A COMBINATORIAL PROOF OF A RECURSIVE FORMULA FOR MULTIPARTITIONS

Oleg Lazarev[1]
Department of Mathematics, Princeton University, Princeton, New Jersey, USA
olazarev@princeton.edu

Holly Swisher[2]
Department of Mathematics, Oregon State University, Corvallis, Oregon, USA
swisherh@math.oregonstate.edu

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Abstract

For $k \geq 1$, let $p_k(n)$ count the number of k-component multipartitions of a nonnegative integer n, and let $\sigma(n) = \sum_{d|n} d$ be the usual divisor function. In this paper, we give a combinatorial proof of the recursive formula

$$p_k(n) = \frac{k}{n} \sum_{r=1}^{n} p_k(n-r)\sigma(r),$$

both for $k \geq 1$, where $p_k(n)$ is defined as above, and also for $k < 0$, which requires a subtler approach.

This formula was used by Gandhi in 1963 to prove several theorems, which yield numerous Ramanujan type congruences for $p_k(n)$, including some well-known congruences for Ramanujan's τ-function.

1. Introduction

The subject of partitions has a long fascinating history, including connections to several areas of mathematics, and mathematical physics (see [5], [3] for a glimpse into some of this history). In particular, the generalization of partitions to k-component multipartitions (also known as k-colored partitions) has been a rich subject in its own right (see [6] for a nice survey of this area). We begin by reviewing partitions and multipartitions.

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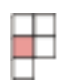
1.1. Partitions

We recall that a *partition* of a positive integer n is defined to be a nonincreasing sequence of positive integers called *parts* that sum to n (these are often written as a sum). For $n = 0$ we consider the empty set the unique "empty partition" of 0. We write $\lambda \vdash n$ to denote that λ is a partition of n, and also say that λ has *size* n, written $|\lambda| = n$. For example, the following gives all the partitions $\lambda \vdash 5$:

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

A partition $\lambda \vdash n$ with parts $l_1 \geq l_2 \geq \cdots \geq l_k$ can be represented graphically by a *Ferrers diagram* $F(\lambda)$, which consists of a left-justified array of n cells, where the ith row contains l_i cells corresponding to the ith part of λ. We refer to each cell in $F(\lambda)$ with an ordered pair (i, j) representing the position of the cell in the ith row and jth column of $F(\lambda)$. For example, the Ferrers diagram of the partition $2+2+1$ of 5 with marked cell $(2, 1)$ is given by the following.



We define the *partition function* $p(n)$ to count the total number of partitions of n. In order to define $p(n)$ on all integers we make the further definition that $p(n) = 0$ when $n < 0$. We see from our example above that $p(5) = 7$.

The generating function for $p(n)$ has the following infinite product form due to Euler,

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.$$

One of the most celebrated results in partition theory is the following list of Ramanujan's congruences for $p(n)$. For all integers $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Work of Ono and Ahlgren [12], [4], [1] has shown that for any m coprime to 6 there exist infinitely many nonnested arithmetic progressions for which $p(an+b) \equiv 0 \pmod{m}$. However, it has been shown by Ahlgren and Boylan [2] that the three Ramanujan congruences above are the only congruences for $p(n)$ of the form

$$p(qn+b) \equiv 0 \pmod{q},$$

for q prime.

1.2. Multipartitions

Partitions can be easily generalized in the following way. We define a *k-component multipartition* of a positive integer n to be a k-tuple of partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that

$$\sum_{i=1}^{k} |\lambda_i| = n.$$

We write $\lambda \vdash_k n$ if λ is a k-component multipartition of n. The following gives all the multipartitions $\lambda \vdash_2 3$:

$$(3, \emptyset), (2{+}1, \emptyset), (1{+}1{+}1, \emptyset), (2, 1), (1{+}1, 1), (1, 2), (1, 1{+}1), (\emptyset, 3), (\emptyset, 2{+}1), (\emptyset, 1{+}1{+}1).$$

We define $p_k(n)$ to count the number of k-component multipartitions of n, again defining $p_k(0) = 1$ and $p_k(n) = 0$ for $n < 0$. We note that ordering does matter in this definition, so a rearrangement of distinct λ_i yields a distinct multipartition. In addition, we note that since the empty set is a partition of 0, some λ_i may equal $\emptyset$. From our example above, we see that $p_2(3) = 10$.

The generating function for $p_k(n)$ is seen to follow from the generating function for $p(n)$ by taking the kth power. Namely,

$$\sum_{n=0}^{\infty} p_k(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k}. \tag{1}$$

For this reason, multipartitions are often referred to as *powers of the partition function*.

This generating function provides a definition of $p_k(n)$ for $k < 0$. We will give a combinatorial interpretation of $p_k(n)$ for these cases in Section 4.

1.3. Congruences for Multipartitions

Much work has been done on the study of Ramanujan type congruences for multipartitions. This can be seen in papers by Andrews [6], Atkin [7], Kiming and Olsson [10], Serre [13], Newman [11], Boylan [8], and Gandhi [9], to name a few.

For example, in [9], Gandhi establishes several theorems which yield numerous Ramanujan type congruences for $p_k(n)$, for various k. For example,

$$\begin{aligned} p_6(5n+4) &\equiv 0 \pmod{5} \\ p_8(7n+5) &\equiv 0 \pmod{7} \\ p_{12}(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

$$\begin{aligned} p_{-4}(5n+4) &\equiv 0 \pmod{5} \\ p_{-6}(7n+5) &\equiv 0 \pmod{7} \\ p_{-10}(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These theorems also yield some well-known congruences for Ramanujan's τ-function, due to the fact that $\tau(n) = p_{-24}(n-1)$.

These results stem from the following important recursive formula for $p_k(n)$.

Proposition 1. *Fix an integer $k \neq 0$. For any integer $n \geq 0$, we have that*

$$p_k(n) = \frac{k}{n}\sum_{r=1}^{n} p_k(n-r)\sigma(r),$$

where $\sigma(r) = \sum_{d|r} d$ is the usual divisor function.

Proposition 1 can be derived quickly from (1) using logarithmic differentiation. However, this sheds little light into this recursive relationship. In this paper, we give a combinatorial proof of Proposition 1 for $k \geq 1$, in terms of k-component multipartitions. In addition, we provide a combinatorial proof of Proposition 1 for $k < 0$, where a subtler interpretation is needed.

2. The Case $k = 1$

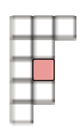
We first demonstrate the proof when $k = 1$, the case involving usual partitions. When $k = 1$, Proposition 1 states that for all integers $n \geq 0$,

$$n \cdot p(n) = \sum_{r=1}^{n} p(n-r)\sigma(r). \tag{2}$$

When $n = 0$ we see that (2) holds trivially, interpreting the empty sum as 0. Fix $n \geq 1$. The left hand side of (2) can be interpreted as the number of partitions of n such that exactly one square of its Ferrers diagram is marked. I.e., let

$$S_n : \{(\lambda, (i,j)) : \lambda \vdash n, (i,j) \in F(\lambda)\}.$$

For example, the element $(3+2+2+2+1, (3,2))$ of S_{10} is represented by the following.



Since each partition λ of n contains exactly n squares in its Ferrers diagram, we see that

$$|S_n| = n \cdot p(n).$$

To interpret the right hand side, we first consider each summand separately. For each $1 \leq r \leq n$, let

$$T_{n,r} := \{(\lambda, \mu, j) : \lambda \vdash n-r,\ \mu \vdash r,\ \mu \text{ rectangular},\ j \text{ a column of } F(\mu)\},$$

where when we say μ is rectangular, we mean that all parts of μ are equal, so that $F(\mu)$ has a rectangular shape.

For example, $(3+2+1, 2+2, 2)$ is an element of $T_{10,4}$. It is represented by the following pair.

Notice that for each rectangular partition μ of r, the number of columns of μ is a distinct divisor of r. Thus, the total number of columns of rectangular partitions μ of r is $\sigma(r)$. Thus we see that $|T_{n,r}| = p(n-r)\sigma(r)$.

Since the $T_{n,r}$ are clearly disjoint for distinct r, if we define $T_n := \cup_{r=1}^{n} T_{n,r}$, then

$$|T_n| = \sum_{r=1}^{n} p(n-r)\sigma(r).$$

We thus have a combinatorial interpretation for the right hand side of (2).

Example 2. Let $n = 3$. Then S_3 contains the following 9 elements

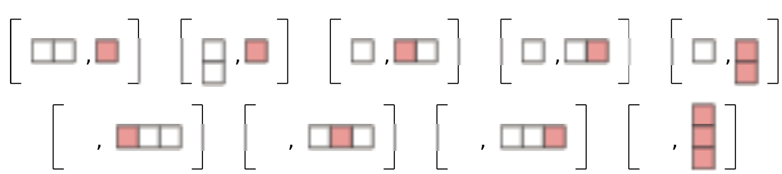
$$\begin{aligned} S_3 = \{&(3,(1,1)),(3,(1,2)),(3,(1,3)),(2+1,(1,1)),(2+1,(1,2)),\\ &(2+1,(2,1)),(1+1+1,(1,1)),(1+1+1,(2,1)),(1+1+1,(3,1))\}, \end{aligned}$$

which correspond to the following Ferrers diagrams:

In addition, T_3 contains the following 9 elements

$$T_3 = \{(2,1,1),(1{+}1,1,1),(1,2,1),(1,2,2),(1,1{+}1,1),(\emptyset,3,1),(\emptyset,3,2),(\emptyset,3,3),(\emptyset,1{+}1{+}1,1)\},$$

which correspond to the following pairs of Ferrers diagrams.

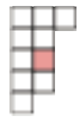


Here the first two are in $T_{3,1}$, the next three are in $T_{3,2}$, and the last four are in $T_{3,3}$.

We prove (2) by constructing a bijection $\Phi_n : S_n \to T_n$. We can describe the map Φ_n most easily by stating what it does to an element of S_n in terms of Ferrers diagrams.

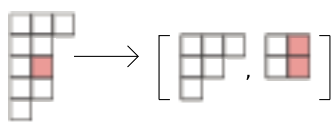
Consider an arbitrary element of S_n, say $(\lambda, (i, j))$. The marked square (i, j) lies in a particular row of $F(\lambda)$, i.e., in a particular part of λ of size s. More specifically, (i, j) lies in the mth occurrence of the part s in λ.

For example, in the element $(3 + 2 + 2 + 2 + 1, (3, 2)) \in S_{10}$, the marked square $(3, 2)$ lies in the second occurrence of the part 2 in $\lambda = 3 + 2 + 2 + 2 + 1$.



The map Φ separates the first m occurrences of the part s to form a rectangular partition $\mu = s+s+\cdots+s$, where the part s is repeated m times. Since ms squares were removed from $F(\lambda)$, $|\mu| = ms$. The resulting partition λ' that remains after the m copies of s are removed must then satisfy $|\lambda'| = n - ms$. Finally, we mark the jth column of μ in correspondence to the marking of the square (i, j) that was used to create μ. In this way, we define $\Phi_n : S_n \to T_n$.

Example 3. We've seen that for $(3 + 2 + 2 + 2 + 1, (3, 2)) \in S_{10}$, the square $(3, 2)$ lies in the second occurrence of the part 2. Thus $s = 2$, and $\mu = 2 + 2$. Removing $2 + 2$ from λ leaves $\lambda' = 3 + 2 + 1$. Finally, since $(3, 2)$ was marked in λ, we mark column 2 in μ. Thus we have that $\Phi_{10}(3+2+2+2+1, (3, 2)) = (3+2+1, 2+2, 2)$.



It is easy to see that the map Φ_n is invertible, because this process is completely reversible. If we start with an element $(\lambda, \mu, j) \in T_n$, then by our construction of T_n, $(\lambda', \mu, j) \in T_{n,k}$ for some $1 \le k \le n$. We can thus simply insert the rectangular partition $\mu = s + s + \cdots + s$ of $k = ms$ into the partition λ' after any parts of size greater than s, and before any parts of size s or less. This creates a new partition λ, and since $|\lambda'| = n - k$, we have that $|\lambda| = n$. The last square of the marked column of μ now becomes the marked (i, j) square of λ.

Thus Φ_n is a bijection, and we have established our combinatorial proof of (2).

3. The Case $k > 1$

In this section, we generalize the ideas from Section 2 to the case when $k > 1$, the case involving k-component multipartitions. When $k > 1$, Proposition 1 states that for all integers $n \ge 0$,

$$n \cdot p_k(n) = k \sum_{r=1}^{n} p_k(n - r)\sigma(r). \tag{3}$$

Again, when $n = 0$ we see that (3) holds trivially. Fix $n \geq 1$.

A k-component multipartition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n can also be viewed as a partition of n for which each part is allowed to be one of k colors, one color for each component. We then must not count rearrangements of colors, so if we label the k colors by $\{1, \ldots, k\}$, we require repeated parts to occur in nondecreasing colors. Such a partition is called a *k-colored partition* of n. Thus we will write $\lambda \vdash_k n$ to denote both that λ is a k-component multipartition of n and also that λ is a k-colored partition of n. For the remainder of the paper we will view our multipartitions in this way. In addition, we will now use colors for parts in our Ferrers diagrams, so we will mark individual cells or columns with crossed lines.

Example 4. The following gives all the 2-colored partitions of 3 (we use boldface to denote the second color):

$$3,\ \mathbf{3},\ 2+1,\ 2+\mathbf{1},\ \mathbf{2}+1,\ \mathbf{2}+\mathbf{1},\ 1+1+1,\ 1+1+\mathbf{1},\ 1+\mathbf{1}+\mathbf{1},\ \mathbf{1}+\mathbf{1}+\mathbf{1}.$$

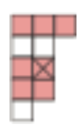
This corresponds to listing the 2-component multipartitions of 3 from our example in Section 1.2 in the following order:

$$(3,\emptyset), (\emptyset,3), (2{+}1,\emptyset), (2,1), (1,2), (\emptyset,2{+}1), (1{+}1{+}1,\emptyset), (1{+}1,1), (1,1{+}1), (\emptyset,1{+}1{+}1).$$

For each color $k \geq 1$, define

$$S_{k,n} := \{(\lambda, (i,j)) : \lambda \vdash_k n, (i,j) \in F_k(\lambda)\}.$$

For example, $(\mathbf{3}+2+\mathbf{2}+\mathbf{2}+1, (3,2)) \in S_{2,10}$ is represented by the following (where shading represents the second color).



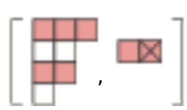
Since each k-colored partition λ of n contains exactly n squares in its Ferrers diagram, we see that

$$|S_{k,n}| = np_k(n).$$

In addition, for each $1 \leq r \leq n$, let

$$T_{k,n,r} := \{(\lambda', \mu, j, c) : \lambda' \vdash_k n-r,\ \mu \vdash_k r \text{ rectangular with all parts color } c,\ j \text{ a column of } F(\mu)\}.$$

For example, $(\mathbf{3}+2+\mathbf{2}+1, \mathbf{2}, 2, 2) \in T_{2,10,2}$ is represented by the following.



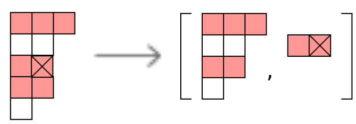
As we noted in Section 2, the total number of columns of rectangular partitions μ of r is $\sigma(r)$. Hence the total number of columns of rectangular partitions μ of r with all parts a single color is $k\sigma(r)$. Therefore, we see that $|T_{k,n,r}| = p_k(n-r) \cdot k\sigma(r)$. Since the $T_{k,n,r}$ are clearly disjoint for distinct r, if we define $T_{k,n} := \cup_{r=1}^{n} T_{k,n,r}$, then

$$|T_{k,n}| = \sum_{r=1}^{n} p_k(n-r) \cdot k\sigma(r) = k \sum_{r=1}^{n} p_k(n-r)\sigma(r).$$

We thus have a combinatorial interpretation for the right hand side of (3).

As in Section 2, we prove (3) by constructing a bijection $\Phi_{k,n} : S_{k,n} \to T_{k,n}$. This map is constructed in an analogous way to the Φ_n in Section 2, however there is one main difference. When removing parts of size s from our k-colored partition λ of n, we remove the first m occurrences of the part s that are the same color as the part in which the marked square (i, j) lies.

Example 5. For $(\mathbf{3} + 2 + \mathbf{2} + \mathbf{2} + 1, (3, 2)) \in S_{2,10}$, although the cell $(3, 2)$ lies in the second occurrence of the part 2, it lies in the first occurrence of the part 2 with color 2. Thus, $s = 2$, $\mu = \mathbf{2}$, and removing μ from $\lambda = \mathbf{3} + 2 + \mathbf{2} + \mathbf{2} + 1$ leaves $\lambda' = \mathbf{3}+2+\mathbf{2}+1$. Thus we have that $\Phi_{2,10}(\mathbf{3}+2+\mathbf{2}+\mathbf{2}+1, (3, 2)) = (\mathbf{3}+2+\mathbf{2}+1, \mathbf{2}, 2)$.



As with Φ_n in Section 2, it is easy to see that the map $\Phi_{k,n}$ is invertible, because this process is completely reversible. If we start with an element $(\lambda', \mu, j, r) \in T_{k,n}$, then by our construction of $T_{k,n}$, $(\lambda', \mu, j, r) \in T_{k,n,r}$ for some $1 \leq r \leq n$. We can thus simply insert the rectangular partition $\mu = s + s + \cdots + s$ of $k = ms$ into the partition λ' after any parts of size greater than s, and before any parts of size s or less, so that the color is in the appropriate order. This creates a new partition λ, and since $|\lambda'| = n - k$, we have that $|\lambda| = n$. The last square of the marked column of μ now becomes the marked (i, j) square of λ.

Thus $\Phi_{k,n}$ is a bijection, and we have established our combinatorial proof of (3).

4. The Case $k < 0$

When $k < 0$, we do not have the definition of $p_k(n)$ as the number of k-component multipartitions of n. However, considering the generating function for $p_k(n)$ as defined for $k \geq 1$ in (1), we define $p_{-k}(n)$ for all $-k < 0$ and $n \geq 0$, by

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^k.$$

Example 6. Let $k = 2$ represent two colors, where as before we use boldface to denote the second color. Then

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^2 = \prod_{n=1}^{\infty}(1-q^n)(1-q^{\mathbf{n}})$$
$$= 1 - q^1 - q^{\mathbf{1}} + q^{1+\mathbf{1}} - q^2 - q^{\mathbf{2}} + q^{1+2} + q^{\mathbf{1}+2} + q^{1+\mathbf{2}} + q^{\mathbf{1}+\mathbf{2}} - q^3 - q^{\mathbf{3}} - q^{1+\mathbf{1}+2} - \cdots$$
$$= (1+q^{1+\mathbf{1}}+q^{1+2}+q^{\mathbf{1}+2}+q^{1+\mathbf{2}}+q^{\mathbf{1}+\mathbf{2}}+\cdots)-(q^1+q^{\mathbf{1}}+q^2+q^{\mathbf{2}}+q^3+q^{\mathbf{3}}+q^{1+\mathbf{1}+2}+\cdots).$$

So for example, $p_{-2}(3) = 4 - 2 = 2$.

We see then that we can give $p_{-k}(n)$ a combinatorial interpretation in the following way. Define $p_{-k}^{\text{even}}(n)$ to count the number of k-colored partitions of n into an even number of parts, where common parts are distinctly colored (thus each part can occur at most k times). Likewise, define $p_{-k}^{\text{odd}}(n)$ to count the number of k-colored partitions of n into an odd number of parts, where common parts are distinctly colored. For short, if a partition has common parts distinctly colored we will call the parts *cpd-colored*. Then for all $n \geq 0$, we have

$$p_{-k}(n) = p_{-k}^{\text{even}}(n) - p_{-k}^{\text{odd}}(n).$$

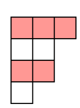
When $-k < 0$, Proposition 1 becomes for all integers $n \geq 0$,

$$n \cdot p_{-k}(n) = -k \sum_{r=1}^{n} p_{-k}(n-r)\sigma(r). \tag{4}$$

Again, when $n = 0$ we see that (4) holds trivially. Fix $n \geq 1$. To interpret the left hand side, we define

$$SD_{k,n}^{\text{even}} := \{(\lambda,(i,j)) : \lambda \vdash_k n \text{ into an even number of cpd-colored parts, } (i,j) \in F_k(\lambda)\},$$
$$SD_{k,n}^{\text{odd}} := \{(\lambda,(i,j)) : \lambda \vdash_k n \text{ into an odd number of cpd-colored parts, } (i,j) \in F_k(\lambda)\}.$$

For example, $(\mathbf{3} + 2 + \mathbf{2} + 1)$ shown below is an element of $SD_{2,8}^{\text{even}}$.



Since each k-colored partition λ of n contains exactly n squares in its Ferrers diagram, we see that $|SD_{k,n}^{\text{even}}| = np_{-k}^{\text{even}}(n)$, and $|SD_{k,n}^{\text{odd}}| = np_{-k}^{\text{odd}}(n)$. Thus

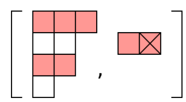
$$n \cdot p_{-k}(n) = |SD_{k,n}^{\text{even}}| - |SD_{k,n}^{\text{odd}}|.$$

In addition, for each $1 \le r \le n$, let

$$TD^{\text{even}}_{k,n,r} := \{(\lambda', \mu, j, c) : \lambda' \vdash_k n - r \text{ into an even number of cpd-colored parts,}\\ \mu \vdash_k r \text{ rectangular with all parts color } c,\ j \text{ a column of } F(\mu)\},$$

$$TD^{\text{odd}}_{k,n,r} := \{(\lambda', \mu, j, c) : \lambda' \vdash_k n - r \text{ into an odd number of cpd-colored parts,}\\ \mu \vdash_k r \text{ rectangular with all parts color } c,\ j \text{ a column of } F(\mu)\}.$$

For example, $(\mathbf{3} + 2 + \mathbf{2} + 1, \mathbf{2}, 2, 2)$ shown below is an element of $TD^{\text{even}}_{2,8,2}$.



As we noted in Section 3, the total number of columns of rectangular partitions μ of r with all parts a single color is $k\sigma(r)$. Therefore, we see that $|TD^{\text{even}}_{k,n,r}| = p^{\text{even}}_{-k}(n-r) \cdot k\sigma(r)$, and $|TD^{\text{odd}}_{k,n,r}| = p^{\text{odd}}_{-k}(n-r) \cdot k\sigma(r)$. Since the $TD^{\text{even}}_{k,n,r}$, and $TD^{\text{odd}}_{k,n,r}$ are clearly disjoint for distinct r, if we define $TD^{\text{even}}_{k,n} := \cup_{r=1}^{n} T^{\text{even}}_{k,n,r}$ and $TD^{\text{odd}}_{k,n} := \cup_{r=1}^{n} T^{\text{odd}}_{k,n,r}$, then

$$|TD^{\text{even}}_{k,n}| = k \sum_{r=1}^{n} p^{\text{even}}_{-k}(n-r)\sigma(r),$$
$$|TD^{\text{odd}}_{k,n}| = k \sum_{r=1}^{n} p^{\text{odd}}_{-k}(n-r)\sigma(r).$$

Thus

$$-k \sum_{r=1}^{n} p_{-k}(n-r)\sigma(r) = -k \sum_{r=1}^{n} \left(p^{\text{even}}_{-k}(n) - p^{\text{odd}}_{-k}(n)\right) \sigma(r) = |TD^{\text{odd}}_{k,n}| - |TD^{\text{even}}_{k,n}|,$$

and we have a combinatorial interpretation for the right hand side of (4).

We will now prove (4) by showing that

$$|TD^{\text{odd}}_{k,n}| - |TD^{\text{even}}_{k,n}| = |SD^{\text{even}}_{k,n}| - |SD^{\text{odd}}_{k,n}|. \tag{5}$$

We note that it is not the case that $|TD^{\text{odd}}_{k,n}| = |SD^{\text{even}}_{k,n}|$ and $|TD^{\text{even}}_{k,n}| = |SD^{\text{odd}}_{k,n}|$. For example, $|SD^{\text{even}}_{2,2}| = 2$, since the only partition of 2 into an even number of parts is $1 + 1$, so the only 2-colored partition of 2 with cpd-colored parts is $1 + \mathbf{1}$. Marking the cells in the Ferrers diagram yields the following elements in $SD^{\text{even}}_{2,2}$.

However, $TD_{2,2}^{\text{odd}} = 4$. This can be seen by first observing that $TD_{2,2,2}^{\text{odd}}$ is empty, since there are no partitions of zero into an odd number of parts. Thus $TD_{2,2}^{\text{odd}} = TD_{2,2,1}^{\text{odd}}$, and as both λ' and μ must partition 1, the four elements are given by the following.

Thus, in order to prove (5), we need more care than in previous sections. We first notice that $TD_{k,n}^{\text{even}}$, $TD_{k,n}^{\text{odd}}$ are subsets of $T_{k,n}$, and $SD_{k,n}^{\text{even}}$, $SD_{k,n}^{\text{odd}}$ are subsets of $S_{k,n}$. Thus we can consider the inverse of our map from Section 3, $\Phi_k^{-1} : T_{k,n} \to S_{k,n}$ restricted to the sets $TD_{k,n}^{\text{even}}$, and $TD_{k,n}^{\text{odd}}$. We use Φ_k^{-1} to partition the sets $TD_{k,n}^{\text{even}}$, and $TD_{k,n}^{\text{odd}}$ in terms of the images of their elements via Φ_k^{-1}. In particular, we define

$$\begin{aligned}
E_{k,n}^{\text{even}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{even}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \in SD_{k,n}^{\text{even}}\},\\
O_{k,n}^{\text{even}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{even}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \in SD_{k,n}^{\text{odd}}\},\\
B_{k,n}^{\text{even}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{even}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \notin SD_{k,n}^{\text{even}} \cup SD_{k,n}^{\text{odd}}\},
\end{aligned}$$

and similarly

$$\begin{aligned}
E_{k,n}^{\text{odd}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{odd}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \in SD_{k,n}^{\text{even}}\},\\
O_{k,n}^{\text{odd}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{odd}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \in SD_{k,n}^{\text{odd}}\},\\
B_{k,n}^{\text{odd}} &:= \{(\lambda', \mu, j, c) \in TD_{k,n}^{\text{odd}} : \Phi_k^{-1}((\lambda', \mu, j, c)) \notin SD_{k,n}^{\text{even}} \cup SD_{k,n}^{\text{odd}}\}.
\end{aligned}$$

Thus

$$\begin{aligned}
TD_{k,n}^{\text{even}} &= E_{k,n}^{\text{even}} \cup O_{k,n}^{\text{even}} \cup B_{k,n}^{\text{even}},\\
TD_{k,n}^{\text{odd}} &= E_{k,n}^{\text{odd}} \cup O_{k,n}^{\text{odd}} \cup B_{k,n}^{\text{odd}},
\end{aligned}$$

and these unions are disjoint.

Lemma 7. *The sets $E_{k,n}^{even}$ and $O_{k,n}^{odd}$ are empty.*

Proof. Recall that elements $(\lambda', \mu, j, c) \in T_{k,n}$ have the property that the parts of μ are all of the same color, c. Thus, if Φ_k^{-1} is to map (λ', μ, j, c) into $SD_{k,n}^{\text{even}} \cup SD_{k,n}^{\text{odd}}$, μ must contain at most one part to ensure cpd-colored parts. Since μ is a partition of $r \geq 1$, μ must then contain exactly one part. Thus by the construction of the map Φ_k, the number of parts in λ' must increase by 1 when combined with μ to form λ. So it will never be the case that Φ_k^{-1} maps an element of $TD_{k,n}^{\text{even}}$ to $SD_{k,n}^{\text{even}}$, or an element of $TD_{k,n}^{\text{odd}}$ to $SD_{k,n}^{\text{odd}}$, and we have established the lemma. $\square$

Since the map Φ_k is a bijection it follows from Lemma 7 that the restrictions $\Phi_k^{-1} : O_{k,n}^{\text{even}} \to SD_{k,n}^{\text{odd}}$ and $\Phi_k^{-1} : E_{k,n}^{\text{odd}} \to SD_{k,n}^{\text{even}}$ are bijections. Thus $|O_{k,n}^{\text{even}}| =$

$|SD_{k,n}^{\text{odd}}|$, $|E_{k,n}^{\text{odd}}| = |SD_{k,n}^{\text{even}}|$, and we have that

$$|TD_{k,n}^{\text{odd}}| - |TD_{k,n}^{\text{even}}| = |SD_{k,n}^{\text{even}}| - |SD_{k,n}^{\text{odd}}| + |B_{k,n}^{\text{odd}}| - |B_{k,n}^{\text{even}}|.$$

It remains only to show the following lemma.

Lemma 8. *The sets $B_{k,n}^{odd}$ and $B_{k,n}^{even}$ have the same cardinality.*

Proof. In order for an element $(\lambda', \mu, j, c) \in TD_{k,n}^{\text{odd}}$, with $\mu \vdash r$ containing m parts of size s and color c, to be in the subset $B_{k,n}^{\text{odd}}$, one of two things must happen. Either λ' already includes a part of size s and color c, so that $\Phi_k^{-1}(\lambda', \mu, j, c) \notin SD_{k,n}^{\text{even}} \cup SD_{k,n}^{\text{odd}}$ for any value of m, or λ' has no parts of size s and color c, but $m \geq 2$, so that $\Phi_k^{-1}(\lambda', \mu, j, c) \notin SD_{k,n}^{\text{even}} \cup SD_{k,n}^{\text{odd}}$. With this in mind, define

$$\begin{aligned} BY_{k,n}^{\text{even}} &:= \{(\lambda', \mu, j, c) \in B_{k,n}^{\text{even}} : \lambda' \text{ has a part of size } s, \text{ color } c\}, \\ BN_{k,n}^{\text{even}} &:= \{(\lambda', \mu, j, c) \in B_{k,n}^{\text{even}} : \lambda' \text{ has no part of size } s, \text{ color } c\}, \end{aligned}$$

and

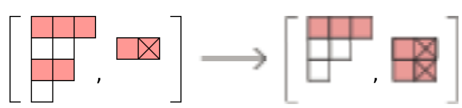
$$\begin{aligned} BY_{k,n}^{\text{odd}} &:= \{(\lambda', \mu, j, c) \in B_{k,n}^{\text{odd}} : \lambda' \text{ has a part of size } s, \text{ color } c\}, \\ BN_{k,n}^{\text{odd}} &:= \{(\lambda', \mu, j, c) \in B_{k,n}^{\text{odd}} : \lambda' \text{ has no part of size } s, \text{ color } c\}. \end{aligned}$$

Thus, since $B_{k,n}^{\text{even}} = BY_{k,n}^{\text{even}} \cup BN_{k,n}^{\text{even}}$ and $B_{k,n}^{\text{odd}} = BY_{k,n}^{\text{odd}} \cup BN_{k,n}^{\text{odd}}$ are disjoint unions, we have that

$$\begin{aligned} |B_{k,n}^{\text{even}}| &= |BY_{k,n}^{\text{even}}| + |BN_{k,n}^{\text{even}}|, \text{ and} \\ |B_{k,n}^{\text{odd}}| &= |BY_{k,n}^{\text{odd}}| + |BN_{k,n}^{\text{odd}}|. \end{aligned}$$

We will show that $|BY_{k,n}^{\text{even}}| = |BN_{k,n}^{\text{odd}}|$ and $|BN_{k,n}^{\text{even}}| = |BY_{k,n}^{\text{odd}}|$, thus establishing the lemma.

Let $(\lambda', \mu, j, c) \in BY_{k,n}^{\text{even}}$. Then λ' is a partition of $n - r$ into an even number of parts, and μ is a partition of $r = ms$ into m parts of size s and color c. We can modify (λ', μ, j, c) by removing the part of size s and color c from λ' that must exist by definition, and attaching it to μ. For example, the following illustrates this process for the element $(\mathbf{3} + 2 + \mathbf{2} + 1, \mathbf{2}, 2, 2) \in BY_{2,10}^{\text{even}}$.



We then obtain an element (λ'', μ', j, c) where λ'' is now a partition of $n - (m-1)s$ into an odd number of parts, that contains no part of size s and color c, and μ' is now a partition of $(m+1)s$ with $m+1$ parts of size s and color c. Thus $(\lambda'', \mu', j, c) \in BN_{k,n}^{\text{odd}}$, because $m + 1 \geq 2$. This process is reversible, since any element of $(\lambda'', \mu', j, c) \in BN_{k,n}^{\text{odd}}$ must have the property that μ' contains at least two parts. Thus $|BY_{k,n}^{\text{even}}| = |BN_{k,n}^{\text{odd}}|$, and with the same process applied to $BY_{k,n}^{\text{odd}}$, we see that $|BN_{k,n}^{\text{even}}| = |BY_{k,n}^{\text{odd}}|$.

$\square$

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