#A43 INTEGERS 11 (2011)

CONVOLUTION AND RECIPROCITY FORMULAS FOR BERNOULLI POLYNOMIALS

Takashi Agoh[1]
Department of Mathematics, Tokyo University of Science, Noda, Chiba, Japan
agoh_takashi@ma.noda.tus.ac.jp

Karl Dilcher[2]
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada
dilcher@mathstat.dal.ca

Received: 10/7/10, Accepted: 5/18/11, Published: 6/27/11

Abstract

We prove a new convolution identity for sums of products of two Bernoulli polynomials. This can be rewritten to obtain a reciprocity relation for a related sum. The proof uses some results on Stirling numbers of both kinds which are of independent interest. In particular, a class of polynomials related to the Stirling numbers of the second kind turns out to be a useful tool.

1. Introduction

For the classical Bernoulli numbers B_n, $n = 0, 1, 2, \ldots$, which can be defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi), \tag{1}$$

a great number of linear and nonlinear recurrence relations are known. For a brief review of the relevant literature see, e.g., [1].

One of the most basic and remarkable identities is the nonlinear or convolution identity

$$\sum_{j=0}^{n} \binom{n}{j} B_j B_{n-j} = -nB_{n-1} - (n-1)B_n \qquad (n \geq 1), \tag{2}$$

[1]Supported in part by a grant of the Ministry ofEducation, Science and Culture of Japan.
[2]Supported in part by the Natural Sciences and Engineering ResearchCouncil of Canada.

which is also known in its equivalent form

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1)B_{2n} \qquad (n \geq 2). \tag{3}$$

These identities, which are usually attributed to Euler, have been extended and generalized in various directions, most recently by the authors [1], [3]. Extensions to Bernoulli *polynomials* are also known, for instance

$$\sum_{j=0}^{n} \binom{n}{j} B_j(y) B_{n-j}(x) = n(x+y-1)B_{n-1}(x+y) - (n-1)B_n(x+y); \tag{4}$$

see, e.g., [11, (50.11.2)], or [7], [9] for further extensions. The Bernoulli polynomials $B_n(x)$ can be defined by the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi), \tag{5}$$

or equivalently

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j, \tag{6}$$

with the obvious connection $B_n(0) = B_n$. The Bernoulli polynomials also satisfy the reflection identity

$$B_n(1-x) = (-1)^n B_n(x), \tag{7}$$

which can be obtained by easy manipulations of (5). Now we let $y = 0$ in (4) and use the fact that $B_1 = -1/2$ and $B_{2j+1} = 0$ for $j \geq 1$. Then

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} B_{2j} B_{n-2j}(x) = n(x - \frac{1}{2})B_{n-1}(x) - (n-1)B_n(x). \tag{8}$$

This has been generalized to the following interesting identity:

$$\begin{aligned} B_k(x)B_m(x) = &\sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \left[\binom{k}{2j} m + \binom{m}{2j} k \right] \frac{B_{2j} B_{k+m-2j}(x)}{k+m-2j} \\ &+ (-1)^{k+1} \frac{k!m!}{(k+m)!} B_{k+m}, \end{aligned} \tag{9}$$

(valid for $k + m \geq 2$); see [14, p. 75] or, in modern notation, [5]. If we set $m = 1$ and $k = n - 1$ in (9), then after some easy manipulations we get (8). Given the identity (9), we may now ask whether there are "easy" evaluations of sums such as the one on the right of (9), but

(i) with $B_{2j}(x)B_{k+m-2j}$, or
(ii) with $B_{2j}(x)B_{k+m-2j}(x)$
in place of $B_{2j}B_{k+m-2j}(x)$. It turns out that, roughly speaking, a sum of type (i) evaluates as one of type (ii), and vice versa. In particular, we will prove the following result.

Theorem 1. *For all $k, m \geq 1$ we have*

$$m\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}}{m+j}B_{m+j}(x)B_{k-j}(x) + k\sum_{j=1}^{m}\binom{m}{j}\frac{1}{k+j}B_{k+j}B_{m-j}(x) = (-1)^k\frac{k!m!}{(k+m)!}B_{k+m}(x) \tag{10}$$

By interchanging m and k and adding the resulting identity thus obtained to the original identity (10), we immediately get the following

Corollary 2. *For all $k, m \geq 1$ we have*

$$\begin{aligned} m\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}B_{m+j}(x)+B_{m+j}}{m+j}B_{k-j}(x) \\ + k\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{m-j}B_{k+j}(x)+B_{k+j}}{k+j}B_{m-j}(x) \\ = B_mB_k(x)+B_kB_m(x)+((-1)^k+(-1)^m)\frac{k!m!}{(k+m)!}B_{k+m}(x). \end{aligned} \tag{11}$$

This can be written as a reciprocity relation, in the spirit of [2]. Indeed, if $b(k, m)$ denotes the first sum in (11), then we have

$$b(k,m)+b(m,k) = B_mB_k(x)+B_kB_m(x)+((-1)^k+(-1)^m)\frac{k!m!}{(k+m)!}B_{k+m}(x).$$

Note the symmetry in k and m.

This paper is structured as follows. In Section 2 we will review some properties of the Stirling numbers of both kinds and define a class of polynomials related to the Stirling number of the second kind. The results obtained in this section can also be considered to be of independent interest. In Section 3 we will then apply these results to the proof of Theorem 1. We conclude this paper with some additional remarks and results, in Section 4, on the Stirling-type polynomials.

2. Stirling Numbers and Related Polynomials

It is well known that Stirling numbers of both kinds and Bernoulli numbers are closely related to each other. In this paper as well, Stirling numbers turn out to be essential tools in the proof of Theorem 1.

While they are often defined in a purely combinatorial way (see, e.g., [10], Section 6.1), for our purposes it is more convenient to consider the Stirling numbers of the first kind, $s(n,k)$, and of the second kind, $S(n,k)$, as coefficients in the change between the two standard bases of the vector space of single-variable polynomials:

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k, \tag{12}$$

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1). \tag{13}$$

It follows directly from these definitions that

$$S(n,0) = s(n,0) = 0 \qquad \text{for} \quad n \geq 1, \tag{14}$$

$$S(n,1) = s(n,1) = 1 \qquad \text{for} \quad n \geq 1, \tag{15}$$

$$S(n,n-1) = -s(n,n-1) = \tfrac{n(n-1)}{2} \qquad \text{for} \quad n \geq 1, \tag{16}$$

$$S(n,n) = s(n,n) = 1 \qquad \text{for} \quad n \geq 0, \tag{17}$$

and by convention we set

$$s(n,j) = S(n,j) = 0 \quad \text{for} \quad j > n \quad \text{and} \quad j < 0. \tag{18}$$

Another pair of basic and important identities are the *triangular* or *Pascal-type* relations

$$S(n+1,k) = S(n,k-1) + k\,S(n,k), \tag{19}$$

$$s(n+1,k) = s(n,k-1) - n\,s(n,k). \tag{20}$$

By substituting (13) into (12) and (12) into (13), we obtain, respectively, the following two orthogonality relations which connect the Stirling numbers of both kinds:

$$\sum_{j=k}^{m} s(m,j)S(j,k) = \delta_{mk}, \qquad \sum_{j=k}^{m} S(m,j)s(j,k) = \delta_{mk}, \tag{21}$$

where δ_{mk} is the Kronecker symbol. For important combinatorial interpretations of the Stirling numbers see, e.g., [10], where numerous other properties can be found, including connections with Bernoulli numbers. The book [8] is another good reference, as are the on-line resources [16], [17].

In studying convolution identities for Bernoulli numbers, the following derivative expression proved to be essential: For any $m \geq 0$ we have

$$\frac{d^m}{dt^m}\frac{1}{e^t-1} = (-1)^m \sum_{j=1}^{m+1} (j-1)!\frac{S(m+1,j)}{(e^t-1)^j}. \tag{22}$$

It is therefore not surprising that in dealing with Bernoulli *polynomials* we obtain a polynomial analogue of (22).

Indeed, let us define, for each $0 \le k \le n$, the polynomial

$$S(n,k;x) = \sum_{j=0}^{n-k} (-1)^j \binom{n-1}{j} S(n-j,k) x^j. \tag{23}$$

The properties (14), (15) and (17) immediately give

$$S(n,0;x) = 0 \qquad \text{for} \quad n \ge 1, \tag{24}$$
$$S(n,1;x) = (1-x)^{n-1} \qquad \text{for} \quad n \ge 1, \tag{25}$$
$$S(n,n;x) = 1 \qquad \text{for} \quad n \ge 0, \tag{26}$$
$$S(n,k;0) = S(n,k) \qquad \text{for} \quad n \ge k \ge 0. \tag{27}$$

Furthermore, we have the following triangular recurrence relation which is analogous to (19):

Lemma 3. *For all* $1 \le k \le n$ *we have*

$$S(n+1,k;x) = S(n,k-1;x) + (k-x)S(n,k;x). \tag{28}$$

Proof. We use (23) to write

$$\begin{aligned} &S(n+1,k;x) - S(n,k-1;x) \\ &\quad = \sum_{j=0}^{n+1-k} (-1)^j \left[\binom{n}{j} S(n+1-j,k) - \binom{n-1}{j} S(n-j,k-1) \right] x^j. \end{aligned} \tag{29}$$

Now we use the triangular relations $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ and

$$S(n-j,k-1) = S(n+1-j,k) - k\,S(n-j,k),$$

where this last one comes from (19), and we see that the expression in square brackets in (29) becomes

$$\binom{n-1}{j-1} S(n+1-j,k) + k \binom{n-1}{j} S(n-j,k),$$

after two terms have canceled. Hence the right-hand side of (29) becomes, after a shift in summation,

$$\sum_{j=0}^{n-k} (-1)^{j+1} \binom{n-1}{j} S(n-j,k) x^{j+1} + k \sum_{j=0}^{n-k} (-1)^j \binom{n-1}{j} S(n-j,k) x^j,$$

which is $(k-x)S(n,k;x)$, by (23). This completes the proof. $\square$

Further properties of the polynomials $S(n,k;x)$ and connections with known objects, not needed here, can be found in Section 4 below.

We are now ready to prove the main lemma of this section, namely the polynomial analogue of (22).

Lemma 4. *For all $m \geq 0$ we have*

$$\frac{d^m}{dt^m}\frac{e^{xt}}{e^t-1} = (-1)^m e^{xt}\sum_{j=1}^{m+1}(j-1)!\frac{S(m+1,j;x)}{(e^t-1)^j}. \tag{30}$$

Proof. We proceed by induction on m. The base case $m=0$ is obvious by (25). Now assume that (30) holds, and note that

$$\frac{d}{dt}\frac{e^{xt}}{(e^t-1)^j} = -e^{xt}\frac{(j-x)(e^t-1)+j}{(e^t-1)^{j+1}}.$$

Then upon differentiating (30) and multiplying both sides by -1, we get

$$\begin{aligned}
&(-1)^{m+1}\frac{d^{m+1}}{dt^{m+1}}\frac{e^{xt}}{e^t-1}\\
&= e^{xt}\left(\sum_{j=1}^{m+1}(j-1)!\frac{(j-x)S(m+1,j;x)}{(e^t-1)^j} + \sum_{j=1}^{m+1} j!\frac{S(m+1,j;x)}{(e^t-1)^{j+1}}\right)\\
&= e^{xt}\left(\sum_{j=1}^{m+1}(j-1)!\frac{(j-x)S(m+1,j;x)}{(e^t-1)^j} + \sum_{j=2}^{m+2}(j-1)!\frac{S(m+1,j-1;x)}{(e^t-1)^j}\right)\\
&= e^{xt}\sum_{j=1}^{m+2}(j-1)!\frac{S(m+2,j;x)}{(e^t-1)^j},
\end{aligned}$$

where we have used (28) in the last equation. This proves (30) by induction. □

The last lemma in this section involves the Stirling numbers of both kinds, as well as the polynomials introduced above, and is reminiscent of the left-hand sides in the orthogonality relations (21).

Lemma 5. *For all $1 \leq k \leq m$ we have*

$$\sum_{j=k}^{m} S(m,j)s(j,k)\frac{1}{j} = (-1)^{m-k}\frac{1}{m}\binom{m}{k}B_{m-k}, \tag{31}$$

$$\sum_{j=k}^{m} S(m,j;x)s(j,k)\frac{1}{j} = (-1)^{m-k}\frac{1}{m}\binom{m}{k}B_{m-k}(x). \tag{32}$$

Proof. We begin with (31). The cases $m = k$ and $m = k+1$ can be verified directly, using (16), (17) and the fact that $B_0 = 1$ and $B_1 = -1/2$. Since $B_{2j+1} = 0$ for all $j \geq 1$, (31) is seen to be the same as identity (6.100) in [10, p. 290] when $m > k+1$.

Let $S_k^m(x)$ denote the left-hand side of (32). To evaluate it, we use (23) and first change the orders of summation:

$$\begin{aligned}
S_k^m(x) &= \sum_{j=k}^{m}\sum_{\nu=0}^{m-j}\binom{m-1}{\nu}(-1)^\nu S(m-\nu,j)s(j,k)x^\nu\frac{1}{j} \\
&= \sum_{j=0}^{m-k}\sum_{\nu=0}^{j}\binom{m-1}{\nu}(-x)^\nu S(m-\nu,m-j)s(m-j,k)\frac{1}{m-j} \\
&= \sum_{\nu=0}^{m-k}\binom{m-1}{\nu}(-x)^\nu\sum_{j=\nu}^{m-k}S(m-\nu,m-j)s(m-j,k)\frac{1}{m-j} \\
&= \sum_{\nu=0}^{m-k}\binom{m-1}{\nu}(-x)^\nu\sum_{j=k}^{m-\nu}S(m-\nu,j)s(j,k)\frac{1}{j}.
\end{aligned}$$

Now we can apply (31), and after manipulating the product of binomial coefficients we get

$$\begin{aligned}
S_k^m(x) &= \sum_{\nu=0}^{m-k}\binom{m-1}{\nu}x^\nu(-1)^{m-k}\binom{m-\nu}{k}\frac{B_{m-\nu-k}}{m-\nu} \\
&= \frac{(-1)^{m-k}}{m}\binom{m}{k}\sum_{\nu=0}^{m-k}\binom{m-k}{\nu}B_{m-\nu-k}x^\nu \\
&= \frac{(-1)^{m-k}}{m}\binom{m}{k}B_{m-k}(x),
\end{aligned}$$

where we have used (6) for the last equation. This completes the proof of (32). $\square$

3. Proof of Theorem 1

We are now ready to prove Theorem 1. Our starting point is the observation that the first sum in (10) is a term in the Cauchy product of two power series. To be more specific, this sum is the coefficient of $t^k/k!$ in the power series expansion of the product

$$\left(\sum_{n=0}^{\infty}\frac{B_{m+n}(x)}{m+n}\frac{t^n}{n!}\right)\left(\sum_{\nu=0}^{\infty}(-1)^\nu B_\nu(x)\frac{t^\nu}{\nu!}\right) =: F_m(t,x)G(t,x). \tag{33}$$

To obtain the generating function $F_m(t,x)$ for the first series, we note that (5) readily gives

$$F_1(t,x) = \frac{e^{tx}}{e^t-1} - \frac{1}{t} = \sum_{n=0}^{\infty} \frac{B_{1+n}(x)}{1+n}\frac{t^n}{n!}$$

(a function analytic at $t=0$), and thus

$$\begin{aligned} F_m(t,x) &= \frac{d^{m-1}}{dt^{m-1}}\left(\frac{e^{xt}}{e^t-1} - \frac{1}{t}\right) \qquad (34)\\ &= (-1)^{m-1}\left(e^{xt}\sum_{j=1}^{m}(j-1)!\frac{S(m,j;x)}{(e^t-1)^j} - \frac{(m-1)!}{t^m}\right), \end{aligned}$$

by Lemma 2. Next, by (7) and (5) we have

$$G(t,x) = \frac{te^te^{-xt}}{e^t-1}.$$

Hence we need to find the coefficient of $t^k/k!$ in the power series expansion of

$$F_m(t,x)G(t,x) = A(t,x) + B(t,x),$$

where

$$A(t,x) := (-1)^{m-1}\sum_{j=1}^{m}(j-1)!S(m,j;x)\frac{te^t}{(e^t-1)^{j+1}}, \qquad (35)$$

$$B(t,x) := (-1)^m(m-1)!\frac{e^te^{-xt}}{t^{m-1}(e^t-1)}. \qquad (36)$$

Both $A(t,x)$ and $B(t,x)$ have poles of order $m-1$ at $t=0$, but this will not affect what follows; we simply consider the corresponding Laurent series. For ease of notation, let $[f(t)]_k$ denote the coefficient of $t^k/k!$ in $f(t)$.

We first determine $[A(t,x)]_k$. To do so, we use the Bernoulli polynomials of order r, defined by the exponential generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \qquad (|t|<2\pi), \qquad (37)$$

and the corresponding Bernoulli numbers of higher order, given by

$$B_n^{(r)} := B_n^{(r)}(0). \qquad (38)$$

For further information on these objects see, e.g., [15] or [12]. Here we take r to be a positive integer parameter. We now have with (37),

$$\frac{te^t}{(e^t-1)^{j+1}} = \frac{1}{t^j}\left(\frac{t}{e^t-1}\right)^{j+1}e^t = \frac{1}{t^j}\sum_{n=0}^{\infty}B_n^{(j+1)}(1)\frac{t^n}{n!},$$

and this means that

$$\left[\frac{te^t}{(e^t-1)^{j+1}}\right]_k = \frac{k!}{(k+j)!}B_{k+j}^{(j+1)}(1). \tag{39}$$

The following lemma now gives an expression of the right-hand side of (39) in terms of Bernoulli numbers and Stirling numbers of the first kind.

Lemma 6. *For all $k \geq 1$ and $n \geq 0$ we have*

$$B_n^{(k)}(1) = k\binom{n}{k}\sum_{r=0}^{k-1}(-1)^{k-1-r}s(k-1,k-r-1)\frac{B_{n-r}}{n-r}. \tag{40}$$

Proof. The higher-order Bernoulli polynomials and numbers satisfy the identities

$$B_n^{(k)}(x+1) - B_n^{(k)}(x) = n\,B_{n-1}^{(k-1)}(x), \tag{41}$$

and

$$B_n^{(k)} = k\binom{n}{k}\sum_{r=0}^{k-1}(-1)^{k-1-r}s(k,k-r)\frac{B_{n-r}}{n-r}; \tag{42}$$

see, e.g., [15, p. 148ff.] or [12]. Now, using (41) with $x=0$ and then (38), we get

$$B_n^{(k)}(1) = B_n^{(k)} + n\,B_{n-1}^{(k-1)}. \tag{43}$$

We consider the last term and obtain with (42),

$$\begin{aligned} n\,B_{n-1}^{(k-1)} &= (k-1)n\binom{n-1}{k-1}\sum_{r=0}^{k-2}(-1)^{k-r}s(k-1,k-1-r)\frac{B_{n-1-r}}{n-1-r} \\ &= (k-1)k\binom{n}{k}\sum_{r=1}^{k-1}(-1)^{k-1-r}s(k-1,k-r)\frac{B_{n-r}}{n-r}. \end{aligned}$$

Now we use the fact that by (20) we have

$$s(k,k-r) + (k-1)s(k-1,k-r) = s(k-1,k-r-1),$$

and therefore by adding the last expression above to (42), we get by (43) the desired formula (40). □

Continuing with the proof of Theorem 1, we use Lemma 4 with (35) and (39) to obtain

$$\begin{aligned} [A(t,x)]_k &= (-1)^{m-1}\sum_{j=1}^{m}(j-1)!S(m,j;x)\frac{k!}{(k+j)!}B_{k+j}^{(j+1)}(1) \\ &= (-1)^{m-1}\sum_{j=1}^{m}\frac{k}{j}S(m,j;x)\sum_{r=0}^{j}(-1)^{j-r}s(j,j-r)\frac{B_{k+j-r}}{k+j-r} \\ &= (-1)^{m-1}\sum_{j=1}^{m}\frac{k}{j}S(m,j;x)\sum_{r=1}^{j}(-1)^{r}s(j,r)\frac{B_{k+r}}{k+r}, \end{aligned}$$

where we have changed the order of summation in the second sum, keeping in mind that $s(j, 0) = 0$ since $j > 0$. Now we interchange the two summations and then use the second part of Lemma 3:

$$\begin{aligned}[A(t,x)]_k &= k\sum_{r=1}^{m}(-1)^{m-1-r}\frac{B_{k+r}}{k+r}\sum_{j=r}^{m}S(m,j;x)s(j,r) \\ &= -\frac{k}{m}\sum_{r=1}^{m}\binom{m}{r}\frac{B_{k+r}}{k+r}B_{m-r}(x).\end{aligned} \tag{44}$$

It remains to deal with $B(t, x)$ in (36). To do this, we note that with (5) we have

$$\frac{e^t e^{-xt}}{t^{m-1}(e^t-1)} = \frac{1}{t^m}\frac{te^{(1-x)t}}{e^t-1} = \frac{1}{t^m}\sum_{n=0}^{\infty}B_n(1-x)\frac{t^n}{n!},$$

so that

$$\left[\frac{e^t e^{-xt}}{t^{m-1}(e^t-1)}\right]_k = \frac{k!}{(k+m)!}B_{k+m}(1-x) = (-1)^{k+m}\frac{k!}{(k+m)!}B_{k+m}(x),$$

where we have used (7) in the second equation. Thus, with (36) we have

$$[B(t,x)]_k = (-1)^k\frac{k!(m-1)!}{(k+m)!}B_{k+m}(x).$$

Finally, this with (44) and the discussion at the beginning of this section immediately gives (10). The proof of Theorem 1 is now complete.

4. Further Remarks

The polynomials $S(n, k; x)$ introduced as a tool in Section 2 are interesting in their own right, and are closely related to some known concepts of generalized Stirling numbers. Indeed, the "non-central Stirling numbers of the second kind", $S_2(n + 1, k; r)$, are defined in [6, p. 314], and it is shown that they satisfy

$$S_2(n+1,k;r) = S_2(n,k-1;r) + (k+r)S_2(n,k;r), \tag{45}$$

with initial conditions $S_2(n, 0; r) = r^n$ for $n \geq 0$, once again with the convention that $S_2(0, 0; r) = 1$. We see that the recurrence (45) is the same as (28), with the exception of the coefficient $(k + x)$. For greater convenience we therefore introduce the class of polynomials $T(n, k; x) := S_2(n, k; -x)$, which can then be defined by

$$T(n+1,k;x) = T(n,k-1;x) + (k-x)T(n,k;x), \tag{46}$$

with
$$T(n, 0, x) := (-x)^n \qquad \text{for all} \quad n \geq 0.$$

Now, the main differences from $S(n, k; x)$ are the different initial conditions (24), namely $S(n, 0; x) = 0$ for $n \geq 1$, and $S(0, 0; x) = 1$. However, it is easily seen that the sum
$$T(n, k; x) + xT(n-1, k; x)$$
satisfies the same recurrence relation as (28), as well as
$$T(n, 0; x) + xT(n-1, 0; x) = (-x)^n + x(-x)^{n-1} = 0 \qquad (n \geq 1),$$
and $T(0, 0; x) = 1$. This uniquely defines these polynomials, and therefore we have
$$S(n, k; x) = T(n, k; x) + xT(n-1, k; x). \tag{47}$$

Now, it is known that the polynomials $S_2(n, k; x)$ satisfy an explicit formula (see, e.g., [6, p. 316]), which translated to the $T(n, k; x)$ is
$$T(n, k; x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j - x)^n. \tag{48}$$

By substituting (48) into (47) we get after some straightforward manipulation with the binomial coefficients,
$$S(n, k; x) = \frac{1}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (k - j - x)^{n-1}. \tag{49}$$

When $x = 0$, this is consistent with a well-known explicit formula for the usual Stirling numbers of the second kind, $S(n, k)$.

Finally, comparing (49) with (48), we get another connection between the two types of polynomials, namely
$$S(n, k; x) = T(n-1, k-1; x-1).$$

We finally remark that the polynomials $T(n, k; x)$ occur as generalized Stirling numbers in [4], where some references and further generalizations can be found. These polynomials are also a special case of a vast and unified generalization in [13]. In fact, these polynomials can be seen as the generalized Stirling numbers belonging to the triple $(0, 1, -x)$ in the notation of [13].

References

[1] T. Agoh and K. Dilcher, *Convolution identities and lacunary recurrences for Bernoulli numbers*, J. Number Theory **124** (2007), 105–122.

[2] T. Agoh and K. Dilcher, *Reciprocity relations for Bernoulli numbers*, Amer. Math. Monthly **115** (2008), 237–244.

[3] T. Agoh and K. Dilcher, *Higher-order recurrences for Bernoulli numbers*, J. Number Theory **129** (2009), 1837–1847.

[4] N. P. Cakić and G. V. Milovanović, *On generalized Stirling numbers and polynomials*, Math. Balkanica (N.S.) **18** (2004), 241–248.

[5] L. Carlitz, *Note on the integral of the product of several Bernoulli polynomials*, J. London Math. Soc. **34** (1959), 361–363.

[6] C. A. Charalambides, *Enumerative combinatorics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[7] K. W. Chen, *Sums of products of generalized Bernoulli polynomials*, Pacific J. Math. **208** (2003), 39–52.

[8] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions*. Revised and enlarged edition. D. Reidel Publ. Co., Dordrecht-Boston, 1974.

[9] K. Dilcher, *Sums of products of Bernoulli numbers*, J. Number Theory **60** (1996), 23–41.

[10] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics, Second Edition*. Addison-Wesley Publ. Co., Reading, MA, 1994.

[11] E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[12] F. T. Howard, *Congruences and recurrences for Bernoulli numbers of higher order*, Fibonacci Quart. **32** (1994), 316–328.

[13] Hsu, Leetsch C. and Shiue, Peter Jau-Shyong, *A unified approach to generalized Stirling numbers*. Adv. in Appl. Math. **20** (1998), no. 3, 366–384.

[14] N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, Gauthier-Villars, Paris, 1923.

[15] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Springer-Verlag, Berlin, 1924.

[16] Weisstein, Eric W., *Stirling Number of the First Kind*. From MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html`.

[17] Weisstein, Eric W., *Stirling Number of the Second Kind*. From MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/StirlingNumberoftheSecondKind.html`.