NORMALITY, PROJECTIVE NORMALITY AND EGZ THEOREM

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Received: 5/18/09, Revised: 1/6/11, Accepted: 2/20/11, Published: 4/16/11

Abstract

In this note, we prove that the projective normality of $(\mathbb{P}(V)/G, \mathcal{L})$, the celebrated theorem of Erdös-Ginzburg-Ziv and normality of an affine semigroup are all equivalent, where V is a finite dimensional representation of a finite cyclic group G over $\mathbb{C}$ and $\mathcal{L}$ is the descent of the line bundle $\mathcal{O}(1)^{\otimes|G|}$.

1. Introduction

Let V be a finite dimensional representation of a finite cyclic group G over the field of complex numbers $\mathbb{C}$. Let $\mathcal{L}$ denote the descent of the line bundle $\mathcal{O}(1)^{\otimes|G|}$ to the GIT quotient $\mathbb{P}(V)/G$. In [4], it is shown that $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal. Proof of this uses the well known arithmetic result due to Erdös-Ginzburg-Ziv (see [2]).

In this note, we prove that the projective normality of $(\mathbb{P}(V)/G, \mathcal{L})$, the Erdös-Ginzburg-Ziv theorem and normality of an affine semigroup are all equivalent.

2. Preliminaries

Normality of a Semigroup: An affine semigroup M is a finitely generated subsemigroup of $\mathbb{Z}^n$ containing 0 for some n. Let N be the subgroup of $\mathbb{Z}^n$ generated by M. Then, M is called normal if it satisfies the following condition: if $kx \in M$ for some $x \in N$ and $k \in \mathbb{N}$, then $x \in M$.

For an affine semigroup M and a field K we can form the affine semigroup algebra $K[M]$ in the following way: as a K-vector space, $K[M]$ has a basis consisting of the symbols X^a, $a \in M$, and the multiplication on $K[M]$ is defined by the K-bilinear extension of $X^a.X^b = X^{a+b}$.

We recall the following theorem from page 141, theorem 4.40 of [1].

Theorem 1. *Let M be an affine semigroup, and K be a field. Then M is normal if and only if $K[M]$ is normal, i.e., it is integrally closed in its field of fractions.*

Projective Normality: A polarized variety $(X, \mathcal{L})$ where $\mathcal{L}$ is a very ample line bundle is said to be projectively normal if its homogeneous coordinate ring $\oplus_{n \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{L}^{\otimes n})$ is integrally closed and is generated as a $\mathbb{C}$-algebra by $H^0(X, \mathcal{L})$ (see Exercise 5.14, Chapter II, Hartshorne [3]).

3. Main Theorem

In this section we will prove our main theorem.

Theorem 2. *The following are equivalent*
1. *Erdös-Ginzburg-Ziv theorem: Let $(a_1, a_2, \cdots, a_m), m \geq 2n-1$ be a sequence of elements of $\mathbb{Z}/n\mathbb{Z}$. Then there exists a subsequence $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ of length n whose sum is zero.*
2. *Let G be a cyclic group of order n and V be any finite dimensional representation of G over $\mathbb{C}$. Let $\mathcal{L}$ be the descent of $\mathcal{O}(1)^{\otimes n}$. Then $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*
$2'$. *Let G be a cyclic group of order n and V be the regular representation of G over $\mathbb{C}$. Let $\mathcal{L}$ be the descent of $\mathcal{O}(1)^{\otimes n}$. Then $(\mathbb{P}(V)/G, \mathcal{L})$ is projectively normal.*
3. *The sub-semigroup M of $\mathbb{Z}^n$ generated by the set $S = \{(m_0, m_1, \cdots, m_{n-1}) \in (\mathbb{Z}_{\geq 0})^n : \sum_{i=0}^{n-1} m_i = n$ and $\sum_{i=0}^{n-1} i m_i \equiv 0 \mod n\}$ is normal.*

Proof. We first prove (1), (2), and ($2'$) are equivalent.

(1) $\Rightarrow$ (2): This follows from the arguments given in page 2, paragraph 6 of [4].

(2) $\Rightarrow$ ($2'$): This is straightforward.

($2'$) $\Rightarrow$ (1): Let $G = \mathbb{Z}/n\mathbb{Z} =< g >$ and let V be the regular representation of G over $\mathbb{C}$. Let ξ be a primitive nth root of unity. Let $\{X_i : i = 0, 1, \cdots, n-1\}$ be a basis of V^* given by:

$$g.X_i = \xi^i X_i, \text{ for every } i = 0, 1, \cdots, n-1.$$

By assumption the algebra $\oplus_{d \in \mathbb{Z}_{\geq 0}} (Sym^{dn} V^*)^G$ is generated by $(Sym^n V^*)^G$ $\quad (*)$

Let $(a_1, a_2, \cdots, a_m), m \geq 2n-1$ be a sequence of elements of G. Consider the subsequence $(a_1, a_2, \cdots, a_{2n-1})$ of length $2n-1$.

Take $a = -(\sum_{i=1}^{2n-1} a_i)$. Then $(\prod_{i=1}^{2n-1} X_{a_i}).X_a$ is a G-invariant monomial of degree $2n$, i.e., $(\prod_{i=1}^{2n-1} X_{a_i}).X_a \in (Sym^{2n}V^*)^G$.

By $(*)$, there exists a subsequence $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ of $(a_1, a_2, \cdots, a_{2n-1}, a)$ of length n such that $\prod_{j=1}^{n} X_{a_{i_j}}$ is G-invariant. So, $\sum_{j=1}^{n} a_{i_j} = 0$. Thus, we have the implication.

We now prove (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2′), which completes the proof of the theorem.

(1) $\Rightarrow$ (3): Let N be the subgroup of $\mathbb{Z}^n$ generated by M. Suppose that $q(m_0, m_1, \ldots, m_{n-1}) \in M$, $q \in \mathbb{N}$ and $(m_0, m_1, \cdots, m_{n-1}) \in N$. We need to show that $(m_0, m_1, \ldots, m_{n-1}) \in M$.

Since $q(m_0, m_1, \cdots, m_{n-1}) \in M$ we have $q.m_i \geq 0 \ \forall\ i$. Hence, $m_i \geq 0 \ \forall\ i$. Since N is the subgroup of $\mathbb{Z}^n$ generated by M and M is the sub-semigroup of $\mathbb{Z}^n$ generated by S, N is generated by S as a subgroup of $\mathbb{Z}^n$. Therefore, the tuple $(m_0, m_1, \cdots, m_{n-1})$ is an integral (not necessarily non-negative) linear combination of elements of S, i.e.,

$$(m_0, m_1, \cdots, m_{n-1}) = \sum_{j=1}^{p} a_j (m_{0,j}, m_{1,j}, \cdots, m_{(n-1),j}),$$

where $a_j \in \mathbb{Z}$ for all $j = 1, 2, \cdots, p$ and $(m_{0,j}, m_{1,j}, \cdots, m_{(n-1),j}) \in S$. Therefore,

$$\sum_{i=0}^{n-1} m_i = \sum_{i=0}^{n-1} \sum_{j=1}^{p} a_j m_{ij} = (\sum_{j=1}^{p} a_j (\sum_{i=0}^{n-1} m_{i,j})) = (\sum_{j=1}^{p} a_j) n = kn$$

for some $k \in \mathbb{Z}$. Moreover $k \geq 0$, since $m_i \geq 0 \ \forall\ i$.

If $k = 1$ then $\sum_{i=0}^{n-1} m_i = n$ and hence, $(m_0, m_1, \cdots, m_{n-1}) \in M$. Otherwise $k \geq 2$ and consider the sequence of integers

$$\underbrace{0, \ldots, 0}_{m_0 \text{ times}},\ \underbrace{1, \ldots, 1}_{m_1 \text{ times}},\ \cdots,\ \underbrace{n-1, \ldots, n-1}_{m_{n-1} \text{ times}}$$

This sequence has atleast $2n$ terms, since $\sum_{i=0}^{n-1} m_i = kn$, $k \geq 2$ and the sum of it's terms is divisible by n by the assumption that $\sum_{i=0}^{n-1} i m_i \equiv 0 \ mod\ n$. So by (1) there exists a subsequence of exactly n terms whose sum is a multiple of n, i.e., there exists $(m'_0, m'_1, \cdots, m'_{n-1}) \in \mathbb{Z}^n_{\geq 0}$ with $m'_i \leq m_i$, $\forall\ i$ such that $\sum_{i=0}^{n-1} m'_i = n$ and $\sum_{i=0}^{n-1} i m'_i$ is a multiple of n. So $(m'_0, m'_1, \cdots, m'_{n-1}) \in M$. Then, by induction $(m_0, m_1, \cdots, m_{n-1}) - (m'_0, m'_1, \cdots, m'_{n-1}) \in M$ and, hence $(m_0, m_1, \cdots, m_{n-1}) \in M$ as required.

(3)$\Rightarrow$(2′): The polarized variety $(\mathbb{P}(V)/G, \mathcal{L})$ is $Proj(\oplus_{d \in \mathbb{Z}_{\geq 0}} (H^0(\mathbb{P}(V), \mathcal{O}(1)^{\otimes d|G|})^G)$ which is the same as $Proj(\oplus_{d \in \mathbb{Z}_{\geq 0}} (Sym^{d|G|} V^*)^G)$. Let $R := \oplus_{d \geq 0} R_d$; $R_d := (Sym^{dn} V^*)^G$. Fix a generator g of G and let ξ be a primitive nth root of unity.

Write $V^* = \oplus_{i=0}^{n-1}\mathbb{C}X_i$, where $\{X_i : i = 0, 1, \cdots, n-1\}$ is a basis of V^* given by: $g.X_i = \xi^i X_i$, for every $i = 0, 1, \cdots, n-1$.

Let R' be the $\mathbb{C}$-subalgebra of $\mathbb{C}[V]$ generated by $R_1 = (Sym^n V^*)^G$. We first note that $\{X_0^{m_0}.X_1^{m_1} \ldots X_{n-1}^{m_{n-1}} : (m_0, m_1, \cdots, m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for R'. We now define the map

$$\Phi : \mathbb{C}[M] \to R' \text{ by extending linearly the map}$$

$$\Phi(X^{(m_0,m_1,\cdots,m_{n-1})}) = X_0^{m_0}.X_1^{m_1} \ldots X_{n-1}^{m_{n-1}} \text{ for } (m_0, m_1, \ldots, m_{n-1}) \in M.$$

Clearly Φ is a homomorphism of $\mathbb{C}$-algebras. Since $\{X^{(m_0,m_1,\ldots,m_{n-1})} : (m_0, m_1, \ldots, m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for $\mathbb{C}[M]$ and $\{X_0^{m_0}.X_1^{m_1} \ldots X_{n-1}^{m_{n-1}} : (m_0, m_1, \ldots, m_{n-1}) \in M\}$ is a $\mathbb{C}$-vector space basis for R', Φ is an isomorphism of $\mathbb{C}$-algebras. Hence R' is the semigroup algebra corresponding to the affine semigroup M. Since by assumption M is a normal affine semigroup, by Theorem 1 the algebra R' is normal. Thus, by Exercise 5.14(a) of [3], the implication (3) $\Rightarrow$ (2′) follows. □

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