ON A THEOREM OF PRACHAR INVOLVING PRIME POWERS

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Received: 1/9/11, Accepted: 10/5/11, Published: 10/10/11

Abstract

Let p, with or without subscripts, always denote a prime number. In this paper we are able to establish two localized results on a theorem of Prachar which states that almost all positive even integers n can be written as $n = p_2^2 + p_3^3 + p_4^4 + p_5^5$. As a consequence of one result, we prove additionally that each sufficiently large odd integer N can be represented as $N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5$ with $\left|p_k^k - \frac{N}{5}\right| \le N^{1-\frac{1}{264}+\varepsilon}$ for $k = 1, \ldots, 5$.

1. Introduction

In 1951, Roth [20] proved that almost all positive integers n can be written in the form

$$n = m_2^2 + m_3^3 + m_4^4 + m_5^5,$$

where m_k are positive integers. Prachar [15] improved the above result two years later by showing that

$$n = p_2^2 + p_3^3 + p_4^4 + p_5^5 \tag{1}$$

is expressible for almost all positive even integers n. Here and in the sequel the letter p, with or without subscript, always stands for a prime number. Let $E(N)$ denote the number of even positive integers n up to N that cannot be written as (1). Then Prachar actually proved that $E(N) \ll N(\log N)^{-\frac{30}{47}+\varepsilon}$ where $\varepsilon > 0$ is arbitrary. In another paper [16], Prachar also considered the following equation

$$n = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5 \tag{2}$$

[1]The first author was partially supported by GIIFSDU (No. yzc09049).

and showed that all large odd integers n can be written in this way. After that, the cardinality of $E(N)$ has been improved by Bauer [1, 2, 3] and Ren and Tsang [18, 19] successively. And the best record at present, $O(N^{\frac{1633}{1680}+\varepsilon})$, was given by Bauer [4].

In this paper, we will investigate Prachar's result in short intervals:

$$\begin{cases} n = p_2^2 + p_3^3 + p_4^4 + p_5^5, \\ \left|p_k^k - \frac{N}{4}\right| \le U, \quad k = 2, \ldots, 5. \end{cases} \tag{3}$$

To state the result precisely, we let $E(N, U)$ be the number of all positive even integers n satisfying $N \le n \le N + U$ which cannot be written as (3). Our interest in $E(N, U)$ is twofold. First, a non-trivial bound of the type $E(N, U) \ll N^{1-\varepsilon}$ implies that almost all even integers n satisfying $N \le n \le N + U$ can be written in the form (3). Namely, we are interested in the bound $N^{1-\varepsilon}$ with U as small as possible. Furthermore, we will not only devote our attention to the size of U, but also be concerned with the cardinality of $E(N, U)$. In particular, we establish the following results in both directions outlined above.

Theorem 1. *For $U = N^{1-\frac{1}{36}+\varepsilon}$, we have*

$$E(N, U) \ll N^{1-\varepsilon}.$$

Theorem 2. *For $U = N^{1-\frac{1}{264}+\varepsilon}$, we have*

$$E(N, U) \ll U^{1-\varepsilon}.$$

Actually our results can be regarded as the localized version of Prachar's theorem involving unlike powers of primes. We remark that the second estimate implies the following theorem, which sharpens substantially the classical result (2), because one can combine it with known results on the distribution of primes in short intervals to deduce the statement in Theorem 3.

Theorem 3. *For every sufficiently large odd integer N, the equation*

$$\begin{cases} N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5, \\ \left|p_k^k - \frac{N}{5}\right| \le U, \quad k = 1, \ldots, 5 \end{cases}$$

is solvable in primes p_k for $U = N^{1-\frac{1}{264}+\varepsilon}$.

Theorems 1 and 2 will be proved by the circle method. A result of this strength needs efforts in three aspects. When treating the major arcs, we will apply the iterative method in Liu [12] and the mean-value estimate for Dirichlet polynomials in Choi and Kumchev [6] to establish the asymptotic formulae for the number of solutions to the problem. In fact we are able to control quite large major arcs for the problem, which will occupy a great portion of this paper (see Sections 3–5).

On the other hand, in handling the minor arcs, the new estimate for exponential sums over primes in short intervals in the recent work of the second author [21] in combination with the estimate in Liu, Lü and Zhan [13] will play an important role. Additionally, we will need the mean-value estimate for exponential sums over the minor arcs. And as usual, the joint contribution of these two aspects on the minor arcs is expected to be as "small" as possible. Combining this with the argument on the major arcs, we can finally determine and calculate the magnitude of the main parameter U in the theorems. The full details will be explained in the following relevant places.

Notation. As usual, $\varphi(n)$, $\mu(n)$ and $\Lambda(n)$ stand for the functions of Euler, Möbius and von Mangoldt respectively. We use $\chi \bmod q$ to denote a Dirichlet character modulo q, and $\chi^0 \bmod q$ the principal character; and use the notation $\sum^*$ to denote sums over all primitive characters. N is a large integer and n satisfies $n \asymp N$, and thus we use L to denote both $\log N$ and $\log n$. $r \sim R$ means $R < r \leq 2R$. The letters ε and A denote positive constants which are arbitrarily small and sufficiently large respectively, c denotes a positive constant which may vary at different places.

2. Outline of the Method and Proof of Theorem 1

We shall concentrate on proving Theorem 1, and then describe the straightforward modifications needed for Theorem 2 and present the proof of Theorem 3 at the end of the paper.

We first introduce the notation

$$\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \qquad \bar{\mu} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Then for even integer n satisfying $N \leq n \leq N + U$ and $U = N^{1-\frac{1}{36}+\varepsilon}$, consider

$$r(n,U) = \sum_{\substack{n=p_2^2+p_3^3+p_4^4+p_5^5 \\ \left|p_k^k - \frac{N}{4}\right| \leq U,\ k=2,\ldots,5}} (\log p_2)\cdots(\log p_5)$$

which is the weighted number of representations of (3). For

$$Y = \frac{N}{4} - U, \qquad X = \frac{N}{4} + U,$$

we define the exponential sums

$$S_k(\alpha) = \sum_{Y \leq p^k \leq X} (\log p) e(p^k \alpha), \quad k = 2, \ldots, 5.$$

Then we have

$$r(n,U)=\int_0^1 S_2(\alpha)\cdots S_5(\alpha)e(-n\alpha)\mathrm{d}\alpha. \tag{4}$$

In order to apply the circle method, we set

$$P=N^{\varepsilon}, \qquad Q=N^{\frac{29}{36}}.$$

By Dirichlet's lemma on rational approximation, each $\alpha\in[\frac{1}{Q},1+\frac{1}{Q}]$ may be written in the form $\alpha=\frac{a}{q}+\lambda$ with $1\le q\le Q$ and $|\lambda|\le\frac{1}{qQ}$. We define $\mathfrak{n}$ to be the subset of α with

$$P\le q\le Q, \qquad |\lambda|\le\frac{1}{qQ}.$$

To define the major arcs, let

$$P_*=N^{\frac{\varepsilon}{2}}, \qquad Q^*=N^{\frac{11}{12}+2\varepsilon}. \tag{5}$$

Then the major arcs $\mathfrak{M}$ are defined as the union of all intervals

$$\left[\frac{a}{q}-\frac{1}{qQ^*},\frac{a}{q}+\frac{1}{qQ^*}\right]$$

with $1\le a\le q\le P_*$ and $(a,q)=1$. Obviously $\mathfrak{M}$ and $\mathfrak{n}$ are disjoint. Let $\mathfrak{k}$ be the complement of $\mathfrak{M}$ and $\mathfrak{n}$ in $[\frac{1}{Q},1+\frac{1}{Q}]$, so that $[\frac{1}{Q},1+\frac{1}{Q}]=\mathfrak{M}\cup\mathfrak{n}\cup\mathfrak{k}$. Consequently, the formula (4) becomes

$$r(n,U)=\left\{\int_{\mathfrak{M}}+\int_{\mathfrak{n}\cup\mathfrak{k}}\right\}S_2(\alpha)\cdots S_5(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

To handle the integral on the major arcs, we have the following asymptotic formula which will be carried out in Sections 3–5.

Lemma 4. *Let $\mathfrak{M}$ be as above. Then for $N\le n\le N+U$ and any $A>0$,*

$$\int_{\mathfrak{M}} S_2(\alpha)S_3(\alpha)S_4(\alpha)S_5(\alpha)e(-n\alpha)\mathrm{d}\alpha=\frac{1}{120}\mathfrak{S}(n)\mathfrak{J}(n,U)+O\big(U^3N^{\mu-4}L^{-A}\big). \tag{6}$$

Here $\mathfrak{S}(n)$ is the singular series defined in (15), for which there exists an absolute positive constant c_0 such that

$$\mathfrak{S}(n)\gg(\log\log N)^{-c_0} \tag{7}$$

for any even integer n; while $\mathfrak{J}(n,U)$ is defined by (19) and satisfies

$$U^3N^{\mu-4}\ll\mathfrak{J}(n,U)\ll U^3N^{\mu-4}. \tag{8}$$

Next we estimate $S_4(\alpha)$ on $\mathfrak{n}\cup\mathfrak{k}$. We first estimate $S_4(\alpha)$ on $\mathfrak{n}$, and this has been done recently by the second author [21, Lemma 2]. This explains why we define $\mathfrak{n}$ in such a way.

Lemma 5. *Let $k \geq 2$, $K = 2^{k-1}$ and $\alpha \in \mathfrak{n}$. Then*

$$\sum_{x \leq p \leq x+y} (\log p) e\big(p^k \alpha\big) \ll y^{1+\varepsilon} \left(\frac{1}{P} + \frac{x^{\frac{1}{2}}}{y} + \frac{x^{\frac{K^2}{K+1}}}{y^K} + \frac{Qx^{k-1}}{y^{2k-1}} \right)^{\frac{1}{K^2}}.$$

Then taking $x = Y^{\frac{1}{4}}$ and $y = X^{\frac{1}{4}} - Y^{\frac{1}{4}}$ in Lemma 5, we have

$$S_4(\alpha) \ll UN^{-\frac{3}{4}+\frac{\varepsilon}{128}} \left(\frac{1}{P} + \frac{N^{\frac{7}{8}}}{U} + \frac{N^{\frac{70}{9}}}{U^8} + \frac{QN^6}{U^7} \right)^{\frac{1}{64}} \ll UN^{-\frac{3}{4}-\frac{\varepsilon}{128}}. \tag{9}$$

To bound $S_4(\alpha)$ on $\mathfrak{k}$, we need the following estimate for exponential sums over primes in short intervals established by Liu, Lü and Zhan [13, Theorem 1.1].

Lemma 6. *For $k \geq 1$, $2 \leq y \leq x$ and $\alpha = \frac{a}{q} + \lambda$, define*

$$\Xi = |\lambda| x^k + x^2 y^{-2}.$$

Then

$$\sum_{x \leq p \leq x+y} (\log p) e\big(p^k \alpha\big) \ll (qx)^{\varepsilon} \left\{ \frac{q^{\frac{1}{2}} y \Xi^{\frac{1}{2}}}{x^{\frac{1}{2}}} + q^{\frac{1}{2}} x^{\frac{1}{2}} \Xi^{\frac{1}{6}} + y^{\frac{1}{2}} x^{\frac{3}{10}} + \frac{x^{\frac{4}{5}}}{\Xi^{\frac{1}{6}}} + \frac{x}{q^{\frac{1}{2}} \Xi^{\frac{1}{2}}} \right\}.$$

Now we estimate $S_4(\alpha)$ on $\mathfrak{k}$. To this end, we further write $\mathfrak{k} = \mathfrak{k}_1 \cup \mathfrak{k}_2$, where

$$\mathfrak{k}_1 = \left\{ \alpha : 1 \leq q \leq P_*,\ \frac{1}{qQ^*} \leq |\lambda| \leq \frac{1}{qQ} \right\}$$

and

$$\mathfrak{k}_2 \subset \left\{ \alpha : P_* < q < P,\ |\lambda| \leq \frac{1}{qQ} \right\}.$$

For $\alpha \in \mathfrak{k}_1$, we have $|\lambda| \geq \frac{1}{qQ^*} \geq \frac{N}{U^2}$, and thus

$$\Xi \asymp |\lambda| N + \frac{N^2}{U^2} \asymp |\lambda| N.$$

Then Lemma 6 gives

$$\begin{aligned} S_4(\alpha) &\ll N^{\frac{\varepsilon}{4}} \left\{ \frac{UN^{-\frac{3}{4}} \sqrt{q|\lambda|N}}{N^{\frac{1}{8}}} + N^{\frac{1}{8}} q^{\frac{1}{2}} (|\lambda| N)^{\frac{1}{6}} + N^{-\frac{3}{10}} U^{\frac{1}{2}} + \frac{N^{\frac{1}{5}}}{(|\lambda|N)^{\frac{1}{6}}} + \frac{N^{\frac{1}{4}}}{\sqrt{q|\lambda|N}} \right\} \\ &\ll UN^{-\frac{3}{4}-\frac{\varepsilon}{128}}. \end{aligned} \tag{10}$$

If $\alpha \in \mathfrak{k}_2$, then

$$P_* < q < P, \quad \Xi \gg N^2 U^{-2}, \quad q\Xi \ll NQ^{-1} + PN^2U^{-2} \ll N^{\frac{7}{36}}.$$

Now Lemma 6 gives

$$S_4(\alpha) \ll N^{\frac{\varepsilon}{4}} \left\{ \frac{UN^{-\frac{3}{4}}(q\Xi)^{\frac{1}{2}}}{N^{\frac{1}{8}}} + N^{\frac{1}{8}} q^{\frac{1}{3}} (q\Xi)^{\frac{1}{6}} + N^{-\frac{3}{10}} U^{\frac{1}{2}} + \frac{N^{\frac{1}{5}}}{\Xi^{\frac{1}{6}}} + \frac{N^{\frac{1}{4}}}{(P_*\Xi)^{\frac{1}{2}}} \right\}$$
$$\ll UN^{-\frac{3}{4}-\frac{\varepsilon}{128}}. \tag{11}$$

From (9), (10) and (11), we obtain

Lemma 7. *Let* $\mathfrak{n}$ *and* $\mathfrak{k}$ *be as above. Then we have*

$$\sup_{\alpha \in \mathfrak{n} \cup \mathfrak{k}} |S_4(\alpha)| \ll UN^{-\frac{3}{4}-\frac{\varepsilon}{128}}.$$

In fact, in the argument above the magnitude of the main parameters U, P_*, Q^*, P, Q was determined simultaneously. We present more detailed explanation at this point. We first show the details for U. There are altogether five terms that restrict the lower bound of U, i.e. the 2nd & 3rd terms in (9), the 3rd term in (10) and the 3rd & 4th terms in (11). After calculation we can give the magnitude of U as shown in the beginning of the section from the 3rd term in (9) which is the worst one among these five terms. Since now U is fixed, the choice of P and Q will be in a matter of course. Note that we need $q\Xi \ll NQ^{-1} + PN^2U^{-2}$ in the first two terms of (11), and hence this makes it better to take the size of Q as large as possible and of P as small as possible. As for P, the 1st term in (9) gives the required lower bound immediately. While for Q, we can fix it from the 4th term in (9) by comparing with the 1st term in (10). Then one can find that the contribution of this choice for P and Q to the first two terms in (11) is also acceptable. Now it remains to show the choice of P_* and Q^*. Since P_* is the size of the major arcs, there will be other restrictions for its upper bound in the following proof of Lemma 4. This implies that it is reasonable to choose a good lower bound of P_* here. Then compared with the 2nd & 4th terms in (10), we take P_* as in (5) from the last term of (11). We will see in Sections 4 and 5 that this choice for P_* also meets all the demands there. For Q^*, the 4th & 5th terms in (10) give the restriction of its upper bound, and one finds easily that the choice in (5) is acceptable.

To bound the contribution of mean-value estimate for exponential sums over the minor arcs, we shall need the following lemma. Note that the argument can also be used in other similar problems (see [4] for example).

Lemma 8. *For* $U \gg N^{1-\frac{1}{30}}$*, we have*

$$\int_0^1 |S_2(\alpha)S_3(\alpha)S_5(\alpha)|^2 \mathrm{d}\alpha \ll U^4 N^{-\frac{44}{15}+\frac{\varepsilon}{128}}. \tag{12}$$

Proof. Denote the left-hand side of (12) by H, and let $\mathcal{N}_k := \left[Y^{\frac{1}{k}}, X^{\frac{1}{k}}\right]$, $N_k := \left|\mathcal{N}_k\right| \asymp UN^{\frac{1}{k}-1}$. Then the value of H equals the number of integer solutions of the

equation
$$a_1^2 - a_2^2 + b_1^3 - b_2^3 + c_1^5 - c_2^5 = 0,$$
where $a_i \in \mathcal{N}_2, b_i \in \mathcal{N}_3, c_i \in \mathcal{N}_5$. The solutions are of the three kinds:

1. $a_1 \neq a_2$;

2. $a_1 = a_2$, $b_1 \neq b_2$;

3. $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

Thus we have
$$H \ll N_3^2 N_5^{2+\frac{\varepsilon}{30}} + N_2 N_5^{2+\frac{\varepsilon}{30}} + N_2 N_3 N_5 \ll U^4 N^{-\frac{44}{15}+\frac{\varepsilon}{128}},$$
as required. □

Now we can establish Theorem 1.

Proof of Theorem 1. Applying Bessel's inequality, we have
$$\sum_{\substack{N \le n \le N+U \\ n \equiv 0 \pmod 2}} \left| \int_{\mathfrak{n} \cup \mathfrak{k}} S_2(\alpha) S_3(\alpha) S_4(\alpha) S_5(\alpha) e(-n\alpha) \mathrm{d}\alpha \right|^2$$
$$\ll \int_{\mathfrak{n} \cup \mathfrak{k}} |S_2(\alpha) S_3(\alpha) S_4(\alpha) S_5(\alpha)|^2 \mathrm{d}\alpha.$$
By Lemmas 7 and 8, the last integral is
$$\ll \sup_{\alpha \in \mathfrak{n} \cup \mathfrak{k}} |S_4(\alpha)|^2 \int_0^1 |S_2(\alpha) S_3(\alpha) S_5(\alpha)|^2 \mathrm{d}\alpha \ll (U^3 N^{\mu-4})^2 N^{1-\varepsilon/128}.$$
Thus, for all but $O(N^{1-\varepsilon/256})$ even integers $n \in [N, N+U]$, one has
$$\left| \int_{\mathfrak{n} \cup \mathfrak{k}} S_2(\alpha) S_3(\alpha) S_4(\alpha) S_5(\alpha) e(-n\alpha) \mathrm{d}\alpha \right| \ll U^3 N^{\mu-4-\varepsilon/512}.$$
Then Theorem 1 follows by Lemma 4. □

3. An Explicit Expression

The purpose of this section is to establish in Lemma 9 an explicit expression for the left-hand side of (6).

For $k = 2, \ldots, 5$ and Dirichlet character χ mod q, we define
$$C_k(\chi, a) = \sum_{h=1}^{q} \bar{\chi}(h) e\left(\frac{ah^k}{q}\right) \quad \text{and} \quad C_k(q, a) = C_k(\chi^0, a), \tag{13}$$

where χ^0 is the principal character modulo q, and $C_k(q,a)$ is the Ramanujan sum. For $\alpha = \frac{a}{q} + \lambda$ with $(a,q) = 1$, we have

$$S_k(\alpha) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{ah^k}{q}\right) \sum_{\substack{Y \le p^k \le X \\ p \equiv h \pmod q}} (\log p) e(p^k \lambda).$$

Note that for $q \le P_*$ and $Y \le p^k \le X, k = 2, \ldots, 5$, we have $(q,p) = 1$. Then by introducing Dirichlet characters to the above sum over p, one can rewrite $S_k(\alpha)$ as

$$\frac{C_k(q,a)}{\varphi(q)} V_k(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda),$$

where

$$V_k(\lambda) = \sum_{Y \le m^k \le X} e(m^k \lambda),$$

$$W_k(\chi, \lambda) = \sum_{Y \le p^k \le X} (\log p) \chi(p) e(p^k \lambda) - \delta_\chi \sum_{Y \le m^k \le X} e(m^k \lambda). \tag{14}$$

Here and throughout, δ_χ is 1 or 0 according as χ is principal or not.

For $j = 0, 1, \ldots, 15$, if we set

$$\mathcal{S}_j = \begin{cases} \{2,3,4,5\}, \text{ if } j=0; & \{2,3,4\}, \text{ if } j=1; \quad \{2,3,5\}, \text{ if } j=2; \quad \{2,3\}, \text{ if } j=3; \\ \{2,4,5\}, \quad \text{ if } j=4; & \{2,4\}, \quad \text{ if } j=5; \quad \{2,5\}, \quad \text{ if } j=6; \quad \{2\}, \quad \text{ if } j=7; \\ \{3,4,5\}, \quad \text{ if } j=8; & \{3,4\}, \quad \text{ if } j=9; \quad \{3,5\}, \quad \text{ if } j=10; \quad \{3\}, \quad \text{ if } j=11; \\ \{4,5\}, \quad \text{ if } j=12; & \{4\}, \quad \text{ if } j=13; \quad \{5\}, \quad \text{ if } j=14; \quad \varnothing, \quad \text{ if } j=15 \end{cases}$$

and $\bar{\mathcal{S}}_j = \{2,3,4,5\} \backslash \mathcal{S}_j$, then we have

$$\int_{\mathfrak{M}} S_2(\alpha) \cdots S_5(\alpha) e(-n\alpha) \mathrm{d}\alpha =: I_0 + I_1 + \cdots + I_{15},$$

where

$$I_j = \sum_{q \le P_*} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left\{ \prod_{k \in \mathcal{S}_j} C_k(q,a) \right\} e\left(-\frac{an}{q}\right)$$
$$\times \int_{-1/(qQ^*)}^{1/(qQ^*)} \left\{ \prod_{k \in \mathcal{S}_j} V_k(\lambda) \right\} \left\{ \prod_{k \in \bar{\mathcal{S}}_j} \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda) \right\} e(-n\lambda) \mathrm{d}\lambda.$$

Therefore we obtain

Lemma 9. *We have*

$$\int_{\mathfrak{M}} S_2(\alpha) \cdots S_5(\alpha) e(-n\alpha) \mathrm{d}\alpha = \sum_{j=0}^{15} I_j.$$

In the following sections we shall prove that I_0 produces the main term, while the others contribute the error term.

4. Treatment of the Major Arcs

We need some more notation. Let $\chi_2, \ldots, \chi_5$ be characters mod q, $C_k(\chi, a)$ be defined by (13). Then we take

$$B(n, q, \chi_2, \ldots, \chi_5) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \prod_{k=2}^{5} C_k(\chi_k, a),$$

$$B(n, q) = B(n, q, \chi^0, \ldots, \chi^0),$$

and write

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{15}$$

Lemma 10. *The singular series $\mathfrak{S}(n)$ satisfies* (7).

For the proof, one may see [15, Lemma 16] or [18, Lemma 5.3].

The following lemma, for which the proof is implied in Leung and Liu [11], plays an important role when we prove Lemma 4.

Lemma 11. *Let χ_k mod r_k with $k = 2, \ldots, 5$ be primitive characters, $r_0 = [r_2, \ldots, r_5]$, and χ^0 the principal character mod q. Then*

$$\sum_{\substack{q \le x \\ r_0 | q}} \frac{1}{\varphi^4(q)} \left| B(n, q, \chi_2\chi^0, \ldots, \chi_5\chi^0) \right| \ll r_0^{-1+\varepsilon} \log^c x.$$

Now we can establish the following asymptotic formula of I_0.

Lemma 12. *Let I_0 be as given in Lemma* 9*. Then for $N \le n \le N + U$ and any $A > 0$,*

$$I_0 = \frac{1}{120} \mathfrak{S}(n) \mathfrak{J}(n, U) + O(U^3 N^{\mu-4} L^{-A}),$$

where $\mathfrak{J}(n, U)$ is defined by (19) *and satisfies* (8).

Proof. Applying [10, Lemma 8.8] to $V_k(\lambda)$, we get for $k = 2, \ldots, 5$,

$$\begin{aligned} V_k(\lambda) &= \int_{Y^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(u^k \lambda) \mathrm{d}u + O(1) = \frac{1}{k} \int_{Y}^{X} v^{\frac{1}{k}-1} e(v\lambda) \mathrm{d}v + O(1) \\ &= \frac{1}{k} \sum_{Y \le m \le X} e(m\lambda) m^{\frac{1}{k}-1} + O(1). \end{aligned} \tag{16}$$

Substituting this into I_0 we see that

$$I_0 = \frac{1}{120} \sum_{q \le P_*} \frac{B(n,q)}{\varphi^4(q)} \int_{-1/(qQ^*)}^{1/(qQ^*)} \left\{ \prod_{k=2}^{5} \sum_{Y \le m \le X} e(m\lambda) m^{\frac{1}{k}-1} \right\} e(-n\lambda) \mathrm{d}\lambda$$
$$+ O\left(\sum_{q \le P_*} \frac{|B(n,q)|}{\varphi^4(q)} \int_{-1/(qQ^*)}^{1/(qQ^*)} \left| \prod_{k=2}^{4} \sum_{Y \le m \le X} e(m\lambda) m^{\frac{1}{k}-1} \right| \mathrm{d}\lambda \right). \tag{17}$$

Using the elementary estimate

$$\sum_{Y \le m \le X} e(m\lambda) m^{\frac{1}{k}-1} \ll N^{\frac{1}{k}-1} \min\left(U, \frac{1}{|\lambda|} \right) \tag{18}$$

and Lemma 11 with $r_0 = 1$, the O-term in (17) can be estimated as

$$\ll \sum_{q \le P_*} \frac{|B(n,q)|}{\varphi^4(q)} \left\{ \int_0^{U^{-1}} U^3 N^{\bar{\mu}-3} \mathrm{d}\lambda + \int_{U^{-1}}^{\infty} N^{\bar{\mu}-3} \frac{\mathrm{d}\lambda}{\lambda^3} \right\}$$
$$\ll U^2 N^{\bar{\mu}-3} L^c \ll U^3 N^{\mu-4} L^{-A}.$$

Now we extend the integral in the main term of (17) to $[-1/2, 1/2]$. Then by a similar argument we see that the resulting error is

$$\ll L^c \int_{1/(P_* Q^*)}^{1/2} N^{\mu-4} \frac{\mathrm{d}\lambda}{\lambda^4} \ll N^{\mu-4} (P_* Q^*)^3 L^c \ll U^3 N^{\mu-4} L^{-A},$$

where we have used (5). Thus by Lemma 10, (17) becomes

$$I_0 = \frac{1}{120} \mathfrak{S}(n) \mathfrak{J}(n,U) + O(U^3 N^{\mu-4} L^{-A}),$$

where

$$\mathfrak{J}(n,U) := \sum_{\substack{m_2 + \cdots + m_5 = n \\ Y \le m_k \le X,\ k=2,\ldots,5}} \left\{ \prod_{k=2}^{5} m_k^{\frac{1}{k}-1} \right\} \asymp U^3 N^{\mu-4}. \tag{19}$$

This finishes the proof of Lemma 12. □

To bound the contribution of I_j for $j = 1, 2, \ldots, 15$, we shall need the following preliminary Lemmas 13–15. In view of this, recall the definition of $W_k(\chi, \lambda)$ in (14) and further, for $k = 2, \ldots, 5$, write

$$J_k(g) = \sum_{r \le P_*} [g,r]^{-1+\varepsilon} \sum_{\chi \bmod r}^{*} \max_{|\lambda| \le 1/(rQ^*)} |W_k(\chi, \lambda)|,$$

$$K_k(g) = \sum_{r \le P_*} [g,r]^{-1+\varepsilon} \sum_{\chi \bmod r}^{*} \left(\int_{-1/(rQ^*)}^{1/(rQ^*)} |W_k(\chi, \lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}.$$

Here and throughout, $\sum^*$ indicates that the summation is taken over all primitive characters.

Lemma 13. *Let P_*, Q^* be as in* (5). *We have*

$$J_k(g) \ll g^{-1+\varepsilon} U N^{\frac{1}{k}-1} L^c.$$

Lemma 14. *Let P_*, Q^* be as in* (5). *For $g = 1$, Lemma* 13 *can be improved to*

$$J_k(1) \ll U N^{\frac{1}{k}-1} L^{-A},$$

where $A > 0$ is arbitrary.

Lemma 15. *Let P_*, Q^* be as in* (5). *We have*

$$K_k(g) \ll g^{-1+\varepsilon} U^{\frac{1}{2}} N^{\frac{1}{k}-1} L^c.$$

The proof of Lemmas 13–15 will be postponed to the next section. From the proof, one will actually find that the magnitude of P_* can be larger than that in (5). We should point, however, that the choice of P_* in (5) is merely convenient for our purpose to get the main parameter U in the theorem. Since more technical argument would be required, we select simplicity here and will not show how large we can take P_* to be in the three lemmas.

With these lemmas ready, we can now use the iterative method of [12] to give the upper bound of I_j for $j = 1, 2, \ldots, 15$. At this stage we shall point out that, in contrast to the previous problems we have investigated, we will be embedded in a slightly new situation in the present paper when applying the iterative procedures to choose the employ of maximizing or integrating $W_k(\chi, \lambda)$ for $k = 2, \ldots, 5$ from a set of multiple sums. Arising from the fact that the exponents of the prime variables in our problem vary from one another, we will have more than one choice. These different procedures could have been expected to be no essentially distinct, however the choice of the iterative procedures definitely results in whether we will appeal to J or K. Note that in the argument below, we will rely on J_2, J_3 and K_4, K_5.

Lemma 16. *Let $I_j, j = 1, \ldots, 15$ be as in Lemma* 9. *Then for $N \le n \le N + U$ and any $A > 0$,*

$$\sum_{j=1}^{15} I_j \ll U^3 N^{\mu-4} L^{-A}.$$

Proof. We begin with I_{15} which is the most complicated one. Reducing the characters in I_{15} into primitive characters, we have

$$\begin{aligned}|I_{15}| &= \Bigg| \sum_{q\le P_*} \sum_{\chi_2 \bmod q} \cdots \sum_{\chi_5 \bmod q} \frac{B(n,q,\chi_2,\ldots,\chi_5)}{\varphi^4(q)} \\ &\quad \times \int_{-1/(qQ^*)}^{1/(qQ^*)} W_2(\chi_2,\lambda)\cdots W_5(\chi_5,\lambda)e(-n\lambda)\mathrm{d}\lambda \Bigg| \\ &\le \sum_{r_2\le P_*} \cdots \sum_{r_5\le P_*} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \cdots \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \sum_{\substack{q\le P_*\\ r_0|q}} \frac{|B(n,q,\chi_2\chi^0,\ldots,\chi_5\chi^0)|}{\varphi^4(q)} \\ &\quad \times \int_{-1/(qQ^*)}^{1/(qQ^*)} \big|W_2(\chi_2\chi^0,\lambda)\big|\cdots\big|W_5(\chi_5\chi^0,\lambda)\big|\,\mathrm{d}\lambda,\end{aligned}$$

where χ^0 is the principal character modulo q and $r_0 = [r_2,\ldots,r_5]$. For $q \le P_*$ and $Y \le p^k \le X, k = 2,\ldots,5$, we have $(q,p) = 1$. Using this and (14), we have $W_k(\chi_k\chi^0,\lambda) = W_k(\chi_k,\lambda)$ for the primitive characters χ_k above. Thus by Lemma 11, we obtain

$$\begin{aligned}|I_{15}| &\le \sum_{r_2\le P_*} \cdots \sum_{r_5\le P_*} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \cdots \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \int_{-1/(r_0Q^*)}^{1/(r_0Q^*)} \big|W_2(\chi_2,\lambda)\big|\cdots\big|W_5(\chi_5,\lambda)\big|\,\mathrm{d}\lambda \\ &\quad \times \sum_{\substack{q\le P_*\\ r_0|q}} \frac{|B(n,q,\chi_2\chi^0,\ldots,\chi_5\chi^0)|}{\varphi^4(q)} \\ &\ll L^c \sum_{r_2\le P_*} \cdots \sum_{r_5\le P_*} r_0^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \cdots \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \\ &\quad \times \int_{-1/(r_0Q^*)}^{1/(r_0Q^*)} \big|W_2(\chi_2,\lambda)\big|\cdots\big|W_5(\chi_5,\lambda)\big|\,\mathrm{d}\lambda.\end{aligned}$$

In the last integral, we take out $|W_2(\chi_1,\lambda)|$ and $|W_3(\chi_3,\lambda)|$, and then use Cauchy's inequality to get

$$\begin{aligned}|I_{15}| &\ll L^c \Bigg\{ \prod_{k=2}^{3} \sum_{r_k\le P_*} \sideset{}{^*}\sum_{\chi_k \bmod r_k} \max_{|\lambda|\le 1/(r_kQ^*)} |W_k(\chi_k,\lambda)| \Bigg\} \\ &\quad \times \sum_{r_4\le P_*} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \left(\int_{-1/(r_4Q^*)}^{1/(r_4Q^*)} |W_4(\chi_4,\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \\ &\quad \times \sum_{r_5\le P_*} r_0^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \left(\int_{-1/(r_5Q^*)}^{1/(r_5Q^*)} |W_5(\chi_5,\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}. \qquad (20)\end{aligned}$$

Now we introduce the iterative procedure to bound the above sums over $r_5,\ldots,r_2$ consecutively.

We first estimate the above sum over r_5 in (20) via Lemma 15. Since $r_0 = [r_2, \ldots, r_5] = [[r_2, \ldots, r_4], r_5]$, the sum over r_5 in (20) is

$$\begin{aligned} &= \sum_{r_5 \le P_*} [[r_2, \ldots, r_4], r_5]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_5 \bmod r_5} \left(\int_{-1/(r_5 Q^*)}^{1/(r_5 Q^*)} |W_5(\chi_5, \lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \\ &= K_5([r_2, \ldots, r_4]) \ll [r_2, \ldots, r_4]^{-1+\varepsilon} U^{\frac{1}{2}} N^{\frac{1}{5}-1} L^c. \end{aligned}$$

This contributes to the sum over r_4 in (20) in amount

$$\begin{aligned} &\ll U^{\frac{1}{2}} N^{\frac{1}{5}-1} L^c \sum_{r_4 \le P_*} [[r_2, r_3], r_4]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_4 \bmod r_4} \left(\int_{-1/(r_4 Q^*)}^{1/(r_4 Q^*)} |W_4(\chi_4, \lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \\ &= U^{\frac{1}{2}} N^{\frac{1}{5}-1} L^c \, K_4([r_2, r_3]) \ll [r_2, r_3]^{-1+\varepsilon} U N^{\frac{1}{4}+\frac{1}{5}-2} L^c, \end{aligned}$$

by Lemma 15 again. The contribution of this quantity to the sum over r_3 in (20) is

$$\begin{aligned} &\ll U N^{\frac{1}{4}+\frac{1}{5}-2} L^c \sum_{r_3 \le P_*} [r_2, r_3]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda| \le 1/(r_3 Q^*)} |W_3(\chi_3, \lambda)| \\ &= U N^{\frac{1}{4}+\frac{1}{5}-2} L^c \, J_3(r_2) \ll r_2^{-1+\varepsilon} U^2 N^{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-3} L^c, \end{aligned}$$

where we have used Lemma 13. Inserting this last bound into (20), we can bound the sum over r_2 and find that

$$\begin{aligned} I_{15} &\ll U^2 N^{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-3} L^c \sum_{r_2 \le P_*} r_2^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \max_{|\lambda| \le 1/(r_2 Q^*)} |W_2(\chi_2, \lambda)| \\ &= U^2 N^{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-3} L^c \, J_2(1) \ll U^3 N^{\mu-4} L^{-A}, \end{aligned} \tag{21}$$

where we have used Lemma 14 in the last step.

For the estimation of the terms $I_1, I_2, \ldots, I_{14}$, by noting (16) and (18) we get

$$\left(\int_{-1/Q^*}^{1/Q^*} |V_k(\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \ll \left(Y^{\frac{2}{k}-2} U \right)^{\frac{1}{2}} \ll U^{\frac{1}{2}} N^{\frac{1}{k}-1}.$$

Using this estimate and the bound of $V_k(\lambda)$ in (16) & (18), we argue similarly to the treatment of I_{15} and obtain

$$\sum_{j=1}^{14} I_j \ll U^3 N^{\mu-4} L^{-A}. \tag{22}$$

Lemma 16 now follows from (21) and (22). □

5. Estimation of J and K

In this section, we will establish Lemmas 13–15. The proofs are similar to those of Lemmas 3.1–3.3 in [5], but with some minor technical modifications to our argument. We first present the details in the proof of Lemma 15. For this purpose, we shall introduce the following mean-value estimate for Dirichlet polynomials established by Choi and Kumchev [6, Theorem 1.1].

Lemma 17. *Let $R \geq 1, X \geq 2, T \geq 2$. Then we have*

$$\sum_{\substack{r\sim R\\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_T^{2T} \Big| \sum_{X \leq n \leq 2X} \Lambda(n)\chi(n)n^{-\mathrm{i}t} \Big| \mathrm{d}t \ll \left(\frac{R^2 T}{d} X^{11/20} + X \right) (\log RTX)^c.$$

5.1. Proof of Lemma 15

We approximate the $W_k(\chi,\lambda)$ in (14) by

$$\hat{W}_k(\chi,\lambda) = \sum_{Y \leq m^k \leq X} (\Lambda(m)\chi(m) - \delta_\chi) e(m^k\lambda).$$

Then the error is

$$W_k(\chi,\lambda) - \hat{W}_k(\chi,\lambda) \ll N^{\frac{1}{2k}}. \tag{23}$$

Therefore we have

$$\begin{aligned}
&\sum_{r \leq P_*} [g,r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ^*)}^{1/(rQ^*)} |W_k(\chi,\lambda) - \hat{W}_k(\chi,\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \\
&\ll N^{\frac{1}{2k}} \sum_{r \leq P_*} [g,r]^{-1+\varepsilon} \left(\frac{r}{Q^*} \right)^{\frac{1}{2}} \ll\ g^{-1+\varepsilon} N^{\frac{1}{2k}} Q^{*-\frac{1}{2}} \sum_{r \leq P_*} \left(\frac{r}{(g,r)} \right)^{-1+\varepsilon} r^{\frac{1}{2}} \\
&\ll g^{-1+\varepsilon} N^{\frac{1}{2k}} Q^{*-\frac{1}{2}} \sum_{\substack{d|g\\ d \leq P_*}} \sum_{\substack{r \leq P_*\\ d|r}} \left(\frac{r}{d} \right)^{-1+\varepsilon} r^{\frac{1}{2}} \ \ll\ g^{-1+\varepsilon} N^{\frac{1}{2k}} P_*^{\frac{1}{2}+\varepsilon} Q^{*-\frac{1}{2}} \\
&\ll\ g^{-1+\varepsilon} U^{\frac{1}{2}} N^{\frac{1}{k}-1} L^c,
\end{aligned}$$

where we have used $[g,r](g,r) = gr$ and (5).

Thus it suffices to show that

$$\sum_{r \sim R} [g,r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ^*)}^{1/(rQ^*)} |\hat{W}_k(\chi,\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \ll g^{-1+\varepsilon} U^{\frac{1}{2}} N^{\frac{1}{k}-1} L^c, \tag{24}$$

where $R \leq P_*$.

By Gallagher's lemma [8, Lemma 1], we have

$$\begin{aligned}\int_{-1/(rQ^*)}^{1/(rQ^*)} |\hat{W}_k(\chi,\lambda)|^2 \mathrm{d}\lambda &\ll \left(\frac{1}{RQ^*}\right)^2 \int_{-\infty}^{+\infty} \Bigg| \sum_{\substack{v<m^k\le v+rQ^* \\ Y\le m^k\le X}} (\Lambda(m)\chi(m)-\delta_\chi)\Bigg|^2 \mathrm{d}v \\ &\ll \left(\frac{1}{RQ^*}\right)^2 \int_{Y-rQ^*}^{X} \Bigg| \sum_{\substack{v<m^k\le v+rQ^* \\ Y\le m^k\le X}} (\Lambda(m)\chi(m)-\delta_\chi)\Bigg|^2 \mathrm{d}v.\end{aligned} \tag{25}$$

Let

$$Y^{1/k} \le y < x \le X^{1/k},$$
$$x-y \ll (v+rQ^*)^{1/k} - v^{1/k} \ll v^{1/k}\{(1+rQ^*/v)^{1/k}-1\} \ll rQ^*Y^{1/k-1}.$$

Then the last sum in (25) can be written as

$$\sum_{y\le m\le x} (\Lambda(m)\chi(m)-\delta_\chi). \tag{26}$$

In the case $R<1$, the quantity in (26) is

$$\ll x-y \ll Q^*Y^{1/k-1}.$$

This contributes to (24) as

$$g^{-1+\varepsilon}\left(\frac{1}{{Q^*}^2}U\frac{{Q^*}^2}{Y^{2-2/k}}\right)^{\frac{1}{2}} \ll g^{-1+\varepsilon}U^{\frac{1}{2}}N^{\frac{1}{k}-1},$$

which is acceptable.

For $R\ge 1$, we have $\chi\ne\chi^0$ and hence $\delta_\chi=0$. Applying Perron's summation formula (see for example, Theorem 2, p. 98 in [14] or Lemma 3.12 in [22]), we see that (26) can be written as

$$S := \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} F(s,\chi)\frac{x^s-y^s}{s}\mathrm{d}s + O(L^2)$$

for $T=N^{\frac{1}{k}}$ and $0<b<L^{-1}$, with

$$F(s,\chi) = \sum_{y\le m\le x} \Lambda(m)\chi(m)m^{-s}.$$

Using trivial estimates, we see that for $0<b<L^{-1}$

$$\frac{x^s-y^s}{s} \ll \min\left(T_0^{-1}, (|t|+b)^{-1}\right)$$

for $T_0 = N(RQ^*)^{-1}$. Thus for $b \to 0$, we have

$$S \ll T_0^{-1} \int_{|t| \le T_0} |F(\mathrm{i}t, \chi)| \mathrm{d}t + \int_{T_0 < |t| \le T} |F(\mathrm{i}t, \chi)| \frac{\mathrm{d}t}{|t|} + L^2. \tag{27}$$

Then from (25) and (27), we find that the left-hand side of (24) is

$$\begin{aligned} &\ll U^{\frac{1}{2}} N^{-1} \sum_{r \sim R} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{|t| \le T_0} |F(\mathrm{i}t, \chi)| \mathrm{d}t \\ &\quad + U^{\frac{1}{2}} (RQ^*)^{-1} \sum_{r \sim R} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_0 < |t| \le T} |F(\mathrm{i}t, \chi)| \frac{\mathrm{d}t}{|t|} \\ &\quad + g^{-1+\varepsilon} L^2 U^{\frac{1}{2}+\varepsilon} Q^{*-1}. \end{aligned}$$

Obviously the third term is acceptable. Therefore it follows that (24) is a consequence of the following two estimates: For $R \le P_*$ and $0 < T_1 \le T_0$, we have

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |F(\mathrm{i}t, \chi)| \mathrm{d}t \ll g^{-1+\varepsilon} N^{\frac{1}{k}} L^c; \tag{28}$$

while for $R \le P_*$ and $T_0 < T_2 \le T$, we have

$$\sum_{r \sim R} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} |F(\mathrm{i}t, \chi)| \mathrm{d}t \ll g^{-1+\varepsilon} R Q^* N^{\frac{1}{k}-1} T_2 L^c. \tag{29}$$

To show (28), we note that $[g, r](g, r) = gr$. Then the left-hand side of (28) is

$$\ll g^{-1+\varepsilon} \sum_{\substack{d|g \\ d \le R}} \left(\frac{R}{d}\right)^{-1+\varepsilon} \sum_{\substack{r \sim R \\ d|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |F(\mathrm{i}t, \chi)| \mathrm{d}t.$$

By Lemma 17, the above quantity can be estimated as

$$\begin{aligned} &\ll g^{-1+\varepsilon} \sum_{\substack{d|g \\ d \le R}} \left(\frac{R}{d}\right)^{-1+\varepsilon} \left(\frac{R^2 T_1}{d} N^{\frac{11}{20k}} + n^{\frac{1}{k}}\right) L^c \\ &\ll g^{-1+\varepsilon} \tau(g) \left(\frac{N^{1+\frac{11}{20k}}}{Q^*} + n^{\frac{1}{k}}\right) L^c \\ &\ll g^{-1+\varepsilon} N^{\frac{1}{k}} L^c. \end{aligned}$$

Similarly, we can prove (29) by taking $T = T_2$ in Lemma 17. Lemma 15 now follows.

5.2. Proof of Lemma 13

The contribution of $N^{\frac{1}{2k}}$ in (23) to $J_k(g)$ is

$$\sum_{r\le P_*}[g,r]^{-1+\varepsilon}rN^{\frac{1}{2k}} \ll g^{-1+\varepsilon}N^{\frac{1}{2k}}\sum_{\substack{d|g\\ d\le P_*}}\sum_{\substack{r\le P_*\\ d|r}}\left(\frac{r}{d}\right)^{-1+\varepsilon}r$$
$$\ll g^{-1+\varepsilon}N^{\frac{1}{2k}}P_*{}^{1+\varepsilon} \ll g^{-1+\varepsilon}UN^{\frac{1}{k}-1}L^{-A}.$$

Hence Lemma 13 is a consequence of the estimate

$$\sum_{r\sim R}[g,r]^{-1+\varepsilon}\sum_{\chi \bmod r}^{*}\max_{|\lambda|\le 1/(rQ^*)}|\hat{W}_k(\chi,\lambda)| \ll g^{-1+\varepsilon}UN^{\frac{1}{k}-1}L^c, \tag{30}$$

where $R \le P_*$.

The case $R<1$ contributes to (30) as $g^{-1+\varepsilon}UN^{\frac{1}{k}-1}L$ which is obviously acceptable. Therefore it remains to show (30) in the case $R\ge 1$.

For $R\ge 1$, we have $\delta_\chi = 0$ for all χ mod r in the definition of $\hat{W}_k(\chi,\lambda)$, and thus

$$\hat{W}_k(\chi,\lambda) = \sum_{Y\le m^k\le X}\Lambda(m)\chi(m)e(m^k\lambda).$$

Then by partial summation and Perron's formula, we get

$$\hat{W}_k(\chi,\lambda) = \frac{1}{2\pi \mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T}H(s,\chi)v(s,\lambda)\mathrm{d}s + O(1),$$

where $0<b<L^{-1}$, $T=(1+|\lambda|N)UN^{\frac{1}{k}-1}L^2$, and

$$H(s,\chi) = \sum_{Y\le m^k\le X}\Lambda(m)\chi(m)m^{-s}, \qquad v(s,\lambda) = \int_{Y^{\frac{1}{k}}}^{X^{\frac{1}{k}}} w^{s-1}e(w^k\lambda)\mathrm{d}w.$$

Using Lemmas 4.3 & 4.4 in [22] together with a trivial estimate, we have

$$v(s,\lambda) \ll N^{b/k}\min\left\{UN^{-1},\ \frac{1}{\sqrt{|t|+1}},\ \max_{Y^{1/k}\le w\le X^{1/k}}\frac{1}{|t+2k\pi\lambda w^k|}\right\}$$
$$\ll \begin{cases} UN^{b/k-1}, & \text{if } |t|\le T_*;\\ N^{b/k}/(|t|+1)^{1/2}, & \text{if } T_*<|t|\le T^*;\\ N^{b/k}/|t|, & \text{if } T^*<|t|\le T,\end{cases}$$

where

$$T_* = \frac{N^2}{U^2}, \qquad T^* = \frac{4k\pi N}{RQ^*}.$$

Here the choice of T^* is to ensure that $|t+2k\pi\lambda w^k| > |t|/2$ whenever $|t|>T^*$; in fact,

$$|t+2k\pi\lambda w^k| \ge |t| - 2k\pi|w^k|/(rQ^*) > |t|/2 + T^*/2 - 2k\pi|w^k|/(RQ^*) \ge |t|/2.$$

Then for $b \to 0$, we obtain

$$\hat{W}_k(\chi,\lambda) \ll UN^{-1} \int_{|t|\le T_*} |H(\mathrm{i}t,\chi)|\mathrm{d}t + \int_{T_*<|t|\le T^*} |H(\mathrm{i}t,\chi)| \frac{\mathrm{d}t}{\sqrt{|t|+1}} + \int_{T^*<|t|\le T} |H(\mathrm{i}t,\chi)| \frac{\mathrm{d}t}{|t|} + O(1).$$

The contribution of the last quantity $O(1)$ to (30) can be checked easily. Therefore, (30) is a consequence of the following three estimates: For $R \le P_*$ and $0 < T_1 \le T_*$, we have

$$\sum_{r\sim R} [g,r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} |H(\mathrm{i}t,\chi)|\mathrm{d}t \ll g^{-1+\varepsilon} N^{\frac{1}{k}} L^c; \tag{31}$$

and for $R \le P_*$ and $T_* < T_2 \le T^*$, we have

$$\sum_{r\sim R} [g,r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} |H(\mathrm{i}t,\chi)|\mathrm{d}t \ll g^{-1+\varepsilon} UN^{\frac{1}{k}-1} (T_2+1)^{\frac{1}{2}} L^c; \tag{32}$$

while for $R \le P_*$ and $T^* < T_3 \le T$, we have

$$\sum_{r\sim R} [g,r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} |H(\mathrm{i}t,\chi)|\mathrm{d}t \ll g^{-1+\varepsilon} UN^{\frac{1}{k}-1} T_3 L^c. \tag{33}$$

The estimates (31), (32) and (33) follow from Lemma 17 via the argument leading to (28) and (29). This proves Lemma 13.

5.3. Proof of Lemma 14

We will first state the following two lemmas which are well-known results in number theory. For the proof of Lemma 18, see for example pp. 669 & 801 in Pan-Pan [14], and for a slightly weak form which suffices for our purposes, see Huxley [9]. For the proof of Lemma 19, see Satz VIII.6.2 in Prachar [17].

Lemma 18. *For $T \ge 2$, let $N^*(\alpha,q,T)$ denote the number of zeros of all the L-functions $L(s,\chi)$ with primitive characters χ mod q in the region $\operatorname{Re} s \ge \alpha, |\operatorname{Im} s| \le T$. Then*

$$\sum_{q\le X} N^*(\alpha,q,T) \ll (X^2T)^{12(1-\alpha)/5} \log^{c_1}(X^2T),$$

where $c_1 > 0$ is an absolute constant.

Lemma 19. *Let $T \ge 2$. There is an absolute constant $c_2 > 0$, such that the product $\prod_{\chi \bmod q} L(s,\chi)$ is zero-free in the region*

$$\operatorname{Re} s \ge 1 - c_2/\max\{\log q, \log^{4/5} T\}, \qquad |\operatorname{Im} s| \le T,$$

except for the possible Siegel zero.

Now we turn to the proof. Clearly, Lemma 14 is the same as that of Lemma 13 except for the saving L^{-A} on its right-hand side. Because of this saving, we have to distinguish two cases according as R is small or large.

By an argument similar to that leading to (30), Lemma 14 is a consequence of the estimate

$$\sum_{r\sim R} r^{-1+\varepsilon} \sum_{\chi \bmod r}^{*} \max_{|\lambda|\le 1/(rQ^*)} |\hat{W}_k(\chi,\lambda)| \ll UN^{\frac{1}{k}-1}L^{-A}, \tag{34}$$

where $R \le P_*$ and $A > 0$ is arbitrary.

In the case $L^C < R \le P_*$ where C is a constant depending on A, we argue similarly in the proof of Lemma 13 for $g = 1$, and use $R > L^C$ to obtain the saving L^{-A}. The details are omitted.

Now we concentrate on the case $R \le L^C$, where $C > 0$ is arbitrary. We use the explicit formula (see [7, pp. 109 & 120], or [14, p. 313])

$$\sum_{m\le u} \Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma|\le T} \frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T}+1\right)\log^2(quT)\right), \tag{35}$$

where $\rho = \beta + \mathrm{i}\gamma$ runs over non-trivial zeros of the function $L(s,\chi)$, and $2 \le T \le u$ is a parameter. Taking $T = N^{\frac{1}{12}-\varepsilon}$ in (35), and then inserting it into $\hat{W}_k(\chi,\lambda)$, we get by $Y^{\frac{1}{k}} \le u \le X^{\frac{1}{k}}$ and (14) that

$$\begin{aligned}
\hat{W}_k(\chi,\lambda) &= \int_{Y^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(u^k\lambda)\mathrm{d}\Big\{\sum_{m\le u}(\Lambda(m)\chi(m)-\delta_\chi)\Big\} \\
&= -\int_{Y^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(u^k\lambda) \sum_{|\gamma|\le N^{\frac{1}{12}-\varepsilon}} u^{\rho-1}\mathrm{d}u + O\Big(\frac{UN^{\frac{1}{k}-1}}{N^{\frac{1}{12}-\varepsilon}}(1+|\lambda|N)L^2\Big) \\
&\ll UN^{\frac{1}{k}-1} \sum_{|\gamma|\le N^{\frac{1}{12}-\varepsilon}} N^{\frac{\beta-1}{k}} + O\big(UN^{\frac{1}{k}-1}L^{-A}\big).
\end{aligned}$$

Note that in the last step, the condition $Q^* \gg N^{\frac{11}{12}+\varepsilon}$ is required.

Now let $\eta(T) = c_3 \log^{-4/5} T$. By Lemma 19, $\prod_{\chi \bmod r} L(s,\chi)$ is zero-free in the region $\mathrm{Re}\, s \ge 1-\eta(T), |\mathrm{Im}\, s| \le T$ except for the possible Siegel zero. But by Siegel's theorem (see, for example, [7, §21]) the Siegel zero does not exist in the present situation, since $r \le L^C$. Thus by Lemma 18,

$$\begin{aligned}
\sum_{|\gamma|\le N^{\frac{1}{12}-\varepsilon}} N^{\frac{\beta-1}{k}} &\ll L^c \int_0^{1-\eta(N^{\frac{1}{12}-\varepsilon})} \big(N^{\frac{1}{12}-\varepsilon}\big)^{12(1-\alpha)/5} N^{\frac{\alpha-1}{k}}\mathrm{d}\alpha \\
&\ll L^c N^{-\eta(N^{\frac{1}{12}-\varepsilon})(\frac{1}{k}-\frac{1}{5})-\varepsilon} \ll \exp\big(-c_4 L^{1/5}\big).
\end{aligned}$$

Consequently,

$$\sum_{r\sim R}\sum_{\chi \bmod r}^{*}\max_{|\lambda|\le 1/(rQ^*)}|\hat{W}_k(\chi,\lambda)| \ll UN^{\frac{1}{k}-1}L^{-A},$$

for any $A>0$. This proves (34) in the second case, and the lemma follows.

6. Proof of Theorems 2 and 3

In this section, we outline the modifications necessary for the proof of Theorem 2 by the previous argument and give a terse proof of Theorem 3. For this purpose, we put

$$P = N^{\frac{4}{33}+\varepsilon}, \quad Q = N^{\frac{225}{264}}, \quad P_* = N^{\frac{1}{264}+\varepsilon}, \quad Q^* = N^{\frac{11}{12}+2\varepsilon}, \tag{36}$$

then define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{n},\mathfrak{k}$ as in Section 2 with P,Q,P_*,Q^* determined by (36). Then for even integer $n\in[N,N+U]$ and $U=N^{1-\frac{1}{264}+\varepsilon}$, consider

$$\begin{aligned}R(n,U) &:= \sum_{\substack{n=p_2^2+p_3^3+p_4^4+p_5^5\\ \left|p_k^k-\frac{N}{4}\right|\le U,\ k=2,\dots,5}}(\log p_2)(\log p_3)(\log p_4)(\log p_5)\\ &= \left\{\int_{\mathfrak{M}}+\int_{\mathfrak{n}\cup\mathfrak{k}}\right\}S_2(\alpha)S_3(\alpha)S_4(\alpha)S_5(\alpha)e(-n\alpha)\mathrm{d}\alpha.\end{aligned}$$

Moreover, define $J_k(g)$ and $K_k(g)$ as in Section 4. By the same treatment, we can estimate J and K for P_* replaced by (36) to get the desired upper bounds as shown in Lemmas 13–15. Then following the proof of Lemma 4, we can get the asymptotic formula of $R(n,U)$ on the major arcs.

Lemma 20. *Let the major arcs $\mathfrak{M}$ be as above. Then for $N\le n\le N+U$ and any $A>0$,*

$$\int_{\mathfrak{M}}S_2(\alpha)S_3(\alpha)S_4(\alpha)S_5(\alpha)e(-n\alpha)\mathrm{d}\alpha = \frac{1}{120}\mathfrak{S}(n)\mathfrak{J}(n,U)+O\big(U^3N^{\mu-4}L^{-A}\big).$$

Here $\mathfrak{S}(n)$ is the singular series defined in (15)*, for which there exists an absolute positive constant c_0 such that*

$$\mathfrak{S}(n) \gg (\log\log N)^{-c_0}$$

for any even integer n; while $\mathfrak{J}(n,U)$ is defined by (19) *and satisfies*

$$U^3N^{\mu-4} \ll \mathfrak{J}(n,U) \ll U^3N^{\mu-4}.$$

By using the similar argument in Section 2, we can estimate $S_4(\alpha)$ on $\mathfrak{n} \cup \mathfrak{k}$ and obtain

Lemma 21. *Let the minor arcs* $\mathfrak{n}$ *and* $\mathfrak{k}$ *be defined as above. Then we have*

$$\sup_{\alpha \in \mathfrak{n} \cup \mathfrak{k}} |S_4(\alpha)| \ll U^{\frac{3}{2}} N^{-\frac{5}{4} - \frac{\varepsilon}{128}}.$$

Here for the choice of the parameters, we only remark that we fix U from the second term in (9) which is different from that in Lemma 7. With Lemmas 20 and 21 known, we find by combining with Lemma 8 that Theorem 2 follows as required.

At the end of this section, we give the proof of Theorem 3.

Proof of Theorem 3. Let N be a sufficiently large odd integer, and $N_1 = \frac{4}{5}N$. Then we consider the subset of primes

$$\mathcal{P} = \left\{ p : \left| p - \frac{N}{5} \right| \leq U = N^{1 - \frac{1}{264} + \varepsilon} \right\}.$$

Obviously the number of elements in $\mathcal{P}$ is $\gg U/L$ by the prime number theorem. And thus there are $\gg U/L$ even integers n such that $n = N - p$ and $N_1 < n \leq N_1 + U$. Then by Theorem 2, we find that there exists a prime $p \in \mathcal{P}$ such that the equation

$$N - p = p_2^2 + p_3^3 + p_4^4 + p_5^5, \qquad \left| p_k^k - \frac{N}{5} \right| \leq U, \quad k = 2, \ldots, 5$$

has solutions. Hence Theorem 3 holds for odd integer N. □

Acknowledgements. The authors are grateful to Professors Jianya Liu and Guangshi Lü for their valuable suggestions and constant encouragement. Part of this work was completed when the first author visited Towson University supported by CSC. He would like to thank Professor Angel Kumchev for the beneficial discussions and his very helpful suggestions. The authors also thank the referee for his/her helpful comments.

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