartitional polynomials and the classical Bell polynomials.

Theorem 2. *We have*

$$\sum_{j=0}^{n}\binom{n}{j}A_{j,n-j,k}\left(x_{I,J}\right)=B_{n,k}\left(\sum_{i=0}^{j}\binom{j}{i}x_{i,j-i}\right), \tag{16}$$

$$A_{0,n,k}\left(x_{I,J}\right)=B_{n,k}\left(x_{0,j}\right), \tag{17}$$

$$A_{m,0,k}\left(x_{I,J}\right)=B_{m,k}\left(x_{j,0}\right), \tag{18}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}A_{j,n-j}\left(x_{I,J}\right)=B_{n}\left(\sum_{i=0}^{j}\binom{j}{i}x_{i,j-i}\right), \tag{19}$$

$$A_{0,n}\left(x_{I,J}\right)=B_{n}\left(x_{0,j}\right), \tag{20}$$

$$A_{m,0}\left(x_{I,J}\right)=B_{m}\left(x_{j,0}\right). \tag{21}$$

Proof. From (8), if we set $t=u$, we obtain

$$\frac{1}{k!}\left(\sum_{i,j\geq 0}x_{i,j}\frac{u^{i+j}}{i!j!}\right)^{k}=\sum_{m+n\geq k}A_{m,n,k}\left(x_{I,J}\right)\frac{u^{m+n}}{m!n!},$$

i.e.

$$\frac{1}{k!}\left(\sum_{s=1}^{\infty}\frac{u^{s}}{s!}\sum_{i=0}^{s}\binom{s}{i}x_{i,s-i}\right)^{k}=\sum_{s\geq k}\frac{u^{s}}{s!}\sum_{m=0}^{s}\binom{s}{m}A_{m,s-m,k}\left(x_{I,J}\right),$$

which, by identification, gives the identity (16). The identities (17) and (18) result by setting respectivelly $t=0$ and $u=0$ in (8).
Summing over all possible values of k on both sides of each of the identities (16), (17) and (18), we obtain identities (19), (20) and (21) respectively. □

For a sequence of real numbers (x_n), we denote by $A_{m,n,k}\left(x_{i+j}\right)$ the quantity obtained by replacing $x_{i,j}$ by x_{i+j} in $A_{m,n,k}\left(x_{i,j}\right)$. The same goes for $A_{m,n}\left(x_{i+j}\right)$.

Theorem 3. *Let (x_n) be a sequence of real numbers. The following holds*

$$A_{m,n,k}\left(x_{i+j}\right)=B_{m+n,k}\left(x_{j}\right), \tag{22}$$

$$A_{m,n}\left(x_{i+j}\right)=B_{m+n}\left(x_{j}\right), \tag{23}$$

$$A_{m+k,n,k}\left(ix_{i+j}\right)=\binom{m+n}{n}\binom{m+n+k}{n}^{-1}B_{m+n+k,k}\left(jx_{j}\right), \tag{24}$$

$$A_{m,n+k,k}\left(jx_{i+j}\right)=\binom{m+n}{m}\binom{m+n+k}{m}^{-1}B_{m+n+k,k}\left(jx_{j}\right). \tag{25}$$

Proof. From (8), we get for $x_{i,j} = x_{i+j}$ that

$$\frac{1}{k!}\left(\sum_{i+j\geq 1} x_{i+j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^k = \sum_{m+n\geq k} A_{m,n,k}\left(x_{i+j}\right)\frac{t^m}{m!}\frac{u^n}{n!};$$

we also get

$$\begin{aligned}
\frac{1}{k!}\left(\sum_{i+j\geq 1} x_{i+j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^k &= \frac{1}{k!}\left(\sum_{s\geq 1}\frac{x_s}{s!}\left(t+u\right)^s\right)^k \\
&= \sum_{s\geq k} B_{s,k}\left(x_j\right)\frac{\left(t+u\right)^s}{s!} \\
&= \sum_{m+n\geq k} B_{m+n,k}\left(x_j\right)\frac{t^m}{m!}\frac{u^n}{n!}.
\end{aligned}$$

By identification, we obtain (22). Identity (23) follows by summing over all possible values of k on both sides of identity (22).We have

$$\begin{aligned}
\frac{1}{k!}\left(\sum_{i+j\geq 1} i x_{i+j}\frac{t^i}{i!}\frac{u^j}{j!}\right)^k &= \frac{1}{k!}\left(\sum_{s\geq 1}\frac{x_s}{s!}\sum_{i+j=s} i\frac{t^i}{i!}\frac{u^j}{j!}\right)^k \\
&= \frac{t^k}{k!}\left(\sum_{s\geq 1}\frac{x_s}{s!}\frac{d}{dt}\left(t+u\right)^s\right)^k \\
&= \frac{1}{k!}\left(\frac{t}{t+u}\right)^k\left(\sum_{s\geq 1} s x_s\frac{\left(t+u\right)^s}{s!}\right)^k \\
&= t^k\sum_{r\geq k} B_{r,k}\left(j x_j\right)\frac{\left(t+u\right)^{r-k}}{r!} \\
&= \sum_{m+n\geq k}\frac{\binom{m+n-k}{m-k}}{\binom{m+n}{n}} B_{m+n-k,k}\left(j x_j\right)\frac{t^m}{m!}\frac{u^n}{n!}.
\end{aligned}$$

Then, by identification with (8) and making use of the symmetry of m, n, we obtain (24) and (25). □

Note that, for Theorem 3, using the identities on partial and complete Bell polynomials given in [3] and [4], we can deduce several identities for the partial and complete bipartitional polynomials.

3. Polynomial Sequences of Trinomial Type

We know that the sequences of binomial type have a strong relationship with the partial and complete Bell polynomials. If $(f_n(x))$ is a sequence of binomial type, the sequence $(h_n(x))$ defined by $h_n(x) := \frac{x}{an+x} f_n(an+x)$ is also of binomial type. For sequences $(f_n(x))$ satisfying the property of convolution, the following is holds: $f_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y)$ with $f_0(x) \neq 1$.

Theorem 4. *Let $(f_{m,n}(x))$ be a sequence of trinomial type and let a, b be real numbers. Then the sequence $(h_{m,n}(x))$ defined by*

$$h_{m,n}(x) := \frac{x}{am+bn+x} f_{m,n}(am+bn+x) \tag{26}$$

is of trinomial type.

Proof. The definition of $(f_{m,n}(x))$, given in (9), states that

$$\left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} f_{i,j}(1)\frac{u^j}{j!}\right)\frac{t^i}{i!}\right)^x = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} f_{m,n}(x)\frac{u^n}{n!}\right)\frac{t^m}{m!},$$

which can also be written as follows:

$$\left(\sum_{i=0}^{\infty} u_i(1)\frac{t^i}{i!}\right)^x = \sum_{m=0}^{\infty} u_m(x)\frac{t^m}{m!} \quad \text{with } u_m(x) := \sum_{n=0}^{\infty} f_{m,n}(x)\frac{u^n}{n!}.$$

This implies that the sequence $(u_m(x))$ satisfies the property of convolution. Hence, the sequence $(U_m(x))$ given by

$$U_m(x) := \frac{x}{am+x} u_m(am+x) = \frac{x}{am+x}\sum_{n=0}^{\infty} f_{m,n}(am+x)\frac{u^n}{n!},$$

has the same property of convolution. We have $\left(\sum_{i=0}^{\infty} U_i(1)\frac{t^i}{i!}\right)^x = \sum_{m=0}^{\infty} U_m(x)\frac{t^m}{m!}$, or similarly

$$\left(\sum_{i=0}^{\infty} v_i(1)\frac{u^i}{i!}\right)^x = \sum_{n=0}^{\infty} v_n(x)\frac{u^n}{n!}, \quad \text{with } v_n(x) := \sum_{m=0}^{\infty}\frac{x}{am+x} f_{m,n}(am+x)\frac{t^m}{m!}.$$

This implies that the sequence $(v_m(x))$ satisfies the property of convolution. Moreover, the sequence $(V_m(x))$ given by

$$V_n(x) := \frac{x}{bn+x} v_n(bn+x) = \sum_{m=0}^{\infty}\frac{x}{am+bn+x} f_{m,n}(am+bn+x)\frac{t^m}{m!},$$

has the same property of convolution. We have $\left(\sum\limits_{j=0}^{\infty} V_j\left(1\right)\frac{u^j}{j!}\right)^x = \sum\limits_{n=0}^{\infty} V_n\left(x\right)\frac{u^n}{n!}$, or similarly

$$\left(\sum_{i,j\geq 0} h_{i,j}\left(1\right)\frac{t^i}{i!}\frac{u^j}{j!}\right)^x = \sum_{m,n\geq 0} h_{m,n}\left(x\right)\frac{t^m}{m!}\frac{u^n}{n!} \quad with \quad h_{0,0}(x) = 1.$$

Consequently, the last identity shows that the sequence $(h_{m,n}\left(x\right))$ is of trinomial type. □

Remark. From (9), we can verify that if a sequence $(f_{m,n}\left(x\right))$ is of trinomial type, then

$$f_{m,n}\left(x+y\right) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j} f_{i,j}\left(x\right) f_{m-i,n-j}\left(y\right).$$

If in (9) we set $t = 0$, we deduce that the sequence $(f_n\left(x\right))$, defined by $f_n\left(x\right) := f_{0,n}\left(x\right)$, is of binomial type.

Theorem 5. *Let $(f_n\left(x\right))$ and $(g_n\left(x\right))$ be two sequences of binomial type with $f_0\left(x\right) = g_0\left(x\right) = 1$. Then the sequences $(p_{m,n}\left(x\right))$ and $(q_{m,n}\left(x\right))$ defined by*

$$p_{m,n}\left(x\right) := f_{m+n}\left(x\right),$$
$$q_{m,n}\left(x\right) := f_m\left(x\right) g_n\left(x\right)$$

are sequences of trinomial type.

Proof. We have $p_{0,0}\left(x\right) = q_{0,0}\left(x\right) = 1$ and

$$\begin{aligned}\sum_{m,n\geq 0} p_{m,n}\left(x\right)\frac{t^m}{m!}\frac{u^n}{n!} &= \sum_{k\geq 0}\frac{f_k\left(x\right)}{k!}\sum_{m=0}^{k}\binom{k}{m}t^m u^{k-m}\\ &= \sum_{k\geq 0} f_k\left(x\right)\frac{\left(t+u\right)^k}{k!}\\ &= \left(\sum_{k\geq 0} f_k\left(1\right)\frac{\left(t+u\right)^k}{k!}\right)^x,\end{aligned}$$

$$\begin{aligned}\sum_{m,n\geq 0} q_{m,n}\left(x\right)\frac{t^m}{m!}\frac{u^n}{n!} &= \left(\sum_{m\geq 0} f_m\left(x\right)\frac{t^m}{m!}\right)\left(\sum_{n\geq 0} g_n\left(x\right)\frac{u^n}{n!}\right)\\ &= \left(\left(\sum_{m\geq 0} f_m\left(1\right)\frac{t^m}{m!}\right)\left(\sum_{n\geq 0} g_n\left(1\right)\frac{u^n}{n!}\right)\right)^x.\end{aligned}$$

□

Remark. Let $(f_n(x))$ and $(g_n(x))$ be two sequences of binomial type with $f_0(x) = g_0(x) = 1$. Then, from [3], the sequences $(F_n(x))$ and $(G_n(x))$ defined by

$$F_n(x) := \frac{x}{(c-a)n+x} f_n((c-a)n+x),$$

and

$$G_n(x) := \frac{x}{(d-b)n+x} g_n((d-b)n+x),$$

are of binomial type for all real numbers a, b, c, d. This implies, according to Theorem 4, that, for all a, b, c, d, the sequences

$$P_{m,n}(x) := x\frac{f_{m+n}(am+bn+x)}{am+bn+x},$$

and

$$Q_{m,n}(x) := x(am+bn+x)\frac{f_m(cm+bn+x)}{cm+bn+x}\frac{g_n(am+dn+x)}{am+dn+x},$$

are of trinomial type.

Example 6. We know that sequences (x^n), $\left(n!\binom{x}{n}\right)$ and $(B_n(x))$ are of binomial type, where $B_n(x)$ is the single variable Bell polynomial. The sequences

$$x(an+x)^{n-1}, \quad \frac{n!x}{an+x}\binom{an+x}{n}, \quad \frac{x}{an+x}B_n(an+x),$$

are also of binomial type.
Combining these sequences, Theorem 5 states that the following sequences are also of trinomial type:

$$x^2(cm+x)^{m-1}(dn+x)^{n-1},$$
$$n!\frac{x^2(cm+x)^{m-1}}{dn+x}\binom{dn+x}{n},$$
$$m!n!\frac{x^2}{(cm+x)(dn+x)}\binom{cm+x}{m}\binom{dn+x}{n},$$

$$\frac{x^2(cm+x)^{m-1}}{dn+x}B_n(dn+x),$$
$$m!\frac{x^2}{(cm+x)(dn+x)}\binom{cm+x}{m}B_n(dn+x),$$
$$(m+n)!\binom{x}{m+n},$$
$$B_{m+n}(x).$$

Combining these sequences and using Theorem 4, we deduce that the following sequences are of trinomial type:

$$a_{m,n}(x) = x(am+bn+x)(cm+bn+x)^{m-1}(am+dn+x)^{n-1},$$
$$b_{m,n}(x) = n!\frac{x(am+bn+x)(cm+bn+x)^{m-1}}{am+dn+x}\binom{am+dn+x}{n},$$
$$c_{m,n}(x) = m!n!\frac{x(am+bn+x)}{(cm+bn+x)(am+dn+x)}\binom{cm+bn+x}{m}\binom{am+dn+x}{n},$$
$$d_{m,n}(x) = \frac{x(am+bn+x)(cm+bn+x)^{m-1}}{am+dn+x}B_n(am+dn+x),$$
$$e_{m,n}(x) = m!\frac{x(am+bn+x)}{(cm+bn+x)(am+dn+x)}\binom{cm+bn+x}{m}B_n(am+dn+x),$$
$$f_{m,n}(x) = \frac{(m+n)!x}{am+bn+x}\binom{am+bn+x}{m+n},$$
$$g_{m,n}(x) = \frac{x}{am+bn+x}B_{m+n}(am+bn+x).$$

4. Bipartitional Polynomials and Polynomials of Binomial and Trinomial Type

Roman [7, p. 82] proves that any polynomial sequence of binomial type $(f_n(x))$, with $f_0(x) = 1$, is connected with the partial Bell polynomials

$$f_n(x) = \sum_{k=1}^{n} B_{n,k}(D_{\alpha=0}f_j(\alpha))x^k = B_n(xD_{\alpha=0}f_j(\alpha)), \quad n \geq 1. \tag{27}$$

Similarly, we establish that any polynomial sequence of trinomial type $(f_{m,n}(x))$, with $f_{0,0}(x) = 1$, admits a connection with the partial bipartitional Bell polynomials.

Theorem 7. *Let $(f_{m,n}(x))$ be a sequence of trinomial type. Then*

$$f_{m,n}(x) = \sum_{k=1}^{m+n} A_{m,n,k}(D_{\alpha=0}f_{i,j}(\alpha))x^k = A_{m,n}(xD_{\alpha=0}f_{i,j}(\alpha)), \quad m+n \geq 1, \tag{28}$$

where $D_{\alpha=0} \equiv \left.\frac{d}{d\alpha}\right|_{\alpha=0}$.

Proof. Since $D_{\alpha=0}f_{0,0}(\alpha)=0$, then for $x_{i,j}=D_{\alpha=0}f_{i,j}(\alpha)$, we have

$$\begin{aligned}\sum_{m,n\geq 0}\frac{t^m}{m!}\frac{u^n}{n!}\sum_{k=0}^{m+n}A_{m,n,k}(x_{i,j})\,x^k &= \sum_{k=0}^{\infty}x^k\sum_{m+n\geq k}A_{m,n,k}(x_{i,j})\frac{t^m}{m!}\frac{u^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{x^k}{k!}\left(\sum_{i,j\geq 0}D_{\alpha=0}f_{i,j}(\alpha)\frac{t^i}{i!}\frac{u^j}{j!}\right)^k\\ &= \exp\left(xD_{\alpha=0}\sum_{i,j\geq 0}f_{i,j}(\alpha)\frac{t^i}{i!}\frac{u^j}{j!}\right)\\ &= \exp\left(xD_{\alpha=0}\left(\sum_{i,j\geq 0}f_{i,j}(1)\frac{t^i}{i!}\frac{u^j}{j!}\right)^{\alpha}\right)\\ &= \exp\left(x\ln\left(\sum_{i,j\geq 0}f_{i,j}(1)\frac{t^i}{i!}\frac{u^j}{j!}\right)\right)\\ &= \left(\sum_{i,j\geq 0}f_{i,j}(1)\frac{t^i}{i!}\frac{u^j}{j!}\right)^x\\ &= \sum_{m,n\geq 0}f_{m,n}(x)\frac{t^m}{m!}\frac{u^n}{n!}.\end{aligned}$$

Therefore, the desired identity follows by identification. □

Some identities related to these polynomials are given by the following theorem.

Theorem 8. *Let $(f_{m,n}(x))$ be a sequence of trinomial type. We have*

$$A_{m+k,n+k,k}\left(xij\frac{f_{i-1,j-1}(a(i-1)+b(j-1)+x)}{a(i-1)+b(j-1)+x}\right) = k!kx\binom{m+k}{k}\binom{n+k}{k}\frac{f_{m,n}(am+bn+kx)}{am+bn+kx}. \quad (29)$$

Proof. We have

$$\sum_{m+n\geq k}A_{m,n,k}(ijf_{i-1,j-1}(x))\frac{t^m}{m!}\frac{t^n}{n!}$$

$$
\begin{aligned}
&= \frac{1}{k!}\left(\sum_{i+j\geq 1} ijf_{i-1,j-1}(x)\frac{t^i}{i!}\frac{u^j}{j!}\right)^k\\
&= \frac{t^k u^k}{k!}\left(\sum_{i,j\geq 0} f_{i,j}(x)\frac{t^i}{i!}\frac{u^j}{j!}\right)^k\\
&= \frac{t^k u^k}{k!}\sum_{m,n\geq 0} f_{m,n}(kx)\frac{t^m}{m!}\frac{u^n}{n!}\\
&= \frac{1}{k!}\sum_{m,n\geq 0}\frac{(m+k)!}{m!}\frac{(n+k)!}{n!}f_{m,n}(kx)\frac{t^{m+k}}{(m+k)!}\frac{u^{n+k}}{(n+k)!}\\
&= k!\sum_{m,n\geq k}\binom{m}{k}\binom{n}{k}f_{m-k,n-k}(kx)\frac{t^m}{m!}\frac{u^n}{n!},
\end{aligned}
$$

which gives

$$
A_{m,n,k}\left(ijf_{i-1,j-1}(x)\right) = k!\binom{m}{k}\binom{n}{k}f_{m-k,n-k}(kx), \quad m,n\geq k. \tag{30}
$$

To finish the proof, we substitute the trinomial type polynomials $(h_{m,n}(x))$ given by (26), instead of $(f_{m,n}(x))$, and replace m by $m+k$ and n by $n+k$. □

Example 9. From relation (29), for $r = i-1$ and $s = j-1$, we have

1. $f_{m,n}(x) = (m+n)!\binom{x}{m+n}$,

   $A_{m+k,n+k,k}\left(x\frac{(r+1)(s+1)(r+s)!}{ar+bs+x}\binom{ar+bs+x}{r+s}\right) = x\frac{\binom{m+n}{m}}{(k-1)!}\frac{(m+k)!(n+k)!}{am+bn+kx}\binom{am+bn+kx}{m+n}$.

2. $f_{m,n}(x) = B_{m+n}(x)$,

   $A_{m+k,n+k,k}\left(x(r+1)(s+1)\frac{B_{r+s}(ar+bs+x)}{ar+bs+x}\right) = x\frac{(m+k)!(n+k)!}{(k-1)!m!n!}\frac{B_{m+n}(am+bn+kx)}{am+bn+kx}$.

3. $f_{m,n}(x) = x^2\left((c-a)m+x\right)^{m-1}\left((d-b)n+x\right)^{n-1}$,

   $A_{m+k,n+k,k}\left(x(r+1)(s+1)(ar+bs+x)(cr+bs+x)^{r-1}(ar+ds+x)^{s-1}\right)$

   $= k!kx\binom{m+k}{k}\binom{n+k}{k}\frac{(am+bn+kx)(cm+bn+kx)^{m-1}}{(am+dn+kx)^{1-n}}$.

4. $f_{m,n}(x) = \frac{m!n!x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}\binom{(d-b)n+x}{n}$,

   $A_{m+k,n+k,k}\left(x\frac{(r+1)!(s+1)!(ar+bs+x)}{(cr+bs+x)(ar+ds+x)}\binom{cr+bs+x}{r}\binom{ar+ds+x}{s}\right)$

   $= \frac{x(m+k)!(n+k)!(am+bn+kx)}{(k-1)!(cm+bn+kx)(am+dn+kx)}\binom{cm+bn+kx}{m}\binom{am+dn+kx}{n}$.

5. $f_{m,n}(x) = n!\frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}\binom{(d-b)n+x}{n}$,

$A_{m+k,n+k,k}\left(\frac{x(r+1)(s+1)!(ar+bs+x)(cr+bs+x)^{i-2}}{(ar+ds+x)}\binom{ar+ds+x}{s}\right)$
$= \frac{x(m+k)!(n+k)!(am+bn+kx)(cm+bn+kx)^{m-1}}{(k-1)!m!(am+dn+kx)}\binom{am+dn+kx}{n}$.

6. $f_{m,n}(x) = \frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}B_n((d-b)n+x)$,

$A_{m+k,n+k,k}\left(\frac{x(r+1)(s+1)(ar+bs+x)(cr+bs+x)^{r-1}}{(ar+ds+x)}B_s(ar+ds+x)\right)$
$= \frac{k!kx(am+bn+kx)}{(am+dn+kx)(cm+bn+kx)^{1-m}}\binom{m+k}{k}\binom{n+k}{k}B_n(am+dn+kx)$.

7. $f_{m,n}(x) = m!\frac{x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}B_n((d-b)n+x)$,

$A_{m+k,n+k,k}\left(\frac{x(s+1)(r+1)!(ar+bs+x)B_s(ar+ds+x)}{(cr+bs+x)(ar+ds+x)}\binom{cr+bs+x}{r}\right)$
$= \frac{x(m+k)!(n+k)!(am+bn+kx)B_n(am+dn+kx)}{(k-1)!n!(cm+bn+kx)(am+dn+kx)}\binom{cm+bn+kx}{m}$.

Theorem 10. *Let $(f_{m,n}(x))$ be a sequence of trinomial type. We have*

$$A_{m+k,n,k}\left(xi\frac{f_{i-1,j}(a(i-1)+bj+x)}{a(i-1)+bj+x}\right) = xk\binom{m+k}{k}\frac{f_{m,n}(am+bn+kx)}{am+bn+kx}, \quad (31)$$

$$A_{m,n+k,k}\left(xj\frac{f_{i,j-1}(ai+b(j-1)+x)}{ai+b(j-1)+x}\right) = xk\binom{n+k}{k}\frac{f_{m,n}(am+bn+kx)}{am+bn+kx}. \quad (32)$$

Proof. Similarly to the proof in Theorem 8, we obtain

$$A_{m,n,k}(if_{i-1,j}(x)) = \binom{m}{k}f_{m-k,n}(kx), \quad m \geq k,\ n \geq 0,$$
$$A_{m,n,k}(jf_{i,j-1}(x)) = \binom{n}{k}f_{m,n-k}(kx), \quad m \geq 0,\ n \geq k.$$

It will be sufficient to use the trinomial type polynomials $(h_{m,n}(x))$ given in (26) instead of $(f_{m,n}(x))$ and replace m by $m+k$ and n by $n+k$ respectively. □

Example 11. From relations (31) and (32), we have the following:

1. $f_{m,n}(x) = (m+n)!\binom{x}{m+n}$,

$A_{m+k,n,k}\left(xi\frac{(i+j-1)!}{a(i-1)+bj+x}\binom{a(i-1)+bj+x}{i+j-1}\right) = \frac{kx(m+n)!}{am+bn+kx}\binom{m+k}{k}\binom{am+bn+kx}{m+n}$,

$A_{m,n+k,k}\left(xj\frac{(i+j-1)!}{ai+b(j-1)+x}\binom{ai+b(j-1)+x}{i+j-1}\right) = \frac{kx(m+n)!}{am+bn+kx}\binom{n+k}{k}\binom{am+bn+kx}{m+n}$.

2. $f_{m,n}(x) = B_{m+n}(x)$,

$$A_{m+k,n,k}\left(xi\frac{B_{i+j-1}(a(i-1)+bj+x)}{a(i-1)+bj+x}\right) = xk\binom{m+k}{k}\frac{Bm+n(am+bn+kx)}{am+bn+kx},$$

$$A_{m,n+k,k}\left(xj\frac{B_{i+j-1}(ai+b(j-1)+x)}{ai+b(j-1)+x}\right) = xk\binom{n+k}{k}\frac{B_{m+n}(am+bn+kx)}{am+bn+kx}.$$

3. $f_{m,n}(x) = x^2\left((c-a)m+x\right)^{m-1}\left((d-b)n+x\right)^{n-1}$,

$$A_{m+k,n,k}\left(xi\left(a(i-1)+bj+x\right)\left(c(i-1)+bj+x\right)^{i-2}\left(a(i-1)+dj+x\right)^{j-2}\right)$$
$$= xk\binom{m+k}{k}\left(am+bn+kx\right)\left(cm+bn+kx\right)^{m-1}\left(am+dn+kx\right)^{n-1},$$

$$A_{m,n+k,k}\left(xj\left(ai+b(j-1)+x\right)\left(ci+b(j-1)+x\right)^{i-2}\left(ai+d(j-1)+x\right)^{j-2}\right)$$
$$= xk\binom{n+k}{k}\left(am+bn+kx\right)\left(cm+bn+kx\right)^{m-1}\left(am+dn+kx\right)^{n-1}.$$

4. $f_{m,n}(x) = m!n!\frac{x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}\binom{(d-b)n+x}{n}$,

$$A_{m+k,n,k}\left(\frac{xi!(j-1)!(a(i-1)+bj+x)}{(c(i-1)+bj+x)(a(i-1)+dj+x)}\binom{c(i-1)+bj+x}{i-1}\binom{a(i-1)+dj+x}{j}\right)$$
$$= x\frac{(m+k)!n!}{(k-1)!}\frac{am+bn+kx}{(cm+bn+kx)(am+dn+kx)}\binom{cm+bn+kx}{m}\binom{am+dn+kx}{n},$$

$$A_{m+k,n,k}\left(\frac{x(i-1)!j!(ai+b(j-1)+x)}{(ci+b(j-1)+x)(ai+d(j-1)+x)}\binom{ci+b(j-1)+x}{i}\binom{ai+d(j-1)+x}{j-1}\right)$$
$$= x\frac{m!(n+k)!}{(k-1)!}\frac{am+bn+kx}{(cm+bn+kx)(am+dn+kx)}\binom{cm+bn+kx}{m}\binom{am+dn+kx}{n}.$$

5. $f_{m,n}(x) = n!\frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}\binom{(d-b)n+x}{n}$,

$$A_{m+k,n,k}\left(\frac{xij!(a(i-1)+bj+x)}{(a(i-1)+dj+x)(c(i-1)+bj+x)^{2-i}}\binom{a(i-1)+dj+x}{j}\right)$$
$$= \frac{x(m+k)!n!(am+bn+kx)(cm+bn+kx)^{m-1}}{(k-1)!m!(am+dn+kx)}\binom{am+dn+kx}{n},$$

$$A_{m,n+k,k}\left(\frac{xi!j(ai+b(j-1)+x)}{(ai+d(j-1)+x)(ci+b(j-1)+x)^{2-j}}\binom{ai+d(j-1)+x}{j-1}\right)$$
$$= \frac{xm!(n+k)!(am+bn+kx)(cm+bn+kx)^{m-1}}{(k-1)!n!(am+dn+kx)}\binom{am+dn+kx}{n}.$$

6. $f_{m,n}(x) = \frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}B_n\left((d-b)n+x\right)$,

$$A_{m+k,n,k}\left(\frac{xi(a(i-1)+bj+x)(c(i-1)+bj+x)^{i-2}}{a(i-1)+dj+x}B_j\left(a(i-1)+dj+x\right)\right)$$
$$= xk\binom{m+k}{k}\frac{(am+bn+kx)(cm+bn+kx)^{m-1}}{am+dn+kx}B_n\left(am+dn+kx\right),$$

$$A_{m,n+k,k}\left(\frac{xj(ai+b(j-1)+x)(ci+b(j-1)+x)^{i-1}}{ai+d(j-1)+x}B_{j-1}\left(ai+d(j-1)+x\right)\right)$$
$$= xk\binom{n+k}{k}\frac{(am+bn+kx)(cm+bn+kx)^{m-1}}{am+dn+kx}B_n\left(am+dn+kx\right).$$

7. $f_{m,n}(x) = m!\frac{x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}B_n((d-b)n+x)$,

$$A_{m+k,n,k}\left(\tfrac{xi!(a(i-1)+bj+x)B_j(a(i-1)+dj+x)}{(c(i-1)+bj+x)(a(i-1)+dj+x)}\tbinom{c(i-1)+bj+x}{i-1}\right)$$
$$= \tfrac{x(m+k)!(am+bn+kx)}{(k-1)!(cm+bn+kx)(am+dn+kx)}\tbinom{cm+bn+kx}{m}B_n(am+dn+kx),$$
$$A_{m,n+k,k}\left(\tfrac{xj!(ai+b(j-1)+x)B_{j-1}(ai+d(j-1)+x)}{(ci+b(j-1)+x)(ai+d(j-1)+x)}\tbinom{ci+b(j-1)+x}{i}\right)$$
$$= \tfrac{x(n+k)!(am+bn+kx)}{(k-1)!(cm+bn+kx)(am+dn+kx)}\tbinom{cm+bn+kx}{m}B_n(am+dn+kx).$$

Theorem 12. *Let $(f_{m,n}(x))$ be a sequence of trinomial type. We have*

$$A_{m,n}\left(\frac{x}{ai+bj}f_{i,j}(ai+bj)\right) = \frac{x}{am+bn+x}f_{m,n}(am+bn+x). \tag{33}$$

Proof. From (28), we get

$$A_{m,n}(xD_{\alpha=0}f_{i,j}(\alpha)) = f_{m,n}(x),$$

which gives the desired identity by replacing $f_{m,n}(x)$ by

$$h_{m,n}(x) := \frac{x}{am+bn+x}f_{m,n}(am+bn+x).$$

given by (26). □

Example 13. From relation (33), we have the following:

1. $f_{m,n}(x) = (m+n)!\binom{x}{m+n}$,

$$A_{m,n}\left(x\frac{(i+j)!}{ai+bj}\binom{ai+bj}{i+j}\right) = x\frac{(m+n)!}{am+bn+x}\binom{(am+bn+x)}{m+n}.$$

2. $f_{m,n}(x) = B_{m+n}(x)$,

$$A_{m,n}\left(\frac{x}{ai+bj}B_{i+j}(ai+bj)\right) = \frac{x}{am+bn+x}B_{m+n}(am+bn+x).$$

3. $f_{m,n}(x) = x^2((c-a)m+x)^{m-1}((d-b)n+x)^{n-1}$,

$$A_{m,n}\left(x(ai+bj)(ci+bj)^{i-1}(ai+dj)^{j-1}\right)$$
$$= x(am+bn+x)(cm+bn+x)^{m-1}(am+dn+x)^{n-1}.$$

4. $f_{m,n}(x) = m!n!\frac{x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}\binom{(d-b)n+x}{n}$,

$$A_{m,n}\left(xi!j!\frac{(ai+bj)}{(ci+bj)(ai+dj)}\binom{ci+bj}{i}\binom{ai+dj}{j}\right)$$
$$= xm!n!\frac{(am+bn+x)}{(cm+bn+x)(am+dn+x)}\binom{cm+bn+x}{m}\binom{am+dn+x}{n}.$$

5. $f_{m,n}(x) = n!\frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}\binom{(d-b)n+x}{n}$,

$$A_{m,n}\left(xj!\frac{(ai+bj)(ci+bj)^{i-1}}{ai+dj}\binom{ai+dj}{j}\right) = xn!\frac{(am+bn+x)(cm+bn+x)^{m-1}}{am+dn+x}\binom{am+dn+x}{n}.$$

6. $f_{m,n}(x) = \frac{x^2((c-a)m+x)^{m-1}}{(d-b)n+x}B_n((d-b)n+x)$,

$$A_{m,n}\left(\frac{x(ai+bj)(ci+bj)^{i-1}}{ai+dj}B_j(ai+dj)\right) = \frac{x(am+bn+x)(cm+bn+x)^{m-1}}{am+dn+x}B_n(am+dn+x).$$

7. $f_{m,n}(x) = m!\frac{x^2}{((c-a)m+x)((d-b)n+x)}\binom{(c-a)m+x}{m}B_n((d-b)n+x)$,

$$A_{m,n}\left(xi!\frac{(ai+bj)\binom{ci+bj}{i}}{(ci+bj)(ai+dj)}B_j(ai+dj)\right) = m!\frac{(am+bn+x)\binom{cm+bn+x}{m}}{(cm+bn+x)(am+dn+x)}B_n(am+dn+x).$$

Remark. For $m = 0$, identity (17) and Theorem 10 give Proposition 1 given in [3], and identity (20) and Theorem 12 give Proposition 3 given in [3].

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