ALIQUOT CYCLES OF REPDIGITS

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Received: 1/17/11, Revised: 6/28/11, Accepted: 8/22/11, Published: 9/19/11

Abstract

Here we show that the only aliquot cycle consisting only of rep-digits in base 10 is the cycle consisting of the perfect number 6. Generally, we show that if g is an even positive integer, then there are only finitely many aliquot cycles consisting entirely of repdigits in base g, which are, at least in principle, effectively computable.

1. Introduction

Let $\sigma(n)$ be the sum of the positive divisors and let $s(n) = \sigma(n) - n$ be the sum of the divisors of n which are less than n. A number is called *perfect* if $\sigma(n) = 2n$, or, equivalently, $s(n) = n$. A pair of distinct positive integers (m, n) is said to form an *amicable pair* if $s(m) = n$ and $s(n) = m$. Many pairs of amicable numbers are known, but it is not known if there exist infinitely many of them. More generally, an *aliquot cycle* of length k is a k-tuple of distinct positive integers $\mathcal{C} = (n_1, \ldots, n_k)$ such that $s(n_i) = n_{i+1}$ for $i = 1, \ldots, k$, where by convention we set $n_{k+1} := n_1$. When $k = 1$, the number n_1 is perfect, and when $k = 2$, the aliquot cycle is just a pair of amicable numbers. As a matter of notation, for a positive integer j we write $s_j(n)$ for the jth fold iteration of the function s applied to the number n. For an extensive list of references regarding works on aliquot cycles, see the webpage [1].

Recently, Pollack [3] proved that the only perfect repdigit in base 10 is $N = 6$. Here we present a slight variation of this result.

Theorem 1. *The only aliquot cycle all whose members are repdigits in base* 10 *is* $\mathcal{C} = (6)$.

Now let $g > 1$ be any integer. One may wonder if in light of Theorem 1 it would be possible to show that given g there are only finitely many aliquot cycles whose members are repdigits in base g. This was proved in [3] for the special case of the perfect numbers. We could not prove that this is the case in general, but we could do so when g is even.

Theorem 2. *If g is even, then there are only finitely many aliquot cycles whose members are repdigits in base g. Moreover, all such cycles are effectively computable.*

Note that $g = 10$ satisfies the condition of Theorem 2, so Theorem 1 is not unexpected, but of course its beauty consists in the fact that one could actually compute all such instances (their finiteness being guaranteed by Theorem 2).

The proof of Theorem 1 is completely elementary. The proof of Theorem 2 uses some considerations from [2].

2. The Proof of Theorem 1

Let $x := a(10^m - 1)/9$, with $a \in \{1, \dots, 9\}$ and m being some positive integer, represent some element of an aliquot cycle $\mathcal{C}$ consisting only of repdigits. By the result from [3], we may assume that $k \geq 2$. We want to prove that there is no such example.

We first ran a computation searching for the aliquot chains containing an element with at most 4 digits. That is, we assumed that $m \leq 4$. It turns out that $s(x)$ is a repdigit in this range only when $s(x) \in \{0, 1, 3, 4, 6, 7\}$. For these last values, unless $s(x) = 6$, iterating s a few more times and evaluating it at x we end up with 0. For example, if $s(x) = 3$, then $s_3(x) = 0$, while when $s(x) = 4$, then $s_4(x) = 0$. Clearly, (6) is an aliquot cycle of length 1.

From now on, we assume that the aliquot cycle contains only repdigits with at least 5 digits.

For a nonzero integer t and a prime p we put $\nu_p(t)$ for the exponent of p in the factorization of t. Write $y := s(x) = b(10^n - 1)/9$, where $b \in \{1, \dots, 9\}$ and $n \geq 5$. We then get that

$$9\sigma(x) = 9(x + y) = 10^m a + 10^n b - (a + b) \equiv -(a + b) \pmod{2^5}$$

because both $m \geq 5$ and $n \geq 5$. Since $a + b \leq 18$, it follows that $\nu_2(\sigma(x)) \in \{0, 1, 2, 3, 4\}$, i.e., $\nu_2(\sigma(x)) < 5$. Note that this inequality is a key part of the proof of Lemma 4.

Next, we ran a computation for $m \leq 51$. That is, we computed $s(x)$ for all $m \in [5, 51]$ and all $a \in \{1, \dots, 9\}$. For the values for which x was prime we got of course $s(x) = 1$, so $s_2(x) = 0$. For all other values of x, we got a value of $s(x) > 10$ which is not a repdigit. So, from now on, we assume that $m > 51$ for all x in the cycle.

Next, we record the following useful observation. We use the symbol $\square$ to denote a number which is a perfect square of an integer.

Lemma 3. *For any positive integer N the inequality*

$$\nu_2(\sigma(N)) \geq \sum_{\substack{p \mid N \\ \nu_p(N) \equiv 1 \pmod 2}} \nu_2(p+1)$$

holds.

Proof. Assume that $N = p_1 \cdots p_k \square$, where $p_1, \ldots, p_k$ are distinct primes. Let α_i be the exponent at which the prime p_i appears in the factorization of N. Then α_i is odd for $i = 1, \ldots, k$. Thus, $\alpha_i + 1$ is even, and therefore $p_i^2 - 1$ divides $p_i^{\alpha_i+1} - 1$. Grouping these accordingly we get

$$\prod_{i=1}^{k} (p_i + 1) \mid \prod_{i=1}^{k} \left(\frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \right) \mid \sigma(N),$$

which implies the desired inequality by comparing the exponents of 2 in the left- and right-hand sides of the above divisibility relation. $\square$

The next result will be used in the proof of Lemma 5 to show that a and $M = (10^m - 1)/9$ are coprime.

Lemma 4. *The number m is coprime to* 3.

Proof. Assume this is false, so $3 \mid m$. Then $3 \mid M$, and in addition $7 \mid M$ if and only if $6 \mid M$.

Assume first that M is a multiple of 3 but not of 7. Then m is an odd multiple of 3. Put $m = 3m_0$, and write

$$\begin{aligned} M &:= \frac{10^m - 1}{9} = \left(\frac{10^{m_0} - 1}{9} \right) (10^{2m_0} + 10^{m_0} + 1) =: A \cdot B, \\ A &:= A_1 \square, \qquad \text{and} \qquad B := B_1 \square, \end{aligned}$$

where A_1 and B_1 are squarefree. In what follows, we use the fact that odd squares are congruent to 1 modulo 8 whenever needed without mentioning it. The greatest common divisor of A and B is 1 or 3. The number B is a multiple of 3 but not of 9. Thus, B_1 is a multiple of 3. Since $m_0 = m/3 > 17$, the number B is congruent to 1 modulo 4, and therefore B_1 is congruent to 1 modulo 4 also. Since $3 \mid B_1$, it follows that there exists a prime $p_1 \geq 11$ dividing B_1 such that $p_1 \equiv 3 \pmod 4$.

Now consider the number A. We have

$$10^{m_0-1} + \cdots + 10^3 + 10^2 + 10 + 1 = A_1 \square.$$

Since $m_0 > 17$, the number on the left above is congruent to 7 modulo 8. Thus, A_1 is congruent to 7 modulo 8.

Suppose first that $3 \nmid A_1$. Then either $A_1 = p_2$ is a prime congruent to 7 modulo 8, or A_1 has at least two odd prime factors q_2 and q_3 such that $q_2 \equiv 3 \pmod 4$. At any rate, we get using Lemma 3 that

$$\begin{aligned} \nu_2(\sigma(x)) &\geq \sum_{\substack{p\mid M\\ \nu_p(M)\equiv 1 \pmod 2\\ p\geq 11}} \nu_2(p+1) \geq \nu_2(p_1+1) + \sum_{\substack{p\mid A_1\\ p\geq 11}} \nu_2(p+1) \\ &\geq 2 + \left\{\begin{matrix} \nu_2(p_2+1) & \text{and} & p_2 \equiv 7 \pmod 8, \text{ or} \\ \nu_2(q_2+1)+\nu_2(q_3+1) & \text{and} & q_2 \equiv 3 \pmod 4 \end{matrix}\right\} \\ &\geq 2+3=5. \end{aligned} \tag{1}$$

Suppose next that $3 \mid A_1$. Then $m_0 = 3m_1$ and

$$\begin{aligned} A &= \left(\frac{10^{m_0}-1}{9}\right) = \frac{10^{m_1}-1}{9}(10^{2m_1}+10^{m_1}+1) =: C \cdot D, \\ C &:= C_1\square, \qquad D := D_1\square, \end{aligned}$$

where C_1 and D_1 are squarefree. Since $m_1 = m_0/3 \geq 6$, we get as before that the greatest common divisor of C and D is 1 or 3, and that D_1 is divisible by a prime $q_2 \geq 11$ which is congruent to 3 modulo 4. Finally, again since $m_1 \geq 6$, we get as before that C_1 is congruent to 7 modulo 8, therefore it must be divisible by some prime $q_3 \geq 11$. Hence, invoking Lemma 3 again, we get that

$$\begin{aligned} \nu_2(\sigma(x)) \geq \sum_{\substack{p\mid M\\ \nu_p(M)\equiv 1 \pmod 2\\ p\geq 11}} \nu_2(p+1) &\geq \nu_2(p_1+1)+\nu_2(q_2+1)+\nu_2(q_3+1) \\ &\geq \quad 2 \quad + \quad 2 \quad + \quad 1 = 5. \end{aligned}$$

This takes care of the case when M is coprime to 7.

Assume next that $7 \mid M$. Then $6 \mid m$. Write $m = 6m_2$ and

$$\begin{aligned} M = \frac{10^{6m_2}-1}{9} &= \left(\frac{10^{m_2}-1}{9}\right)(10^{m_2}+1)(10^{2m_2}-10^{m_2}+1)(10^{2m_2}+10^{m_2}+1) \\ =: &\ ABCD, \quad \text{with} \quad A := A_1\square, \quad B := B_1\square, \quad C := C_1\square, \quad D := D_1\square, \end{aligned}$$

where A_1, B_1, C_1, and D_1 are squarefree. As in the analysis of the case when $7 \nmid M$, the greatest common divisor of any two of the numbers $A,\ B,\ C,\ D$ is 1 or 3.

Since $m_2 \geq 9$, it follows that $D \equiv 1 \pmod 8$, so $D_1 \equiv 1 \pmod 8$ and, since $9 \nmid D$, $3 \mid D_1$. Also, D_1 is a divisor of D which is number coprime to 5 (in fact, congruent to 1 modulo 5), so D_1 is coprime to 5 as well. Since D_1 is squarefree, it follows that if D_1 has no prime factor $p_1 \geq 11$, then $D_1 \in \{1, 3, 7, 21\}$. However, this last set does not contain any multiple of 3 which is congruent to 1 modulo 8. Hence, D_1 has a prime factor $p_1 \geq 11$.

Observe that C is coprime to 3 and $C \equiv 1 \pmod 8$, so $C_1 \equiv 1 \pmod 8$. Also, C is not a perfect square. Indeed, if it were then with $u := 10^{m_2}$, we would get $u^2 - u + 1 = v^2$ with some positive integer v, which can be rewritten as $(2u-1)^2 + 3 = (2v)^2$. Clearly, the only positive integer solution (u, v) of this last equation is $u = v = 1$, which is not convenient for us. Thus, $C_1 \notin \{1, 3, 7, 21\}$, and therefore C_1 has a prime factor $p_2 \geq 11$.

We next consider the number B. Clearly, $B \equiv 1 \pmod 8$; therefore $B_1 \equiv 1 \pmod 8$, and B is coprime to 3. The number B is not a square since if it were, then we would have $10^{m_2} + 1 = v^2$, or $10^{m_2} = (v-1)(v+1)$. Since both $v-1$ and $v+1$ are complementary divisors of the same parity of 10^{m_2} and their greatest common divisor divides their difference which is 2, one sees easily that the only possibility is $v + 1 = 2 \cdot 5^{m_2}$ and $v - 1 = 2^{m_2 - 1}$, which leads to

$$2 = (v+1) - (v-1) = 2(5^{m_2} - 2^{m_2-2}) > 2,$$

a contradiction. Hence, $B_1 \notin \{1, 3, 7, 21\}$, therefore B_1 has a prime factor $p_3 \geq 11$.

Finally, let us consider the number A. Suppose first that $3 \nmid A$. In particular, 7 does not divide A either. Observe that $A \equiv 7 \pmod 8$; therefore $A_1 \equiv 7 \pmod 8$, so A_1 has a prime factor $p_4 \equiv 3 \pmod 4$. Now using Lemma 3, we get

$$\nu_2(\sigma(x)) \geq \sum_{\substack{p \mid M \\ \nu_p(M) \equiv 1 \pmod 2 \\ p \geq 11}} \nu_2(p+1) \geq \sum_{i=1}^{4} \nu_2(p_i + 1) \geq 1 + 1 + 1 + 2 = 5.$$

Assume next that A is a multiple of 3 and write $m_2 = 3m_3$. Then $m_3 \geq 3$ and

$$\begin{aligned} A &= \frac{10^{3m_3} - 1}{9} = \left(\frac{10^{m_3} - 1}{9}\right)(10^{2m_3} + 10^{m_3} + 1) =: EF, \\ E &= E_1 \square, \qquad F := F_1 \square, \end{aligned}$$

where E_1 and F_1 are squarefree. The greatest common divisor of E and F is 1 or 3. Since $m_3 \geq m_2/3 \geq 3$, we have again that F and F_1 are both congruent to 1 modulo 8 and since $3 \mid F_1$, one concludes as in previous arguments there exists a prime $q_5 \geq 11$ dividing F. Returning to B, we have

$$\begin{aligned} B &= 10^{3m_3} + 1 = (10^{m_3} + 1)(10^{2m_3} - 10^{m_3} + 1) =: GH, \\ G &= G_1 \square, \qquad H := H_1 \square, \end{aligned}$$

where G_1 and G_2 are squarefree. The greatest common divisor of G and H is 1 or 3, and since $m_3 \geq 3$, it follows that $H \equiv 1 \pmod 8$; therefore $H_1 \equiv 1 \pmod 8$. A previous argument shows that H is not a perfect square; therefore $H_1 \notin \{1, 3, 7, 21\}$, so H_1 is divisible by some prime $r_6 \geq 11$. Finally, $G \equiv 1 \pmod 8$ and therefore $G_1 \equiv 1 \pmod 8$. Furthermore, a previous argument shows that G is not a perfect square; therefore $G_1 \notin \{1, 3, 7, 21\}$, so G_1 also has a prime factor $r_7 \geq 11$. From the above analysis we get

$$\begin{aligned}
\nu_2(\sigma(x)) &\geq \sum_{\substack{p \mid M \\ \nu_p(M) \equiv 1 \pmod 2 \\ p \geq 11}} \nu_2(p+1) \\
&\geq \nu_2(p_1+1) + \nu_2(p_2+1) + \nu_2(q_5+1) + \nu_2(r_6+1) + \nu_2(r_7+1) \\
&\geq 1 + 1 + 1 + 1 + 1 = 5.
\end{aligned}$$

In all possible cases, we have obtained that $\nu_2(\sigma(x)) \geq 5$, which is impossible. Hence, m is coprime to 3. □

The next lemma is an easy consequence of Lemma 4.

Lemma 5. *We have* $\gcd(a, M) = 1$. *In particular,* $\nu_2(\sigma(M)) \geq 3$.

Proof. Since M is coprime to both 2 and 5, it follows that if a and M are not coprime, then M is a multiple of either 3 or 7. In both cases, $3 \mid m$, which is not allowed by Lemma 4. Since a and M are coprime, we have that $\sigma(x) = \sigma(a)\sigma(M)$. Furthermore, since $m \geq 3$, we have that $M \equiv 7 \pmod 8$. Write $M := M_1\Box$, where M_1 is squarefree. Since M is coprime to 3 and 7, we get that either M_1 has a prime factor (≥ 11) congruent to $7 \pmod 8$, or M_1 has at least two prime factors (both ≥ 11), one of which is congruent to 3 modulo 4. The argument used to derive (1) based on Lemma 3 shows here that $\nu_2(\sigma(M)) \geq 3$, which is what we wanted to prove. □

It is now time to continue with the proof of Theorem 1. Recall that $y = s(x) = b(10^n - 1)/9$. We put $N = (10^n - 1)/9$. Lemma 5 tells us that $\nu_2(\sigma(N)) \geq 3$. Since we now know that $\nu_2(\sigma(x)) \geq \nu_2(\sigma(M)) \geq 3$, by Lemma 5, we get that $a + b \in \{8, 16\}$.

Suppose first that $a + b = 16$. Then $\{a, b\} = \{7, 9\}$, or $\{8, 8\}$. In the first case, assuming say that $a = 7$, we get $\nu_2(\sigma(x)) = \nu_2(\sigma(7M)) = \nu_2(8\sigma(M)) \geq 6$, which is impossible. In the second case, we get that $5 \mid 15 = \sigma(8) \mid \sigma(x) = x + y$, therefore

$$5 \mid x + y = \frac{8(10^m + 10^n) - 16}{9},$$

which is also impossible.

Hence, $a + b = 8$, therefore $\nu_2(\sigma(x)) = 3$. Since $\nu_2(\sigma(x)) = \nu_2(\sigma(a)) + \nu_2(\sigma(M))$, we get, by Lemma 5, that $\sigma(a)$ is odd, therefore $a \in \{1, 2, 4, 9\}$. A similar argument applies to b. Since $a + b = 8$, the only possibility is $a = b = 4$.

Let us now prove that m is odd. Indeed, if not, then $m = 2m_0$ and

$$M = \frac{10^{2m_0} - 1}{10 - 1} = \left(\frac{10^{m_0} - 1}{9}\right)(10^{m_0} + 1) := AB.$$

The two factors above A and B are coprime and B is not a square by an argument from the proof of Lemma 4. Now Lemma 5 shows that since $m_0 \geq 26$ and A is coprime to both 3 and 7, we have that $\nu_2(\sigma(A)) \geq 3$. Hence, $\nu_2(\sigma(M)) = \nu_2(\sigma(A)) + \nu_2(\sigma(B)) \geq 4$, which is a contradiction. A similar argument applies to n. Thus, both m and n are invertible modulo 6.

Now

$$7 = \sigma(4) \mid \sigma(x) = x + y = \frac{4(10^m + 10^n) - 8}{9},$$

giving that $10^m + 10^n \equiv 2 \pmod 7$. But m and n are congruent to $\pm 1 \pmod 6$. The case $m \equiv n \equiv 1 \pmod 6$ leads to $20 \equiv 2 \pmod 7$, which is false. The case $m \equiv n \equiv -1 \pmod 6$ leads to $2 \times 10^{-1} \equiv 2 \pmod 7$, which is again false. Finally, the case when one of m and n is congruent to 1 and the other is congruent to -1 modulo 6 leads to $10 + 10^{-1} \equiv 2 \pmod 7$, which is again false.

The theorem is therefore proved.

3. The Proof of Theorem 2

As in the proof of Theorem 1, we use $x = a(g^m - 1)/(g - 1)$ for some element of the aliquot cycle. We write $c_1, c_2, \ldots$ for possive computable constants which depend on g. They are labelled increasingly in their order of appearance. We also use the Landau symbol O and the Vinogradov symbol $\ll$ with their usual meaning. The constants implied by them also depend on g. For a positive integer m we use the standard notations $\tau(m)$, $\omega(m)$ and $\Omega(m)$ for the total number of divisors of m, the number of distinct prime divisors of m, and the number of prime power (> 1) divisors of m (or the number of primes appearing in the factorization of m counted with the appropriate multiplicity).

Assume that $\{n_1, \ldots, n_k\}$ is the set of components of an aliquot cycle $\mathcal{C}$, where we order these numbers as $n_1 < n_2 < \cdots < n_k$. By the result from [3], we may assume that $k \geq 2$. There exists $j \in \{1, \ldots, k-1\}$ such that $s(n_j) = n_k$. In particular, n_j is abundant. Put $x := n_j$. Then it suffices to show that x is bounded by some constant c_1. We proceed as follows. As in the proof of Theorem 1, put $y := s(x)$. Then $y > x$, therefore if we write $y = b(g^n - 1)/(g - 1)$, then $n \geq m$. Put $c_2 := \lfloor \log(2(g-1))/(\log 2) \rfloor + 1$ and assume that $x > g^{c_2}$. Then $m \geq c_2$, so $n \geq c_2$. The equation $\sigma(x) = x + y$ together with the fact that $m \geq c_2$, $n \geq c_2$, and g is even, implies that

$$(g-1)\sigma(x) = a(g^m - 1) + b(g^n - 1) \equiv -(a + b) \pmod{2^{c_2}}.$$

Since $a+b \le 2(g-1) < 2^{c_2}$, it follows that $\nu_2(\sigma(x)) \le c_3 := c_2 - 1$. Lemma 3 in [2] shows that there exists a constant c_4 depending on g, such that $(g^m-1)/(g-1)$ has in its prime factorization at least $\Omega(m) - c_4$ prime factors p appearing at odd exponents. Up to replacing c_4 by $c_4+\pi(g)$, we may assume that all these primes are greater than g. In particular, there are at least $\Omega(m)-c_4$ prime factors appearing at odd exponents in the factorization of x. Together with the present Lemma 3, it follows that $\nu_2(\sigma(x)) \ge \Omega(m) - c_4$.

This inequality is a key part of the proof. Combining these two facts, we get that $\Omega(m) \le c_5 := c_3 + c_4$. Put again $M := (g^m-1)/(g-1)$ and observe that

$$\frac{\sigma(x)}{x} \ll \frac{\sigma(M)}{M}. \tag{2}$$

Lemma 2 in [2], shows that

$$\frac{\sigma(M)}{M} \ll \log(e\omega(m))^2. \tag{3}$$

Since $\omega(m) \le \Omega(m) \le c_5$, it follows that $\sigma(x)/x \le c_6$. Now

$$c_6 \ge \frac{\sigma(x)}{x} = 1 + \frac{y}{x} = 1 + \left(\frac{b}{a}\right)\left(\frac{g^n-1}{g^m-1}\right) \ge 1 + \frac{g^{n-m}}{g-1},$$

showing that $n - m \le c_7$. Since all three parameters a, b and $n-m$ are at this point bounded, we may assume that a and b are fixed and that $n - m = c$ is also fixed. So, we need to study the equation

$$\sigma(x) = \sigma\left(a\left(\frac{g^m-1}{g-1}\right)\right) = \left(\frac{a+bg^c}{g-1}\right)g^m - \frac{a+b}{g-1}. \tag{4}$$

To proceed, we use the information that $\Omega(m) \le c_5$ and successively bound the possible prime factors of m. We first bound the smallest prime factor of m, let's call it $p(m)$. Well, let us assume that $p(m) > g$. It is easy to see, invoking Fermat's Little Theorem for example, that all prime factors p of M are congruent to 1 modulo some divisor $d > 1$ of m. In particular, they are all $> p(m) > g$. Hence, a and M are coprime. We get

$$\frac{\sigma(x)}{M} = \sigma(a)\left(\frac{\sigma(M)}{M}\right) = (a+bg^c)\left(\frac{g^m}{g^m-1}\right) - \frac{a+b}{(g-1)M} = a+bg^c+O\left(\frac{1}{g^m}\right). \tag{5}$$

The proof of Lemma 2 in [2] shows that

$$\log\left(\frac{\sigma(M)}{M}\right) \ll \sum_{\substack{d\mid m\\ d>1}} \frac{\log(ed)}{d} \ll \sum_{\substack{d\mid m\\ d>1}} \frac{\log d}{d},$$

where the right–most inequality follows because $3d \le d^3$ for all $d \ge 2$ (hence, $\log(ed) \le 3\log d$).

The function $d \mapsto (\log d)/d$ is decreasing for all $d \geq 3$ (note that $p(m) \geq 3$ since $p(m) > g \geq 2$). Furthermore, since all divisors $d > 1$ of m are at least $p(m)$, we get that

$$\sum_{\substack{d|m \\ d>1}} \frac{\log d}{d} \leq \frac{(\tau(m)-1)\log p(m)}{p(m)} < \frac{2^{\Omega(m)}\log p(m)}{p(m)} \leq \frac{c_8 \log p(m)}{p(m)},$$

where we can take $c_8 := 2^{c_5}$. If $p(m) > c_9$, where $c_9 > g$ is so large such that the inequality $c_8 \log p(m)/p(m) < 1/2$ holds, we then get that

$$\frac{\sigma(M)}{M} \leq \exp\left(\frac{c_8 \log p(m)}{p(m)}\right) < 1 + \frac{2c_8 \log p(m)}{p(m)}.$$

Returning to equation (5), we get that

$$\sigma(a) + O\left(\frac{\log p(m)}{p(m)}\right) = a + bg^c + O\left(\frac{1}{g^m}\right).$$

If $\sigma(a) \neq a + bg^c$, we see that the above estimate implies that $p(m)$ is bounded. Let us now treat the case when $\sigma(a) = a + bg^c$. If M is not a prime, then the smallest prime factor of M is $\leq M^{1/2} \ll g^{m/2}$, and therefore

$$\frac{\sigma(M)}{M} \geq 1 + \frac{c_{10}}{g^{m/2}}.$$

Hence, returning to equation (5), we get that

$$\sigma(a) + \frac{c_{10}}{g^{m/2}} < a + bg^c + O\left(\frac{1}{g^m}\right),$$

which via the fact that $\sigma(a) = a + bg^c$ gives $g^{m/2} \ll 1$, so m is bounded. Finally, assume that M is prime. Then equation (4) becomes

$$\sigma(a)\left(1 + \frac{1}{M}\right) = (a + bg^c)\left(1 + \frac{1}{(g-1)M}\right) - \frac{a+b}{(g-1)M},$$

giving

$$\sigma(a) = \frac{a + bg^c}{g-1} - \frac{a+b}{g-1}.$$

Since also $\sigma(a) = a + bg^c$, we get

$$\frac{a + bg^c}{g-1} - \frac{a+b}{g-1} = a + bg^c, \qquad \text{or} \qquad (a + bg^c)\left(1 - \frac{1}{g-1}\right) = -\frac{a+b}{g-1},$$

but this last relation is impossible since its left–hand side is ≥ 0 while its right–hand side is < 0.

And so, we have bounded $p(m)$.

We next use induction to bound successively the other prime factors of m. Namely, fix some positive integer $s \le c_5$ and assume that $m = p_1 p_2 \ldots p_s$, where $p_1 \le p_2 \le \cdots \le p_s$. Assume further that we have showed that for some $j \in \{1, \ldots, s-1\}$ the prime p_j is bounded by some constant depending on g. Observe that we have just shown such a statement with $j = 1$. Write $m = p_1 p_2 \cdots p_j m_1$, assume that $p_1 p_2 \cdots p_j$ is fixed, and all we need to do is to bound p_{j+1}, which is now the smallest prime factor of m_1. We use a similar argument as before. Put $g_1 := g^{p_1 \cdots p_j}$, $M_1 := (g_1^{m_1} - 1)/(g_1 - 1)$, $a_1 := a(g_1 - 1)/(g - 1)$, and observe that relation (4) can be rewritten as

$$\sigma(a_1 M_1) = \left(\frac{a + bg^c}{g-1}\right) g_1^{m_1} - \frac{a+b}{g-1}.$$

Assume that $p_{j+1} > g_1$. Then, since all prime factors of $M_1 = (g_1^{m_1} - 1)/(g_1 - 1)$ are congruent to 1 modulo some divisor $d > 1$ of m_1, it follows, in particular, that they are at least as large as $g_1 > a_1$. Hence, M_1 and a_1 are coprime, and so we get that

$$\sigma(a_1)\left(\frac{\sigma(M_1)}{M_1}\right) = \left(\frac{(a + bg^c)(g_1 - 1)}{g-1}\right)\left(\frac{g_1^{m_1}}{g_1^{m_1} - 1}\right) - \frac{a+b}{(g-1)M_1}. \tag{6}$$

The right–hand side above is

$$\frac{(a + bg^c)(g_1 - 1)}{g-1} + O\left(\frac{1}{g^m}\right). \tag{7}$$

On the left–hand side in relation (6) above, we have again that

$$\log\left(\frac{\sigma(M_1)}{M_1}\right) \ll \sum_{\substack{d \mid m_1 \\ d > 1}} \frac{\log d}{d} \ll \frac{\log p_{j+1}}{p_{j+1}}.$$

So, if $p_{j+1} > c_{11}$ is sufficiently large, then

$$\frac{\sigma(M_1)}{M_1} = \exp\left(\log\left(\frac{\sigma(M_1)}{M_1}\right)\right) = \exp\left(O\left(\frac{\log p_{j+1}}{p_{j+1}}\right)\right) = 1 + O\left(\frac{\log p_{j+1}}{p_{j+1}}\right). \tag{8}$$

Inserting estimates (7) and (8) into equation (6), we get

$$\sigma(a_1) + O\left(\frac{\log p_{j+1}}{p_{j+1}}\right) = \frac{(a + bg^c)(g_1 - 1)}{g-1} + O\left(\frac{1}{g^m}\right).$$

As before, if $\sigma(a_1) \ne (a + bg^c)(g_1 - 1)/(g - 1)$, we then get that $p_{j+1} \ll 1$, which is what we wanted. So, assume that $\sigma(a_1) = (a + bg^c)(g_1 - 1)/(g - 1)$. Again as before, if M_1 is not prime, then $\sigma(M_1)/M_1 \ge 1 + c_{12}/g^{m/2}$. Together with equation (6), we get that

$$\sigma(a_1) + \frac{c_{12}}{g^{m/2}} \le \frac{(a + bg^c)(g_1 - 1)}{g-1} + O\left(\frac{1}{g^m}\right),$$

which implies, via the fact that $\sigma(a_1) = (a + bg^c)(g_1 - 1)/(g - 1)$, that $g^{m/2} \ll 1$, so $m \ll 1$. Finally, if M_1 is prime, we then get that equation (6) is

$$\sigma(a_1)\left(1 + \frac{1}{M_1}\right) = \left(\frac{(a + bg^c)(g_1 - 1)}{g - 1}\right)\left(1 + \frac{1}{(g_1 - 1)M_1}\right) - \frac{a + b}{(g - 1)M_1},$$

which implies via the fact that $\sigma(a_1) = (a + bg^c)(g_1 - 1)/(g - 1)$, that the relation

$$\sigma(a_1) = \frac{a + bg^c}{g - 1} - \frac{a + b}{g - 1}$$

also holds. Since also $\sigma(a_1) = (a + bg^c)(g_1 - 1)/(g - 1)$, we get that

$$\frac{(a + bg^c)(g_1 - 1)}{g - 1} = \frac{a + bg^c}{g - 1} - \frac{a + b}{g - 1},$$

or

$$\frac{(a + bg^c)(g_1 - 2)}{g - 1} = -\frac{a + b}{g - 1}.$$

However, this is impossible since its left–hand side is ≥ 0, while its right hand side is < 0. This finishes the proof of the fact that $p_{j+1} \ll 1$, and of Theorem 2.

We conclude with a couple of open problems.

Problem 6. Extend Theorem 2 to the case of an odd base g.

Problem 7. Show that if $g > 1$ is fixed, then there are only finitely many repdigits in base g which are part of an amicable pair (with the other member of the amicable pair not necessarily a repdigit).

Acknowledgements. We thank the referee for a careful reading of the original manuscript and for suggestions which improved the quality of this paper. F. L. worked on this project during a visit to CWI in Amsterdam in October 2010 as a Beeger lecturer. During the preparation of this paper, F. L. was also supported in part by Grants SEP-CONACyT 79685 and PAPIIT 100508.

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