ON 3-ADIC VALUATIONS OF GENERALIZED HARMONIC NUMBERS

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Received: 6/28/11, Revised: 8/26/11, Accepted: 10/31/11, Published: 11/2/11

Abstract

We investigate 3-adic valuations of generalized harmonic numbers $H_n^{(m)}$. These valuations are completely determined by the 3-adic expansion of n. Moreover, we also give 3-adic valuations of generalized alternating harmonic numbers.

1. Introduction

Harmonic numbers H_n $(n \geq 1)$ are rational numbers defined by partial sums of harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}. \tag{1}$$

These numbers appear in various areas in mathematics and have been investigated. It is well known that H_n goes to infinity as fast as the logarithmic function when n tends to infinity, i.e.,

$$\lim_{n\to\infty} \frac{H_n}{\log n} = 1.$$

This fact implies that the number H_n can be larger than any real numbers, but the following theorem holds.

Theorem 1. *The number H_n is never an integer for $n \geq 2$.*

It seems that this theorem was first proved by Theisinger [7], but there is a simple proof using the 2-adic valuation, as stated below. For any prime p and rational number x, we denote the p-adic order of x by $v_p(x)$ and use the notation $|x|_p = p^{-v_p(x)}$. For $n \geq 2$, let k be the unique integer such that $2^k \leq n < 2^{k+1}$. Then the right-hand side of (1) has the term $1/2^k$ and denominators of other terms have 2-adic order less than k. Therefore we have $|H_n|_2 = 2^k$ and this shows that H_n is never an integer for $n \geq 2$ (cf. [2, Theorem 1], [3, p. 258]).

n	$\lvert H_n^{(1)}\rvert_3$
$1*\cdots*$	3^k
$20*\cdots*$ $22*\cdots*$	3^{k-1}
$210*\cdots*$ $212*\cdots*$	3^{k-2}
$211*\cdots*$	3^{k-3}

Table 1: 3-adic valuations of $H_n^{(1)}$

For a positive integer m, generalized harmonic numbers $H_n^{(m)}$ are defined as follows:

$$H_n^{(m)} := \sum_{i=1}^{n} \frac{1}{i^m}$$

(e.g., [6]). It is clear that $H_n^{(1)} = H_n$. By the same argument above, we can obtain that $|H_n^{(m)}|_2 = 2^{km}$ where k is an integer satisfying $2^k \leq n < 2^{k+1}$. This means that $H_n^{(m)}$ is never an integer for any $m \geq 1$ and $n \geq 2$.

In this paper we investigate 3-adic valuations of $H_n^{(m)}$. The following is the main theorem of this paper, which completely determines 3-adic valuations of $H_n^{(m)}$. The results of Theorem 2 (i) are summarized in Table 1.

Theorem 2. *Let n be a positive integer with $a_k 3^k + a_{k-1}3^{k-1} + \cdots + a_1 \cdot 3 + a_0$, $(0 \leq a_i \leq 2,\quad 0 \leq i \leq k)$ being the 3-adic expansion of n. The 3-adic valuations of $H_n^{(m)}$ are determined as follows:*

(i) *For $m = 1$,*

- (a) *if $a_k = 1$, then $|H_n^{(1)}|_3 = 3^k \quad (k \geq 0)$.*
- (b) *if $a_k = 2$ and $a_{k-1} = 0, 2$, then $|H_n^{(1)}|_3 = 3^{k-1} \quad (k \geq 1)$.*
- (c) *if $a_k = 2$, $a_{k-1} = 1$ and $a_{k-2} = 0, 2$, then $|H_n^{(1)}|_3 = 3^{k-2} \quad (k \geq 2)$.*
- (d) *if $a_k = 2$, $a_{k-1} = 1$ and $a_{k-2} = 1$, then $|H_n^{(1)}|_3 = 3^{k-3} \quad (k \geq 2)$.*

(ii) *For $m \geq 2$ and $k \geq 0$, we have*

$$|H_n^{(m)}|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is even or } a_k = 1, \\ 3^{km - v_3(m) - 1} & \text{if } m \text{ is odd and } a_k = 2. \end{cases} \tag{2}$$

Here we explain relationships between this theorem and some known results. Theorem 2 implies that $|H_n^{(m)}|_3$ goes to infinity as n tends to infinity for any fixed m. This fact for $m = 1$ is stated in [2, p. 3] without proof. Eswarathasan and

Levine [4, Sect. 4] showed that $n = 1$,2, 6, 7, 8, 21, 22, 23, 66, 67 and 68 are all the numbers satisfying $|H_n|_3 \leq 1$. This fact can be also derived from Theorem 2 (i). In general, for any prime p, the sets

$$I(p) = \{n \geq 1; |H_n|_p \leq 1\} \quad \text{and} \quad J(p) = \{n \geq 1; |H_n|_p < 1\}$$

have been investigated (e.g., [1], [2] and [4]). The above example shows that $I(3) = \{1, 2, 6, 7, 8, 21, 22, 23, 66, 67, 68\}$ (we note that Eswarathasan et al. defined $H_0 = 1$, hence $0 \in I(3)$ in [4]). It is known that $|J(3)| = 3$, $|J(5)| = 3$, $|J(7)| = 13$ (cf. [4] and [5, Prob. 6.52]) and $|J(11)| = 638$ (see [1]). It seems to be difficult to determine $I(p)$ and $J(p)$ for general p, and it is conjectured that $I(p)$ and $J(p)$ are finite for all primes p ([4, Conjecture A]).

The alternating harmonic series is a simple analogy of the harmonic series, and it is well-known that this series converges to $\log 2$: $\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} = \log 2$. We also define generalized alternating harmonic numbers $A_n^{(m)}$ as follows:

$$A_n^{(m)} := \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i^m} \tag{3}$$

for positive integers m and n. We can also give 3-adic valuations of $A_n^{(m)}$, and we note that the alternating case is simpler than the ordinary one:

Theorem 3. *Let n be a positive integer and $n = a_k 3^k + a_{k-1} 3^{k-1} + \cdots + a_1 \cdot 3 + a_0$ be the 3-adic expansion of n. For $m \geq 1$ and $k \geq 0$, we have*

$$|A_n^{(m)}|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is odd or } a_k = 1, \\ 3^{km - v_3(m) - 1} & \text{if } m \text{ is even and } a_k = 2. \end{cases} \tag{4}$$

2. Proof of Theorem 2

In this section we prove Theorem 2. We first give a proposition and lemmas, which are used in the proof of our main theorems.

Proposition 4. *Let p be a prime, and let $k \geq 0$ and $m \geq 1$ be integers.*

(i) *The following are equivalent ($u_k = \lfloor n/p^k \rfloor$ where $\lfloor \cdot \rfloor$ is the floor function):*

(a) $\left| H_{u_k}^{(m)} \right|_p = 1$,

(b) $|H_n^{(m)}|_p = p^{km}$.

(ii) *If $p^k \leq n < 2p^k$, then $|H_n^{(m)}|_p = p^{km}$.*

Proof. (i) The number of multiples of p^k less than or equal to n is u_k. Hence we obtain that

$$H_n^{(m)} = \frac{1}{p^{km}}\left(\frac{1}{1^m}+\frac{1}{2^m}+\cdots+\frac{1}{u_k^m}\right)+\sum_{\substack{1\le i\le n\\ p^k\nmid i}}\frac{1}{i^m} = \frac{1}{p^{km}}H_{u_k}^{(m)}+\sum_{\substack{1\le i\le n\\ p^k\nmid i}}\frac{1}{i^m}.$$

The second term satisfies

$$\left|\sum_{\substack{1\le i\le n\\ p^k\nmid i^m}}\frac{1}{i^m}\right|_p < p^{km}$$

because of the property $|x+y|_p \le \max\{|x|_p,|y|_p\}$. Therefore we obtain $|H_n^{(m)}|_p = p^{km}|H_{u_k}^{(m)}+\alpha|_p$, where α is a rational number with $|\alpha|_p < 1$. By the property $|x+y|_p = \max\{|x|_p,|y|_p\}$ if $|x|_p \ne |y|_p$, the conditions $|H_{u_k}^{(m)}|_p = 1$ and $|H_n^{(m)}|_p = p^{km}$ are equivalent.

(ii) If $p^k \le n < 2p^k$, then $u_k = \lfloor n/p^k\rfloor = 1$. We note that $H_1^{(m)} = 1$ for any $m \ge 1$. Hence we obtain $|H_n^{(m)}|_p = p^{km}$ by the statement (i). □

Remark 5. The same statement of Proposition 4 holds for generalized alternating harmonic numbers $A_n^{(m)}$. This will be used in the proof of Theorem 3.

Lemma 6. *Let x be a positive integer such that $x \equiv 2, 5 \pmod 9$. Then*

$$v_3(x^{3^m}+1) = m+1 \tag{5}$$

for any integer $m \ge 0$.

Proof. We prove the lemma by induction on m. First we assume $m = 0$. Since $x^1+1 \equiv 3, 6 \pmod 9$, we get $v_3(x^1+1) = 1$. Next we assume (5) holds for some m, i.e., assume $v_3(x^{3^m}+1) = m+1$. We put $x^{3^m} = y$. Then the assumption says that $v_3(y+1) = m+1$. We have

$$\begin{aligned} v_3(x^{3^{m+1}}+1) = v_3((x^{3^m})^3+1) = v_3(y^3+1) &= v_3(y+1)+v_3(y^2-y+1)\\ &= m+1+v_3(y^2-y+1). \end{aligned}$$

We note that $y \equiv 2 \pmod 3$ because $x \equiv 2, 5 \pmod 9$. Hence we can write $y = 3u+2$ $(u \in \mathbb{Z})$. Then

$$y^2-y+1 = (3u+2)^2-(3u+2)+1 = 9u^2+9u+3 = 3(3u^2+3u+1)$$

and we have $v_3(y^2-y+1) = 1$. Therefore $v_3(y^3+1) = m+2$. Then equation (5) holds for $m+1$ and this completes the proof. □

Lemma 7. *For an integer $m \geq 0$, we have*

$$v_3(2^m + 1) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ v_3(m) + 1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. First we assume that m is even, say $m = 2t$ ($t \in \mathbb{Z}_{\geq 0}$). Since $2^2 \equiv 1 \pmod 3$, we have $2^m + 1 = (2^2)^t + 1 \equiv 2 \pmod 3$. This implies that $v_3(2^m + 1) = 0$.

Next we assume that m is odd. Then we can write $m = 3^{v_3(m)} q$ where q is an integer satisfying $q \equiv 1, 5 \pmod 6$. Since $2^6 \equiv 1 \pmod 9$, we see that $2^q \equiv 2, 5 \pmod 9$. Therefore, by Lemma 6, we obtain that

$$v_3(2^m + 1) = v_3\left((2^q)^{3^{v_3(m)}} + 1\right) = v_3(m) + 1,$$

which completes the proof. $\square$

Proof of Theorem 2. (i) For $n = a_k 3^k + a_{k-1} 3^{k-1} + \cdots + a_1 \cdot 3 + a_0$, we have

$$\left\lfloor \frac{n}{3^{k-j}} \right\rfloor = a_k 3^j + a_{k-1} 3^{j-1} + \cdots + a_{k-j}$$

for any $j \geq 0$. In the case (a), therefore, the value $\lfloor n/3^k \rfloor$ is equal to 1. In a similar way, we obtain the following values for each case:

(a) $\left\lfloor \frac{n}{3^k} \right\rfloor = a_k = 1$.

(b) $\left\lfloor \frac{n}{3^{k-1}} \right\rfloor = a_k \cdot 3 + a_{k-1} = 6, 8$.

(c) $\left\lfloor \frac{n}{3^{k-2}} \right\rfloor = a_k \cdot 3^2 + a_{k-1} \cdot 3 + a_{k-2} = 21, 23$.

(d) $\left\lfloor \frac{n}{3^{k-3}} \right\rfloor = a_k \cdot 3^3 + a_{k-1} \cdot 3^2 + a_{k-2} \cdot 3 + a_{k-3} = 66, 67, 68$.

Therefore, by Proposition 4, we only have to show that $|H_n|_3 = 1$ for $n = 1$, 6, 8, 21, 23, 66, 67 and 68. This can be checked by numerical calculations. We give the values of H_n below. These numbers satisfy $|H_n|_3 = 1$ and this proves (i).

n	H_n
1	1
6	$\frac{49}{20}$
8	$\frac{761}{280}$
21	$\frac{18858053}{5173168}$
23	$\frac{444316699}{118982864}$
66	$\frac{2090609990055351596677640233}{437876623741786025004208 00}$
67	$\frac{14050874595745034300902316411}{2933773379069966367528193600}$
68	$\frac{14094018321907827923954201611}{2933773379069966367528193600}$

(ii) First we assume that $a_k = 1$. Then $p^k \leq n < 2p^k$. By Proposition 4 (ii), we have $|H_n^{(m)}|_3 = 3^{km}$ for all $m \geq 1$.

Next we assume that $a_k = 2$. Then we have

$$H_n^{(m)} = \frac{1}{(3^k)^m} + \frac{1}{(2 \cdot 3^k)^m} + \sum_{\substack{i=1 \\ 3^k \nmid i}}^{n} \frac{1}{i^m} = \left(1 + \frac{1}{2^m}\right) \frac{1}{3^{km}} + \frac{r}{3^{(k-1)m}},$$

where r is a rational number such that $|r|_3 \leq 1$.

Now we see

$$\left| \frac{1}{3^{(k-1)m}} \right|_3 = 3^{km-m}.$$

On the other hand, by Lemma 7, we see

$$v_3\left(1 + \frac{1}{2^m}\right) = v_3(2^m + 1) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ v_3(m) + 1 & \text{if } m \text{ is odd.} \end{cases}$$

Hence

$$\left| \left(1 + \frac{1}{2^m}\right) \frac{1}{3^{km}} \right|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is even,} \\ 3^{km - v_3(m) - 1} & \text{if } m \text{ is odd.} \end{cases}$$

Because $v_3(m) + 1 < m$ for $m \geq 2$, we obtain that

$$\left| \left(1 + \frac{1}{2^m}\right) \frac{1}{3^{km}} \right|_3 > \left| \frac{r}{3^{(k-1)m}} \right|_3.$$

As a consequence, it follows that

$$|H_n^{(m)}|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is even,} \\ 3^{km - v_3(m) - 1} & \text{if } m \text{ is odd.} \end{cases}$$

The theorem follows. □

3. Generalized Alternating Harmonic Numbers

In this section, we prove Theorem 3. We first give the following lemma which is an analogue of Lemma 7. This lemma can be proved in the same way as the proof of Lemma 7, but we give another proof using Lemma 7.

Lemma 8. *For an integer $m \geq 1$, we have*

$$v_3(2^m - 1) = \begin{cases} 0 & \textit{if } m \textit{ is odd.} \\ v_3(m) + 1 & \textit{if } m \textit{ is even} \end{cases}.$$

Proof. First we assume that m is odd, say $m = 2t+1$ ($t \in \mathbb{Z}_{\geq 0}$). Since $2^2 \equiv 1 \pmod 3$, we have $2^m - 1 = 2\cdot(2^2)^t - 1 \equiv 1 \pmod 3$. This implies that $v_3(2^m-1) = 0$.

Next we assume that m is even, say $m = 2^t u$ where u is an odd integer. Then we have $2^m - 1 = 2^{2^t u} - 1 = (2^u-1)(2^u+1)(2^{2u}+1)\cdots(2^{2^{t-1}u}+1)$. We have $v_3(2^u-1) = 0$ from the odd case above. Moreover, we have $v_3(2^{2u}+1) = v_3(2^{2^2 u}+1) = \cdots = v_3(2^{2^{t-1}u}+1) = 0$ and $v_3(2^u+1) = v_3(u)+1$ by Lemma 7. Therefore we have $v_3(2^m-1) = v_3(u)+1$. By definition of u, it follows that $v_3(u) = v_3(m)$. Hence we obtain that $v_3(2^m-1) = v_3(m)+1$ and this completes the proof. $\square$

Now we prove Theorem 3. This theorem can be proved by exactly the same way as the proof of Theorem 2 (ii), but we give its proof to make the paper self-contained.

Proof of Theorem 3. First we assume that $a_k = 1$. As stated in Remark 5, Proposition 4 hods for $A_n^{(m)}$. Hence we have $|A_n^{(m)}|_3 = 3^{km}$ for all $m \geq 1$.

Next we assume that $a_k = 2$. Then we have

$$A_n^{(m)} = \frac{(-1)^{3^k-1}}{(3^k)^m} + \frac{(-1)^{2\cdot 3^k-1}}{(2\cdot 3^k)^m} + \sum_{\substack{i=1 \\ 3^k \nmid i}}^{n} \frac{(-1)^{i-1}}{i^m} = \left(1 - \frac{1}{2^m}\right)\frac{1}{3^{km}} + \frac{r}{3^{(k-1)m}},$$

where r is a rational number such that $|r|_3 \leq 1$.

Now we see

$$\left|\frac{1}{3^{(k-1)m}}\right|_3 = 3^{km-m}.$$

On the other hand, by Lemma 8, we see

$$v_3\left(1 - \frac{1}{2^m}\right) = v_3(2^m-1) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ v_3(m)+1 & \text{if } m \text{ is even.} \end{cases}$$

Therefore

$$\left|\left(1-\frac{1}{2^m}\right)\frac{1}{3^{km}}\right|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is odd,} \\ 3^{km-v_3(m)-1} & \text{if } m \text{ is even.} \end{cases}$$

Because $v_3(m)+1 < m$ for $m \geq 2$, we obtain that

$$\left|\left(1-\frac{1}{2^m}\right)\frac{1}{3^{km}}\right|_3 > \left|\frac{r}{3^{(k-1)m}}\right|_3$$

for any $m \geq 1$. As a consequence, we have

$$|A_n^{(m)}|_3 = \begin{cases} 3^{km} & \text{if } m \text{ is odd,} \\ 3^{km-v_3(m)-1} & \text{if } m \text{ is even,} \end{cases}$$

and this completes the proof. $\square$

References

[1] D. W. Boyd, A p-adic study of the partial sums of the harmonic series, *Exp. Math.* **3** (1994), 287–302.

[2] K. Conrad, The p-adic growth of harmonic sums, available at
`http://www.math.uconn.edu/~kconrad/blurbs/ugradnumthy/padicharmonicsum.pdf`.

[3] J. H. Conway and R. Guy, *The Book of Numbers*, 2nd edition, Springer, 1996.

[4] A. Eswarathasan and E. Levine, p-integral harmonic sums, *Discrete Math.* **91** (1991), 249–257.

[5] R. Graham, Donald E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley Professional, 1989.

[6] M. Kuba, H. Prodinger and C. Schneider, Generalized reciprocity laws for sums of harmonic numbers, *Integers* **8** (2008), ♯A17.

[7] L. Theisinger, Bemerkung uber die harmonische Reihe (in German), *Monatshefte für Mathematik* **26** (1915), 132–134.