NUMBER OF WEIGHTED SUBSEQUENCE SUMS WITH WEIGHTS IN $\{1, -1\}$

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Abstract

Let G be an abelian group of order n and let it be of the form $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, where $n_i \mid n_{i+1}$ for $1 \leq i < r$ and $n_1 > 1$. Let $A = \{1, -1\}$. Given a sequence S with elements in G and of length $n + k$ such that the natural number k satisfies $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, where $r' = |\{i \in \{1, 2, \cdots, r\} : 2 \mid n_i\}|$, if S does not have an A-weighted zero-sum subsequence of length n, we obtain a lower bound on the number of A-weighted n-sums of the sequence S. This is a weighted version of a result of Bollobás and Leader. As a corollary, one obtains a result of Adhikari, Chen, Friedlander, Konyagin and Pappalardi. A result of Yuan and Zeng on the existence of zero-smooth subsequences and the DeVos-Goddyn-Mohar Theorem are some of the main ingredients of our proof.

1. Introduction

Let G be an abelian group of order n, written additively. The *Davenport constant* $D(G)$ is defined to be the smallest natural number t such that any sequence of elements of G of length t has a non-empty subsequence whose sum is zero (the identity element of the group).

Another interesting constant, $E(G)$, is defined to be the smallest natural number t such that any sequence of elements of G of length t has a subsequence of length n whose sum is zero. A classical result of Erdős, Ginzburg and Ziv [8] says that $E(\mathbb{Z}/n\mathbb{Z}) = 2n - 1$.

The constants $D(G)$ and $E(G)$ were being studied independently until Gao [9] (see also [11], Proposition 5.7.9) established the following result connecting these two invariants:

$$E(G) = D(G) + n - 1. \tag{1}$$

Generalizations of the constants $E(G)$ and $D(G)$ with weights were considered in [2] and [4] for finite cyclic groups and generalizations for an arbitrary finite abelian group G were introduced later [1].

Given an abelian group G of order n, and a finite non-empty subset A of integers, the *Davenport constant of* G *with weight* A, denoted by $D_A(G)$, is defined to be the least positive integer t such that for every sequence $(x_1, \ldots, x_t)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \ldots, x_{j_l})$ and $a_i \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$. Similarly, $E_A(G)$ is defined to be the least positive integer t such that every sequence of elements of G of length t contains a subsequence $(x_{j_1}, \ldots, x_{j_n})$ satisfying $\sum_{i=1}^{n} a_i x_{j_i} = 0$, for some $a_i \in A$. When G is of order n, one may consider A to be a non-empty subset of $\{0, 1, \ldots, n-1\}$ and one avoids the trivial case $0 \in A$.

In several papers (see [2], [15], [12], [3]) the problem of determining the exact values of $E_A(\mathbb{Z}/n\mathbb{Z})$ and $D_A(\mathbb{Z}/n\mathbb{Z})$ has been taken up for various weight sets A.

In the present paper we take up a particular weighted generalization of a result of Bollobás and Leader [6] (see also [19]).

More precisely, we prove the following theorem. For some terminology used in the statement of the theorem, one may look into the next section.

Theorem 1. *Let* G *be a finite abelian group of order* n *and let it be of the form* $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, *where* $1 < n_1 \mid \cdots \mid n_r$. *Let* $A = \{1, -1\}$ *and* k *be a natural number satisfying* $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, *where* $r' = |\{i \in \{1, 2, \cdots, r\} : n_i \text{ is even}\}|$. *Then, given a sequence* $S = (x_1, x_2, \cdots, x_{n+k})$, *with* $x_i \in G$, *if* S *has no* A*-weighted zero-sum subsequence of length* n, *there are at least* $2^{k+1} - \delta$ *distinct* A*-weighted* n*-sums, where* $\delta = 1$, *if* $2 \mid n$ *and* $\delta = 0$, *otherwise.*

For a finite abelian group G of order n, Gao and Leader [10] obtained some result on the description of some sequences which do not have 0 as an n sum and at which the minimum number of n sums is attained.

2. Notations and Preliminaries

Let G be a finite abelian group of order n written additively and let A be a non-empty subset of $\{1, \ldots, n-1\}$. Given a sequence $S = (s_1, s_2, \cdots, s_r)$ of elements of G and $\bar{a} = (a_1, a_2, \cdots, a_r) \in A^r$, we define $\sigma(S) = \sum_{i=1}^{r} s_i$ and $\sigma^{\bar{a}}(S) = \sum_{i=1}^{r} a_i s_i$. If $\sigma(S) = 0$ (resp. $\sigma^{\bar{a}}(S) = 0$ for some $\bar{a} \in A^r$), we say that S is a *zero-sum sequence* (resp. an A*-weighted zero-sum sequence*).

If H is a subgroup of G, then $\phi_H : G \to G/H$ will denote the natural homomorphism and given a sequence $S = (s_1, s_2, \cdots, s_r)$ of elements of G, $\phi_H(S)$ will denote the sequence $(\phi_H(s_1), \phi_H(s_2), \cdots, \phi_H(s_r))$ with elements in G/H.

The length of a sequence S will be denoted by $|S|$; we think that this will not have any confusion with the usual notation $|G|$ used to denote the order of a finite group G.

For a subsequence S' of a sequence S, we use $S \setminus S'$ to denote the sequence obtained by removing the elements of the subsequence S' from S.

Generalizing a definition in [20], we call a sequence S with elements in G an *A-weighted zero-smooth sequence* if for any $1 \le \ell \le |S|$, there exists an A-weighted zero-sum subsequence of S of length ℓ. When $A = \{1\}$, S is simply called a *zero-smooth sequence*.

Remark. We observe that if $U = (u_1, u_2, \cdots, u_r)$ and $V = (v_1, v_2, \cdots, v_s)$ are sequences of elements of G such that U is an A-weighted zero-smooth sequence and V is an A-weighted zero-sum sequence with $|V| \le |U| + 1$, then the sequence $(u_1, u_2, \cdots, u_r, v_1, v_2, \cdots, v_s)$, obtained by appending V to U, is an A-weighted zero-smooth sequence.

We shall need the following result of Yuan and Zeng [20]:

Theorem A (Yuan, Zeng) *Let G be an abelian group of order n and S a sequence with elements in G such that $|S| \ge n + D(G) - 1$. Assume that the element 0 is repeated maximum number of times in S. Then there exists a subsequence S_1 of S which is zero-smooth and $|S_1| \ge |S| - D(G) + 1$.*

Let $\mathcal{A} = (A_1, A_2, \cdots, A_r)$, $r \ge n$, be a sequence of finite non-empty subsets of G. Let $\Sigma_n(\mathcal{A})$ denote the set of all group elements representable as a sum of n elements chosen from distinct terms of $\mathcal{A}$ and let $H = stab(\Sigma_n(\mathcal{A})) = \{g \in G : g + \Sigma_n(\mathcal{A}) = \Sigma_n(\mathcal{A})\}$. The following result of DeVos, Goddyn and Mohar [7] generalizes Kneser's addition theorem [14] (one may also look into [16] or [18]).

Theorem B (DeVos, Goddyn, Mohar) With the above notation, we have

$$|\Sigma_n(\mathcal{A})| \ge |H| \left(1 - n + \sum_{g \in G/H} \min\{n, |\{j : g \cap A_j \ne \emptyset\}|\} \right).$$

3. Proof of Theorem 1

In the case $r' = 0$, it is possible to have $k = 0$. We observe that in this case, $|S| = n$ and if $\sigma(S) = t \ne 0$, then $-\sigma(S) = -t \ne 0$. Again, n being odd, G does not have any element of order 2 and thus there are at least two distinct A-weighted n-sums. So, the result is true in this case and we may assume that $k \ge 1$.

If possible, suppose that the result is not true and choose a counterexample (G, S, k) with $|G| = n$ minimal.

Considering the sequence $\mathcal{A} = (A_1, A_2, \cdots, A_{n+k})$, where $A_i = Ax_i$ for each i,

$1 \le i \le n+k$, we have,

$$\begin{aligned} 0 &\notin \Sigma_n(\mathcal{A}), && (2)\\ |\Sigma_n(\mathcal{A})| &< 2^{k+1}-\delta. && (3)\end{aligned}$$

Let $L = stab(\Sigma_n(\mathcal{A}))$. We claim that $L = \langle 0\rangle$.

If possible, let $L \neq \langle 0\rangle$, so that $|G/L| < n$. Writing the identity element of G/L as $\mathbf{0}$, if for every subsequence $S' = \{x_{i_1}, \cdots, x_{i_d}\}$ of S of length $d = |G/L| + k$, $\mathbf{0}$ is representable as a sum of $|G/L|$ elements from distinct terms of the sequence $(\phi_L(Ax_{i_1}), \phi_L(Ax_{i_2}), \cdots, \phi_L(Ax_{i_d}))$, then we get pairwise disjoint subsequences $S_1, S_2, \cdots, S_{|L|}$, each of length $|G/L|$ and $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_{|L|} \in A^{|G/L|}$ such that $\sigma^{\bar{a}_i}(\phi_L(S_i)) = \mathbf{0}$, for each $i \in \{1, 2, \cdots, |L|\}$.

Therefore, we have

$$\sum_{i=1}^{|L|} \sigma^{\bar{a}_i}(\phi_L(S_i)) = \mathbf{0}.$$

Writing $\theta = \sigma^{\bar{a}_1}(S_1) + \sigma^{\bar{a}_2}(S_2) + \cdots + \sigma^{\bar{a}_{|L|}}(S_{|L|})$, since $-\theta$ also belongs to $L = stab(\Sigma_n(\mathcal{A}))$ and $\theta \in \Sigma_n(\mathcal{A})$, we have $0 \in \Sigma_n(\mathcal{A})$, which contradicts (2).

Hence there exists a subsequence S' of S with length $|G/L|+k$ (observe that a permissible value k for G is obviously permissible for G/L) such that $\mathbf{0} \notin \Sigma_{|G/L|}(\phi_L(\mathcal{A}'))$, where $\mathcal{A}'$ is the subsequence of $\mathcal{A}$ corresponding to the sequence S' and hence by minimality of $|G|$, we have

$$|\Sigma_{|G/L|}(\phi_L(\mathcal{A}'))| \ge 2^{k+1} - \delta' \ge 2^{k+1} - \delta,$$

and hence $|\Sigma_{|G/L|}(\mathcal{A}')| \ge 2^{k+1} - \delta$, where δ' is the parity of $|G/L|$ and δ is that of n.

Since the length of the subsequence $\mathcal{A} \setminus \mathcal{A}'$ is $n + k - (|G/L| + k) = n - |G/L|$,

$$|\Sigma_n(\mathcal{A})| \ge |\Sigma_{|G/L|}(\mathcal{A}')| \ge 2^{k+1} - \delta,$$

– a contradiction to (3).

Therefore, we have $L = \langle 0\rangle$ and hence by Theorem B, we have

$$|\Sigma_n(\mathcal{A})| \ge 1 - n + \sum_{x\in G} \min\{n, |\{i : 1 \le i \le n+k, x \in A_i\}|\}.$$

Since (2) implies that no element of G can be in n distinct A_i's, we have

$$\begin{aligned} |\Sigma_n(\mathcal{A})| &\ge 1 - n + \sum_{x\in G} \min\{n, |\{i : 1 \le i \le n+k, x \in A_i\}|\}\\ &= 1 - n + \sum_{x\in G} |\{i : 1 \le i \le n+k, x \in A_i\}|\\ &= 1 - n + \sum_{i=1}^{n+k} |A_i|.\end{aligned}$$

Writing $t = |\{j : 1 \leq j \leq n+k,\ |A_j| = 1\}|$, from (2) and the above inequality we have,

$$n - 1 \geq |\Sigma_n(\mathcal{A})| \geq 1 - n + 2(n + k - t) + t,$$

and hence,

$$t \geq 2(k+1).$$

Rearranging, if needed, we assume that $(x_1, x_2, \cdots, x_t)$ is the subsequence of S such that $|A_i| = |Ax_i| = 1$ for each i, $1 \leq i \leq t$ and the element x_1 is repeated maximum number of times in $(x_1, x_2, \cdots, x_t)$.

We observe that all the x_i's appearing in $(x_1, x_2, \cdots, x_t)$ are either equal to the zero element of the group or those of order 2, when n is even.

Consider the sequence $S' = (y_1, y_2, \cdots, y_{n+k})$, where $y_i = x_i - x_1$, for each i, $1 \leq i \leq n+k$. Write $\mathcal{B} = (B_1, B_2, \cdots, B_{n+k})$, where $B_i = Ay_i = A(x_i - x_1)$, for each i, $1 \leq i \leq n+k$.

Observing that $|Ax_1| = 1$, if we consider a typical element $\epsilon_{i_1} y_{i_1} + \epsilon_{i_2} y_{i_2} + \cdots + \epsilon_{i_n} y_{i_n}$, of $\Sigma_n(\mathcal{B})$, where $\epsilon_j \in \{1, -1\}$, then it can be written as:

$$\begin{aligned} & \epsilon_{i_1}(x_{i_1} - x_1) + \epsilon_{i_2}(x_{i_2} - x_1) + \cdots + \epsilon_{i_n}(x_{i_n} - x_1) \\ = \quad & \epsilon_{i_1} x_{i_1} + \epsilon_{i_2} x_{i_2} + \cdots + \epsilon_{i_n} x_{i_n}, \end{aligned}$$

since $\sum_{j=1}^{n} \epsilon_{i_j} x_1 = nx_1 = 0$.

Hence, $\Sigma_n(\mathcal{A}) = \Sigma_n(\mathcal{B})$ and from (2) and (3), we have

$$0 \notin \Sigma_n(\mathcal{B}), \tag{4}$$

$$|\Sigma_n(\mathcal{B})| < 2^{k+1} - \delta. \tag{5}$$

By our construction, in the subsequence $S_1 = (y_1, y_2, \cdots, y_t)$ of S', all the elements y_i, $1 \leq i \leq t$, satisfy $2y_i = 0$ and $y_1 = 0$ is repeated maximum number of times.

Depending on the parity of n, we consider the following two cases:

`Case I` (n is odd). We observe that in this case, $y_i = 0$ for all i, $1 \leq i \leq t$. Now, we choose a maximal A-weighted zero-sum subsequence S_2 of $S' \setminus S_1$, possibly empty. If $|(S \setminus S_1) \setminus S_2| \leq k$, then $(n+k) - |S_1| - |S_2| \leq k \quad \Rightarrow \quad n - |S_2| \leq |S_1|$ and hence by appending a subsequence of (zeros) S_1 of length $n - |S_2|$ to S_2 we get an A-weighted zero-sum subsequence of S' of length n, which is a contradiction to (4).

Thus, there exists a subsequence $S_3 = y_{j_1} y_{j_2} \cdots y_{j_{k+1}}$ of $(S' \setminus S_1) \setminus S_2$ which does not have any non-empty A-weighted zero-sum subsequence, by maximality of S_2.

Consider the set

$$X = \left\{ \sum_{i=1}^{k+1} \epsilon_i y_{j_i} : \epsilon_i \in A = \{1, -1\} \right\}.$$

If for $\epsilon_i, \epsilon_i' \in A = \{1, -1\}$, we have

$$\sum_{i=1}^{k+1} \epsilon_i y_{j_i} = \sum_{i=1}^{k+1} \epsilon_i' y_{j_i},$$

then, writing $I = \{i : \epsilon_i \neq \epsilon_i'\}$,

$$2 \sum_{i \in I} \epsilon_i y_{j_i} = 0,$$

which implies, since n is odd, that $\sum_{i \in I} \epsilon_i y_{j_i} = 0$, which leads to a contradiction to the maximality of S_2 if I is non-empty.

Thus, we have $|X| \geq 2^{k+1}$. Now, considering the sum of a fixed subsequence of $S' \setminus S_3$ of length $n - (k+1)$, and adding that to various sums in X, we have $|\Sigma_n(\mathcal{B})| \geq 2^{k+1}$ – a contradiction to (5).

Case II (n is even). Put $H = \langle y_1, y_2, \cdots, y_t \rangle$. As we have already observed, $2y_i = 0$, for all i, $1 \leq i \leq t$. Hence H is a subgroup of $\mathbb{Z}_2^{r'}$.

Thus,

$$|H| \leq 2^{r'} \tag{6}$$

and by a result of Olson [17] on the Davenport constant of p-groups,

$$D(H) \leq D(\mathbb{Z}_2^{r'}) = r' + 1. \tag{7}$$

Since, by our assumption, $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, by (6) we have,

$$|S_1| = t \geq 2(k+1) \geq 2^{r'} + r' \geq |H| + D(H) - 1.$$

Also, 0 is repeated maximum number of times in S_1.

So, we can apply Theorem A and it follows that S_1 has a zero-smooth subsequence T_1 such that $|T_1| \geq |S_1| - D(H) + 1$.

Therefore, from the fact $|S_1| = t \geq 2(k+1)$ and (7) we have

$$|T_1| \geq 2k + 2 - r'.$$

Again, since, $k \geq 2^{r'-1} - 1 + \frac{r'}{2}$, we have $k - r' \geq 2^{r'-1} - 1 - \frac{r'}{2} \geq -1$, we have

$$|T_1| \geq 2k + 2 - r' = k + 2 + k - r' \geq k + 1.$$

We choose a maximal A-weighted zero-smooth subsequence T of S'. We have, $|T| \geq |T_1| \geq k + 1$. Further, (4) implies that $|T| < n$.

Consider the subsequence $S' \setminus T = y_{s_1} y_{s_2} \cdots y_{s_{k+l}}$, say, (since $|T| < n$, $l \geq 1$) and the set

$$Y = \left\{ \sum_{i \in I} y_{s_i} : I \subset \{1, 2, \cdots, k+1\}, I \neq \emptyset \right\}.$$

Now, if for subsets I, J of $\{1, 2, \cdots, k+1\}$, with $I \neq J, I \neq \emptyset, J \neq \emptyset$, we have

$$\sum_{i \in I} y_{s_i} = \sum_{i \in J} y_{s_i},$$

then we have

$$\sum_{i \in I'} \delta_i y_{s_i} = 0, \quad \delta_i \in A,$$

where $I' = (I \cup J) \setminus (I \cap J)$.

Since it is clear that I' is non-empty, and $1 \leq |I'| \leq k+1 \leq |T|$, by the observation made in the Remark in Section 2, appending the subsequence corresponding to I' to T, we get a contradiction to the maximality of T. Therefore, we have $|Y| = 2^{k+1} - 1$. Adding $y_{s_{k+2}} + \cdots + y_{s_{k+l}}$ to each of the distinct sums in Y, we get $2^{k+1} - 1$ distinct sums $y_{s_{k+2}} + \cdots + y_{s_{k+l}} + \sum_{i \in I} y_{s_i} : I \subset \{1, 2, \cdots, k+1\}, I \neq \emptyset$.

Now, for a given $I \subset \{1, 2, \cdots, k+1\}$, as $n - (|I| + l - 1) \leq n - l = |T|$, we can append an $n - (|I| + l - 1)$ length A-weighted zero-sum subsequence of T to $y_{s_{k+2}} + \cdots + y_{s_{k+l}} + \sum_{i \in I} y_{s_i}$ to make an A-weighted n-sum without changing the value of the sum.

Thus, $|\Sigma_n(\mathcal{B})| \geq 2^{k+1} - 1$, which is a contradiction to (5), and completes the proof of Theorem 1.

Remark. It is not difficult to observe that for a finite abelian group $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, $1 < n_1 \mid \cdots \mid n_r$, satisfying $|G| > 2^{(2^{r'-1} - 1 + \frac{r'}{2})}$, where $r' = |\{i \in \{1, 2, \cdots, r\} : 2 \mid n_i\}|$, and $A = \{1, -1\}$, our theorem along with some counter examples like those given in [2] (see also [5]), yields

$$|G| + \sum_{i=1}^{r} \lfloor \log_2 n_i \rfloor \leq E_A(G) \leq |G| + \lfloor \log_2 |G| \rfloor. \tag{8}$$

This gives the exact value of $E_A(G)$ when G is cyclic (thus giving another proof of the main result in [2]) and unconditional bounds in many cases.

However, we mention that when $A = \{1, -1\}$, finding the corresponding bounds for $D_A(G)$ for a finite abelian group G and the exact value of $D_A(G)$ when G is cyclic, is not so difficult (see [2], [5]). Therefore, from the relation

$$E_A(G) = D_A(G) + n - 1,$$

which generalizes (1) for an abelian group G with $|G| = n$ and a non-empty subset A of $\{1, \ldots, n-1\}$, established for cyclic groups by Yuan and Zeng [21] and for general finite abelian groups by Grynkiewicz, Marchan and Ordaz [13], the result (8) follows.

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