A $(-\beta)$-EXPANSION ASSOCIATED TO STURMIAN SEQUENCES

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Received: 1/6/11, Revised: 8/30/11, Accepted: 12/4/11, Published: 1/2/12

Abstract

We consider a $(-\beta)$-expansion which makes use of the structure of the corresponding Sturmian sequences, and study some basic properties.

1. Introduction

Since Renyi [15] first introduced the theory of β-expansion, many aspects of that have been studied, such as the characterization of the admissible sequences and the shift spaces [15, 14], the conditions for finite or periodic expansions [6], the corresponding dynamical system [15, 14], and the self-similar tilings [1]. Recently, Ito-Sadahiro [8] proposed a theory of β-expansion with negative bases (we henceforth call it $(-\beta)_{IS}$-expansion), and studied those properties mentioned above. In this paper we consider another notion of $(-\beta)$-expansion, associated to the Sturmian sequence $v_0 = \{v_0(n)\}_{n\in\mathbf{Z}}$ with rotation β^{-1}. It makes use of the fact that the combinatorial composition of an element v_θ of the hull is equivalent to the approximation of θ in terms of a certain family of interval division of $[0, 1)$. The main feature is that the characterization of admissible sequences is simple and the shift space is that of finite type (SFT), while it can only be defined for particular β satisfying $\beta^2 = k\beta + 1$, $k \in \mathbf{N}$:

$$\beta = \beta_k := \frac{k + \sqrt{k^2 + 4}}{2}, \quad k = 1, 2, \dots$$

For general $\beta > 1$, one has to consider an extended notion of $(-\beta)$-expansion. As for related works, Góra [7] considered the transformations given by piecewise linear maps and computed the invariant densities. Dajani and Kalle [3] studied a family of transformations generating $(-\beta)$-expansions. Our $(-\beta)$-expansion belongs to this family after some rescaling and translations.

This paper is organized as follows. In Section 2, we recall basic facts on the Sturmian sequences. In Section 3, we consider the case $k = 1$ where $\beta = \beta_1 = \tau$ is the golden number. We separate the discussion since $\beta = \tau$ is the simplest case to consider, and the definition of the embedding operation is slightly different from the other cases. We define the $(-\tau)$-expansion and study the characterization of the admissible sequence and of the shift space. In Section 4, we consider the case of arbitrary $\beta = \beta_k$ ($k \in \mathbf{N}$), and study the same properties as well as the invariant measure of the corresponding shift map. In Section 5, we introduce the $(-\beta)$-expansion in an extended sense for general irrational $\beta > 1$. In Section 6, we consider arbitrary $(-\beta_k)$-expansions and show that they can be transformed, by successive application of local flips, to the $(-\beta_k)_{IS}$-expansion and that defined in this paper. In the Appendix, we review the main results in Ito-Sadahiro's paper [8] to compare with those obtained here.

2. Sturmian Sequences

As a preliminary, we recall basic facts on Sturmian sequences. Let $\alpha \in (0,1) \cap \mathbf{Q}^c$ and let

$$\begin{aligned} v_\theta(n) &:= 1_{[1-\alpha,1)}(\alpha n + \theta, \quad (\mathrm{mod}\ 1)), \\ v'_\theta(n) &:= 1_{(1-\alpha,1]}(\alpha n + \theta, \quad (\mathrm{mod}\ 1)), \quad n \in \mathbf{Z}, \quad \theta \in \mathbf{T} := [0,1) \end{aligned}$$

be the Sturmian sequences of rotation α. Let

$$\Omega := \overline{\{v_0(\cdot - m)\}_{m \in \mathbf{Z}}}$$

be the hull: the closure of the set $\{v_0(\cdot - m)\}_{m\in\mathbf{Z}}$ of translates of v_0 under the topology of pointwise convergence. It is known that

$$\Omega \;=\; \{v_\theta\}_{\theta \in \mathbf{T}} \cup \{v'_0(\cdot - m)\}_{m \in \mathbf{Z}} = \{v'_\theta\}_{\theta \in \mathbf{T}} \cup \{v_0(\cdot - m)\}_{m \in \mathbf{Z}}.$$

Let $\alpha = [a_1, a_2, \dots]$, $a_n \in \mathbf{N}$ be the continued fraction expansion of α. Let $\{s_n\}_{n=-1}^{\infty}$ be the sequence of words defined recursively by

$$\begin{aligned} s_{-1} &= 1, \quad s_0 = 0, \quad s_1 = s_0^{a_1 - 1} s_{-1} \\ s_{n+1} &= s_n^{a_{n+1}} s_{n-1}, \quad n \geq 1. \end{aligned} \tag{1}$$

Then the word $\{v_0(n)\}_{n=1}^{\infty}$ is equal to the limit r of $\{s_n\}$ in the sense that each s_n is the prefix of $\{v_0(n)\}_{n=1}^{\infty}$. And the word $\{v_0(-n)\}_{n=0}^{\infty}$ (resp. $\{v'_0(-n)\}_{n=0}^{\infty}$) is equal to the limit l of $\{s_{2n}\}$ (resp. limit l' of $\{s_{2n+1}\}$) in the sense that each s_{2n} (resp. s_{2n+1}) is the suffix of $\{v_0(-n)\}_{n=0}^{\infty}$ (resp. $\{v_0'(-n)\}_{n=0}^{\infty}$). That is,

$$\begin{aligned}
&r := \lim_{n\to\infty} s_n,\\
&l := \lim_{n\to\infty} s_{2n}, \quad l' := \lim_{n\to\infty} s_{2n+1},\\
&v_0 = lr, \quad v_0' = l'r.
\end{aligned}$$

It is also known that $l = \overline{r}(10)$, $l' = \overline{r}(01)$ where $\overline{r}$ is the reflection of r. We recall the results in [4]. The $(n-1,n)$-*partition* of $v \in \Omega$ is the non-overlapping covering of the sequence $\{v(n)\}_{n\in\mathbf{Z}}$ by two words s_{n-1}, s_n.

Lemma 1. ([4]) *For any $n \geq 0$, $v \in \Omega$ has the unique $(n-1,n)$-partition.*

Corollary 2. ([4]) *Let $n \geq 1$. In the $(n-1,n)$-partition of $v \in \Omega$,*
(1) s_{n-1} does not appear consecutively (s_{n-1} is always isolated), and
(2) s_n always appears a_{n+1} or $(a_{n+1}+1)$ times successively.

For instance, in the Fibonacci word ($k = 1$), $v_0 = \ldots 10110 \ldots$ has the unique $(0,1)$-partition $\ldots s_1 s_0 s_1 s_1 s_0 \ldots$ where s_0 is always isolated and s_1 appears at most twice successively.

3. Golden Number Case

In this section, $\beta = \tau$ is the golden number and $\alpha = \tau^{-1}$. Then $\{s_n\}_{n=0}^{\infty}$ satisfies the following recursion relation.

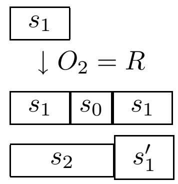
$$\begin{aligned}
s_0 &= 0, \quad s_1 = 1,\\
s_{n+1} &= s_n\, s_{n-1}, \quad n \geq 1. \qquad (2)
\end{aligned}$$

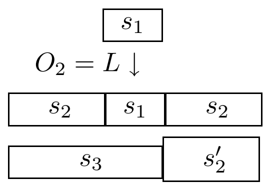
3.1. R, L-Construction of the Fibonacci Word

We recall a procedure discussed in [12] to construct combinatorially an element of Ω. By (2), one can consider the two operations $R : s_n \mapsto s_n s_{n-1} = s_{n+1}$, $L : s_n \mapsto s_{n+1} s_n = s_{n+2}$, to embed s_n into $s_{n'}$ ($n < n'$). They are the special cases of the de-substitution[11]. We start at s_0 or s_1 and the following argument shows that successive application of operations R or L gives us an element v in Ω. Let $W := \{R, L\}^{\mathbf{N}}$ be the set of operations and let $(O_1, O_2, \ldots) \in W$.

`Case (1)` $O_1 = R$: We put $s_1 = 1$ at 0 and let $v(0) = 1$. We add blocks s_k's to s_1

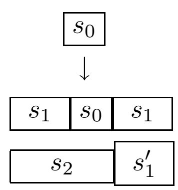
depending on whether $O_2 = R$ or $O_2 = L$, as are shown in the following figures.



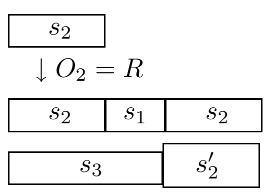


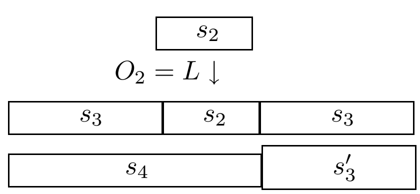
The dash s'_1 in $s_2s'_1$ in the first figure means that this part in v is not determined yet in the $(1,2)$-partition (and the same for s'_2 in the second figure). In fact, if this s'_1 was followed by s_0, it would be covered by s_2 in the $(1,2)$-partition.

`Case (2)` $O_1 = L$: We put $s_0 = 0$ at 0 and let $v(0) = 0$. Since s_0 is isolated in the $(0,1)$-partition by Corollary 2, its neighbor is uniquely determined as follows.



Hence we regard $s_2s'_1$ as the initial state and add blocks s_k's depending on whether $O_2 = R$ or L.

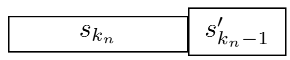




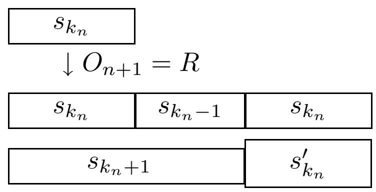
We continue this procedure. Set

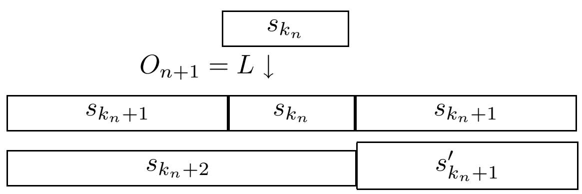
$$k_n := n + \sharp\{l \le n \,|\, O_l = L\}, \quad n = 1, 2, \dots$$

After carrying out $O_1, O_2, \dots, O_n$, we have



We then add blocks s_k's depending on whether $O_{n+1} = R$ or L as follows.





By repeating this procedure, we obtain $v \in \Omega$ for a given sequence of operations $(O_1, O_2, \dots) \in W$. In fact, if $\sharp\{l \,|\, O_l = L\} = \infty$, we have a double-sided sequence belonging to Ω. Otherwise, $O_j = R$ for all but finitely many j's and we obtain a single-sided sequence which coincides with a translation of $r := \lim_{n\to\infty} s_n = \{v_0(n)\}_{n\ge1}$. In this case we regard that $\{O_n\} \in W$ corresponds to the following two elements

$$\begin{aligned}\overline{r}\,10\,r &= v_0(\cdot + m)\\ \overline{r}\,01\,r &= v_0'(\cdot + m)\end{aligned}$$

for some $m \in \mathbf{N}$. Hence we have defined a correspondence $\tilde{\Phi} : W \to \Omega$. Conversely, for any $v \in \Omega$ we can construct corresponding sequence of operations $(O_1, O_2, \dots) \in W$ uniquely[12]: the inverse correspondence $\Phi (:= (\tilde{\Phi})^{-1}) : \Omega \to W$ is a well-defined map, which is two-to-one on $\Omega_R := \{v_0(\cdot + m), v_0'(\cdot + m)\}_{m\ge1}$.

3.2. $(-\tau)_S$-Expansion

Let $\Psi : \mathbf{T} \to \Omega$ be the map $\theta \in \mathbf{T} \overset{\Psi}{\mapsto} v_\theta \in \Omega$. The composition map $\Phi \circ \Psi : \mathbf{T} \overset{\Psi}{\to} \Omega \overset{\Phi}{\to} W$ corresponds to a sequence of interval division of $\mathbf{T}$ [12] as is explained below. We first decompose $\mathbf{T}$ into two intervals of ratio $1 : \tau$

$$\mathbf{T} \quad = \quad I_L \cup I_R := [0, 1-\alpha) \cup [1-\alpha, 1).$$

Then they are the inverse images of the cylinder set

$$\begin{aligned} I_L &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid O_1 = L\}) \\ I_R &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid O_1 = R\}). \end{aligned}$$

We further divide I_L in the same ratio

$$I_L = I_{LL} \cup I_{LR} := [0, \alpha^4) \cup [\alpha^4, \alpha^2),$$

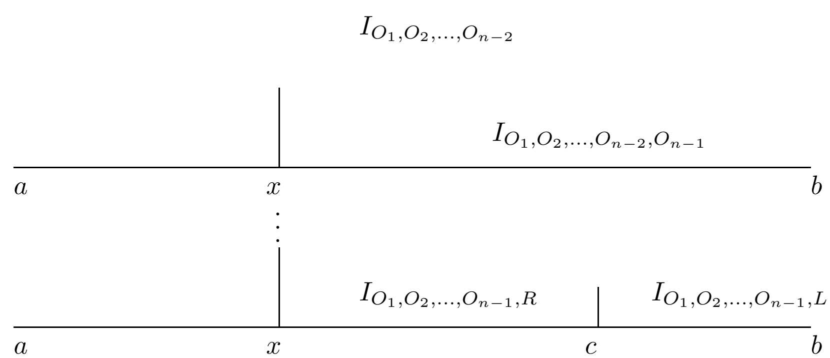
and we have

$$\begin{aligned} I_{LL} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (L, L)\}) \\ I_{LR} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (L, R)\}). \end{aligned}$$

We similarly divide I_R and have the same consequence:

$$\begin{aligned} I_R &= I_{RR} \cup I_{RL} := [\alpha^2, 2\alpha^2) \cup [2\alpha^2, 1) \\ I_{RR} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (R, R)\}) \\ I_{RL} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (R, L)\}). \end{aligned}$$

We repeat this procedure and define inductively the right-open half interval $I_{O_1,O_2,\dots,O_n} (\subset \mathbf{T})$ for a given sequence of operations $(O_1, O_2, \dots, O_n) \in \{R, L\}^n$. Suppose an interval $I_{O_1,O_2,\dots,O_{n-1}} (\subset [0, 1))$ is given by dividing its "parent" interval $I_{O_1,O_2,\dots,O_{n-2}}$ at x. We divide $I_{O_1,O_2,\dots,O_{n-1}}$ into two intervals such that the ratio of them is $\tau : 1$ from x, and let $I_{O_1,O_2,\dots,O_{n-1},R}$ (resp. $I_{O_1,O_2,\dots,O_{n-1},L}$) be the longer (resp. shorter) interval. For instance, in the figure below, $I_{O_1,O_2,\dots,O_{n-2}} = [a, b)$, $I_{O_1,O_2,\dots,O_{n-2},O_{n-1}} = [x, b)$, $I_{O_1,O_2,\dots,O_{n-2},O_{n-1},R} = [x, c)$, and $I_{O_1,O_2,\dots,O_{n-2},O_{n-1},L} = [c, b)$.



The interval division to define $I_{O_1,O_2,\dots,O_n}$

Then for any $(P_1, P_2, \dots, P_n) \in \{R, L\}^n$ we have

$$I_{P_1,P_2,\dots,P_n} = (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots, O_n) \in W \mid O_j = P_j,\ j = 1, 2, \dots, n\}).$$

Thus, constructing $v_\theta \in \Omega$ by a sequence of operations $(O_1, O_2, \dots) \in W$ is equivalent to approximating $\theta \in [0,1]$ by the corresponding sequence of intervals $\{I_{O_1,O_2,\dots,O_n}\}_{n=1}^{\infty}$.

Remark 3. Let $D_- := \{-n\alpha \pmod 1 \,|\, n \geq 1\}$ be the set of division points. For $\theta \in D_-$, taking the sequence of intervals is equivalent to approximating θ from above and thus $\Psi(\theta) = v_0(\cdot - m)$ for some $m \geq 1$. If these intervals $I_{O_1,O_2,\dots,O_n}$ were left-open, it would be equivalent to approximating θ from below and we would have $\lim_{\epsilon \downarrow 0} \Psi(\theta - \epsilon) = v_0'(\cdot - m)$.

We shall have a representation $\theta = \sum_{j=0}^{\infty} y_j(-\alpha)^j$ for given $\theta \in \mathbf{T}$ using this interval division. Let $(\Phi \circ \Psi)(\theta) = (O_1, O_2, \dots) \in W$ be the corresponding sequence of operations. We start from the point $1 + (-\alpha)$ which divides $\mathbf{T}$ into I_L and I_R.

(1) If $O_1 = R$, we add $(-\alpha)^2$ to go to $1 + (-\alpha) + (-\alpha)^2$ which divides I_R into I_{RR} and I_{RL}.

(2) If $O_1 = L$, we add $(-\alpha)^3$ to go to $1 + (-\alpha) + (-\alpha)^3$ which divides I_L into I_{LR} and I_{LL}.

We repeat this procedure. Set

$$p_n := \begin{cases} n + \sharp\{k \leq n-1 \,|\, O_k = L\} & (n \geq 2) \\ 1 & (n = 1) \end{cases}$$

then we have the expansion $\theta = 1 + \sum_{n=1}^{\infty}(-\alpha)^{p_n}$. Equivalently, let $\{y_j\}_{j=1}^{\infty}$ be the sequence obtained by applying the substitution $R \mapsto 1$, $L \mapsto 10$ to the sequence $(\Phi \circ \Psi)(\theta) = (O_1, O_2, \dots) \in W$. Then we have a power series expansion of θ in terms of $(-\alpha) = (-\beta)^{-1}$

$$\theta = 1 + \sum_{j=1}^{\infty} y_j(-\alpha)^j.$$

This definition is natural in the sense that this representation of θ is related to the combinatorial structure of the corresponding Sturmian sequence v_θ. However, if $\theta \in D_-$ is in the set of division points, the characterization of the admissible sequences would be complicated and, to be seen later, it would be impossible to define the shift map. This is mainly because the intervals $I_{O_1,O_2,\dots,O_n}$ are half-open. Therefore we slightly modify the intervals $I_{O_1,O_2,\dots,O_n}$ and consider the division of $[0,1]$ by another family of intervals $\{J_{O_1,O_2,\dots,O_n} \,|\, n \in \mathbf{N}, (O_1, O_2, \dots, O_n) \in \{R, L\}^n\}$ defined as follows. The interior is the same: $J^\circ_{O_1,\dots,O_n} = I^\circ_{O_1,\dots,O_n}$, but the division points always belong to the longer interval, that is, the one corresponding to $O_n = R$. For instance,

$$\begin{aligned} \mathbf{T} &= J_L \cup J_R := [0, 1-\alpha) \cup [1-\alpha, 1] \\ J_L &= J_{LL} \cup J_{LR} := [0, \alpha^4) \cup [\alpha^4, \alpha^2) \\ J_R &= J_{RR} \cup J_{RL} := [\alpha^2, 2\alpha^2] \cup (2\alpha^2, 1]. \end{aligned}$$

Definition 4. ($(-\beta)_S$-expansion) For a given $\theta \in [0,1]$, let $(O_1, O_2, \dots) \in W$ be the sequence of operations corresponding to the interval division $\{J_{O_1,O_2,\dots,O_n} \mid n \in \mathbf{N}, (O_1, O_2, \dots, O_n) \in \{R, L\}^n\}$ to approximate θ. Let $\{y_j\}_{j=1}^{\infty}$ be the sequence given by applying the substitution S defined by $S : R \mapsto 1$, $L \mapsto 10$ to the sequence $(O_1, O_2, \dots) \in W$. Then the power series expansion

$$\theta = 1 + \sum_{j=1}^{\infty} y_j(-\alpha)^j = 1 + \sum_{j=1}^{\infty} y_j(-\tau)^{-j} \tag{3}$$

of θ in terms of $(-\alpha)$ is called the $(-\tau)_S$-*expansion* of $\theta \in [0,1]$. We write $d_S(\theta, -\tau) = \{y_j\}_{j=1}^{\infty}$.

We always have $y_1 = 1$ in this expansion. By definition, we have infinitely many 1's in $\{y_j\}_{j=1}^{\infty}$; indeed 0 is isolated, so that we do not have finite expansion.

Remark 5. (1) Since we adopt $\{J_{O_1,\dots,O_n}\}$ as the interval division, the tails of $(\Phi \circ \Psi)(\theta)$ for $\theta \in D_-$ is always equal to $RR\overline{L} = (1, 1, (\overline{10}))$, while $LR\overline{L} = (1, 0, 1, (\overline{10}))$ does not appear[1]. (2) For $\theta \in D_-$, the relation to the Sturmian sequence v_θ is not simple anymore: some $\theta \in D_-$ corresponds to $v_0(\cdot - m)$ while others to $v'_0(\cdot - m')$.

Remark 6. It is possible to define the $(+\tau)$-expansion by the method above. To construct the corresponding intervals $\{J^+_{O_1,O_2,\dots,O_n} \mid n \in \mathbf{N}, (O_1, O_2, \dots, O_n) \in \{R, L\}^n\}$, we divide $J^+_{O_1,O_2,\dots,O_{n-1}}$ into two intervals by the ratio $\tau : 1$ such that the one closer to the origin is longer.

3.3. $(-\tau)_S$-Admissibility

Definition 7. ($(-\tau)_S$-admissibility) We say a sequence $\{y_j\}_{j=2}^{\infty} \in \{0,1\}^{\mathbf{N}}$ is $(-\tau)_S$-*admissible* if and only if it corresponds to the $(-\tau)_S$-expansion for some $\theta \in [0,1]$.

This is a condition for the $j \geq 2$ part of the sequence $\{y_j\}_{j=1}^{\infty}$, since we always have $y_1 = 1$. The consideration in the former subsection gives us the following characterization of $(-\tau)_S$-admissible sequences.

Theorem 8. *Let*

$$\begin{aligned} X &= \{\{y_j\}_{j=2}^{\infty} \in \{0,1\}^{\mathbf{N}} \mid \ 0 \textit{ is isolated } \} \\ Y &= \{\{y_j\}_{j=2}^{\infty} \in \{0,1\}^{\mathbf{N}} \mid \ \textit{tail is equal to } 101\overline{10} \ \}. \end{aligned}$$

Then we have $\{\{y_j\}_{j=2}^{\infty} \in \{0,1\}^{\mathbf{N}} \mid \ \{y_j\}_{j=2}^{\infty}$ *is* $(-\tau)_S$*-admissible* $\} = X \setminus Y$.

We end this subsection by a brief remark on the $(-\tau)_S$-admissibility. The condition that 0 is isolated is equivalent to

$$(010101\dots) \preceq_{lex} (x_n, x_{n+1}, \dots) \quad \text{for any } \ n \geq 2,$$

[1] $\overline{L} = LL\dots$ denotes the infinite repetition of L.

where $\preceq_{lex}$ denotes the lexicographic order. This is similar to the case for the $(+\tau)$-expansion, where the $(+\tau)$-admissibility of $\{x_n\}_{n=1}^{\infty}$ is equivalent to

$$(x_n, x_{n+1}, \dots) \preceq_{lex} (101010\dots) \quad \text{for any } \ n \geq 2 \tag{4}$$

and that the tail is not equal to $\overline{10}$. Condition (4) is natural since $(\overline{10})$ is the $(+\tau)$-expansion of 1 and $x < y$ is equivalent to $\{x_n\} \prec_{lex} \{y_n\}$ in the β-expansion (apart from some exceptional points). However, this is not the case for the $(-\tau)_S$-expansion. In fact, $d_S(\tau^{-1}, -\tau) = (\overline{1})$ is the maximum in $\{0,1\}^{\mathbf{N}}$ in the lexicographic order, while $\tau^{-1} \in [0,1]$ lies in the interior.

Thus we consider another ordering $\prec_{IS}$, which was introduced in [8].

Definition 9. (IS-ordering) For two sequences $\{c_k\}_{k=1}^{\infty}$, $\{d_k\}_{k=1}^{\infty} \in \{0,1\}^{\mathbf{N}}$, we define the ordering $\prec_{IS}$ as follows.
(1) $\{c_k\}_{k=1}^{\infty} \prec_{IS} \{d_k\}_{k=1}^{\infty} \overset{def}{\Longleftrightarrow} \{(-1)^k c_k\}_{k=1}^{\infty} \prec_{lex} \{(-1)^k d_k\}_{k=1}^{\infty}$.
(2) $\{c_k\}_{k=1}^{\infty} \preceq_{IS} \{d_k\}_{k=1}^{\infty} \overset{def}{\Longleftrightarrow} \{c_k\}_{k=1}^{\infty} \prec_{IS} \{d_k\}_{k=1}^{\infty}$ or $\{c_k\}_{k=1}^{\infty} = \{d_k\}_{k=1}^{\infty}$.

Since $d_S(0, -\tau) = (1,0,1,0,\dots)$ and $d_S(1,-\tau) = (1,1,0,1,\dots)$, it is reasonable to expect that $(-\tau)_S$-admissibility is equivalent to:

$$\text{for any } \ k \geq 2, \quad (1,0,1,0,\dots) \preceq_{IS} (d_k, d_{k+1}, \dots) \preceq_{IS} (0,1,0,1,\dots). \tag{5}$$

However this equivalence is not true. In fact, $\overline{01}$, (resp. $\overline{10}$) is the maximum (resp. the minimum) in $\{0,1\}^{\mathbf{N}}$ in the IS-ordering so that the condition (5) imposes no restriction on the sequences in $\{0,1\}^{\mathbf{N}}$. The reason is that we fix the expansion for $x \in D^-$ so that we would not have a statement like (5). Nevertheless, since we define the $(-\tau)_S$-expansion by interval division, the IS-ordering $\prec_{IS}$ preserves the order of θ.

Proposition 10. *We have: $\theta < \theta'$ if and only if $d_S(\theta, -\tau) \prec_{IS} d_S(\theta', -\tau)$.*

3.4. Shift Map

Let $T_{-\tau,S} : [0,1] \to [0,1]$ be the shift map sending $\theta = 1 + (-\alpha) + \sum_{y=2}^{\infty} y_j (-\alpha)^j$ to $\theta' = 1 + (-\alpha) + \sum_{y=2}^{\infty} y_{j+1} (-\alpha)^j$. By a direct computation,

$$T_{-\tau,S}(\theta) = \begin{cases} -\alpha^{-1}\theta + 1 & (\theta \in [0, 1-\alpha)) \\ -\alpha^{-1}\theta + \alpha^{-1} & (\theta \in [1-\alpha, 1]) \end{cases}$$

and we can rephrase the definition of the $(-\tau)_S$-expansion as

$$\theta = 1 + (-\alpha) + \sum_{j=2}^{\infty} y_j (-\alpha)^j, \quad y_j := 1_{[1-\alpha,1]}((T_{-\tau,S})^{j-2}(\theta)), \quad j \geq 2.$$

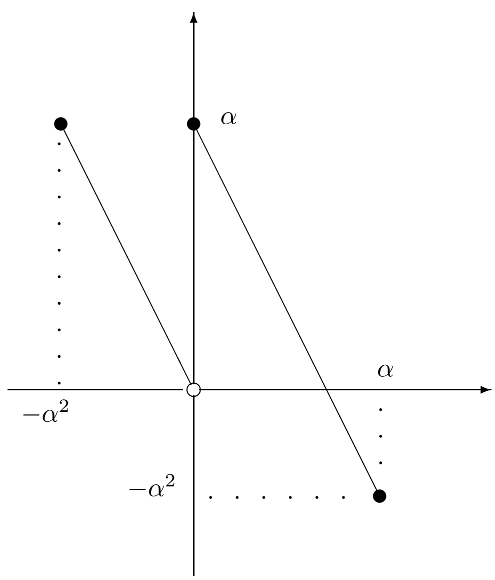
Remark 11. It is also possible to regard equation (3) as the $(-\tau)_S$-expansion of $\theta-(1+(-\alpha)) \in [-\alpha^2, \alpha]$: $\theta-(1+(-\alpha))=\sum_{k=2}^{\infty} y_k\,(-\alpha)^k$. Under this point of view, we translate $T_{-\tau,S}$:

$$\begin{aligned}\hat{T}_{-\tau,S}(\theta) &:= T_{-\tau,S}(\theta+\alpha^2)-\alpha^2\\ &= -\alpha^{-1}\theta+\alpha 1_A(\theta), \quad A=[0,\alpha]\,, \quad \theta\in[-\alpha^2,\alpha]\end{aligned}$$

and the definition of the $(-\tau)_S$-expansion becomes

$$\theta=\sum_{j=2}^{\infty} y_j(-\alpha)^j, \quad y_j := 1_A((\hat{T}_{-\tau,S})^{j-2}(\theta)), \quad j\ge 2, \quad \theta\in[-\alpha^2,\alpha].$$

Then, as is done for the $(-\beta)_{IS}$-expansion (Definition 23), we can expand any $\theta\in\mathbf{R}$.



The graph of $\hat{T}_{-\tau,S}$

3.5. Shift Space

Let

$$S_{-\tau,S} := \{\{x_n\}_{n\in\mathbf{Z}} \mid \text{ any finite subword of } \{x_n\} \text{ appears in } (-\tau)_S \text{ -admissible sequences }\}$$

be the shift-invariant set of double-sided sequences obtained by taking translations of $(-\tau)_S$-admissible sequences. Sequences $\{y_j\}_{j=2}^{\infty}$ whose tails are equal to $101\overline{10}$ can be approximated by $(-\tau)_S$-admissible sequences so that we have

Theorem 12. *We have* $S_{-\tau,S} = \{\{x_n\}_{n\in\mathbf{Z}} \mid 0$ *is isolated, i.e.,* 00 *is prohibited* $\}$.

Hence $S_{-\tau,S}$ is SFT.

4. General k Case

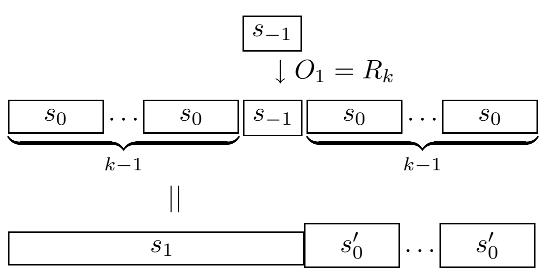
In this section we develop the notion of $(-\beta)_S$-expansion for $\beta = \beta_k$, which is the positive root of the equation $\beta^2 = k\beta + 1 (k \in \mathbf{N})$, along the discussion in the previous section. Let $\alpha = \beta_k^{-1}$. The recursion relation (1) of words $\{s_n\}_{n=-1}^{\infty}$ becomes

$$
\begin{aligned}
s_{-1} &= 1, \quad s_0 = 0, \quad s_1 = s_0^{k-1} s_{-1}, \\
s_{n+1} &= s_n^k \, s_{n-1}, \quad n \geq 1. \qquad (6)
\end{aligned}
$$

4.1. R, L-Construction of Sturmian Words

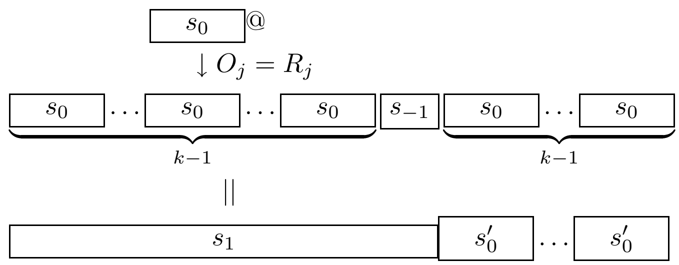
Equation (6) implies that there are k operations $R_1, R_2, \ldots, R_k$ to embed s_n into s_{n+1} so that we take $W := \{L, R_1, R_2, \ldots, R_k\}^{\mathbf{N}}$ as the set of operations. We define the correspondence $\tilde{\Phi} : W \to \Omega$ obtaining $v \in \Omega$ from $(O_1, O_2, \ldots) \in W$, as follows.

`Case (1)` $O_1 = R_k$: We put $s_{-1} = 1$ at 0. By Corollary 2, we have at least $(k-1)$ s_0's on both sides of s_{-1} and hence this s_{-1} is always embedded into the rightmost position of $s_1 = s_0^{k-1} s_{-1}$. Thus we have $s_1(s_0')^{k-1}$. As in Section 2.1, the dash in s_0' means that this is not determined in the $(0,1)$-partition.

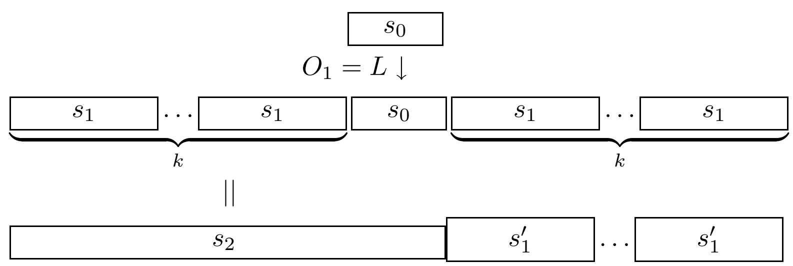


`Case (2)` $O_1 = R_j$ $(j = 1, 2, \ldots, k-1)$: We put $s_0 = 0$ at 0. Since $s_1 = s_0^{k-1}\ s_{-1}$, there are $(k-1)$ ways to embed s_0 into s_1 which correspond to operations $R_1, \ldots, R_{k-1}$ respectively. For instance, if $O_1 = R_j$ we embed s_0 into the

jth position counted from the left in s_1. We then have $s_1(s'_0)^{k-1}$.

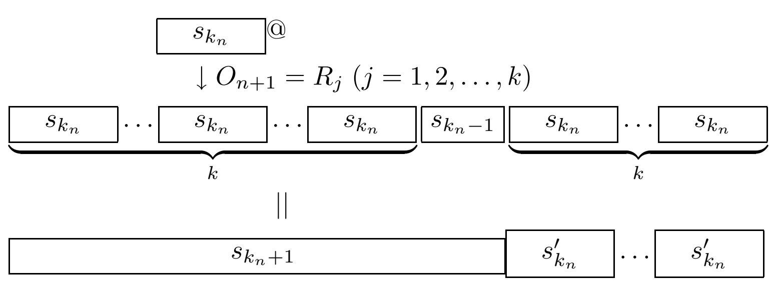


`Case (3)` $O_1 = L$: We put s_0 at 0 and embed s_0 into the rightmost position of $s_2 = s_1^k s_0$. By Corollary 2, s_0 is isolated in the $(0,1)$-partition and there are at least k s_1's on both sides of s_0. Hence we have $s_2(s'_1)^k$.

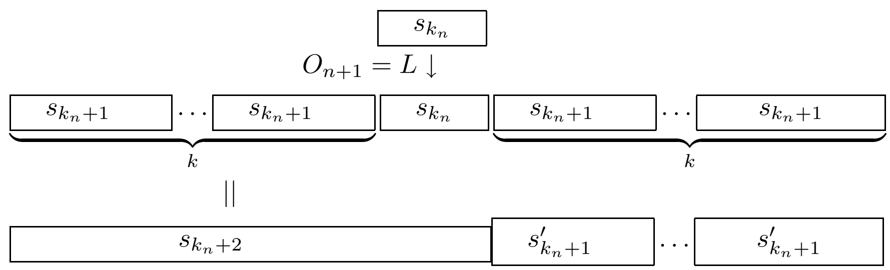


We repeat this procedure. After carrying out $O_1, O_2, \dots, O_n$, we have $s_{k_n}(s'_{k_n-1})^k$, where $k_n = n + \sharp\{l \leq n \mid O_l = L\}$. Depending on O_{n+1}, we embed it into larger s_k's as follows.

`Case (1)` $O_{n+1} = R_j$ $(j = 1, 2, \dots, k)$:



`Case (2)` $O_{n+1} = L$:



If the tail of $(O_1, O_2, \dots) \in W$ is equal to $\overline{R_1}$, we have a single-sided sequence and we regard that as corresponding to two elements $v_0(\cdot + m)$, $v'_0(\cdot + m)$ for some $m \in \mathbf{N}$. Thus we obtain the correspondence $\tilde{\Phi} : W \to \Omega$. As in Section 2.1, the inverse $\Phi := (\tilde{\Phi})^{-1} : \Omega \to W$ is a well-defined map.

4.2. $(-\beta)_S$-Expansion

Let $\Psi : \mathbf{T} \to \Omega$ be the map given by $\Psi(\theta) := v_\theta$. As in Section 2.2, we can explicitly derive the composition map $\theta \overset{\Psi}{\mapsto} v_\theta \overset{\Phi}{\mapsto} (O_1, O_2, \dots) \in W$ by the division of $[0, 1)$ by right-open half intervals. We first divide $[0, 1)$ into an interval I_0 of length α^2 and k intervals $I_1, \dots, I_k$ of length α:

$$\begin{aligned} [0,1) &= I_0 \cup I_1 \cup \cdots \cup I_k \\ I_0 &= [0, \alpha^2), \\ I_j &= \alpha^2 + (j-1)\alpha + [0, \alpha), \quad j = 1, 2, \dots, k. \end{aligned}$$

Then I_0 corresponds to the operation $O_1 = L$ and $I_1, I_2, \dots, I_k$ correspond to the operations $O_1 = R_1, R_2, \dots, R_k$ respectively:

$$\begin{aligned} I_0 &:= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid O_1 = L\}) \\ I_j &:= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid O_1 = R_j\}), \quad j = 1, \dots, k. \end{aligned}$$

The interval I_j $(j = 0, 1, \dots, k)$ is further divided, from the division point, into k intervals $I_{j,k}, I_{j,k-1}, \dots, I_{j,2}, I_{j,1}$ of length multiplied by α and an interval $I_{j,0}$ of length multiplied by α^2:

(1) $j = 0$

$$\begin{aligned} I_{0,0} &:= [0, \alpha^4) \\ I_{0,l} &:= \alpha^4 + (l-1)\alpha^3 + [0, \alpha^3) \quad (l = 1, \dots, k) \end{aligned}$$

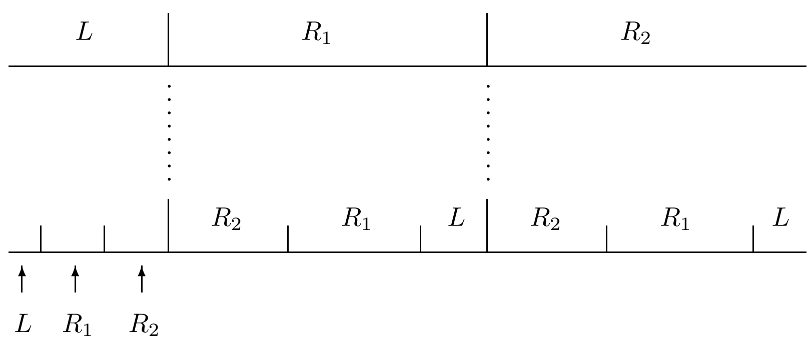
(2) $j = 1, \dots, k$

$$\begin{aligned} I_{j,0} &:= \alpha^2 + (j-1)\alpha + [k\alpha^2, \alpha) \\ I_{j,l} &:= \alpha^2 + (j-1)\alpha + (k-l)\alpha^2 + [0, \alpha^2) \quad (l = 1, 2, \dots, k). \end{aligned}$$

And the operation R_k (resp. L) corresponds to $I_{j,k}$ (resp. $I_{j,0}$). For instance

$$\begin{aligned} I_{0,0} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (L, L)\}) \\ I_{0,j} &= (\Phi \circ \Psi)^{-1}(\{(O_1, O_2, \dots) \in W \mid (O_1, O_2) = (L, R_j)\}), \quad j = 1, \dots, k. \end{aligned}$$

The figure below is the example for $k = 2$ where the intervals I_{ij} $(i, j = 0, 1, 2)$ and the corresponding operations are shown.



We repeat this procedure. Suppose we have an interval $J = I_{j_1, j_2, \dots, j_n}$ after applying the operation O_n, by dividing the parent interval $I_{j_1, j_2, \dots, j_{n-1}}$ at x. The length of J is equal to α^{k_n}. We divide J, from x, into k intervals $I_k, I_{k-1}, \dots, I_1$ of length α^{k_n+1} and an interval I_0 of length α^{k_n+2}. Then each interval corresponds to the operations $O_{n+1} = R_k, R_{k-1}, \dots, R_1, L$, respectively. Thus we have defined the interval division $\{I_{j_1, j_2, \dots,} \mid j_l = 0, 1, \dots, k\}$ each of which corresponds to the cylinder set of these operations.

To have a simple characterization of the $(-\beta)_S$-admissibility and the shift map, we slightly modify the definition and consider the division of $[0, 1]$ by the family of intervals $\{J_{j_1, j_2, \dots, j_n} \mid j_l = 0, \dots, k\}$ as follows. They have the same interior as $\{I_{j_1, j_2, \dots, j_n} \mid j_l = 0, \dots, k\}$ and in dividing an interval J, from the former division point x, the first $(k-1)$ long intervals $I_{j_1, j_2, \dots, j_{n-1}, k}$, $I_{j_1, j_2, \dots, j_{n-1}, k-1}$, $\dots$, $I_{j_1, j_2, \dots, j_{n-1}, 2}$ are closed-open, and the k-th long interval $I_{j_1, j_2, \dots, j_{n-1}, 1}$ is closed. The short interval $I_{j_1, j_2, \dots, j_{n-1}, 0}$ is open-closed. For instance $I_0, I_1, \dots, I_k$ are replaced by $J_0, J_1, \dots, J_k$ given below (in this case, $x = 1$):

$$\begin{aligned} [0,1] &= J_0 \cup J_1 \cup \cdots \cup J_k \\ J_0 &= [0, 1-k\alpha) = [0, \alpha^2) \\ J_1 &= [1-k\alpha, 1-(k-1)\alpha] \\ J_j &= (1-(k-j+1)\alpha, 1-(k-j)\alpha] \quad (j = 2, \dots, k). \end{aligned}$$

For given $\theta \in [0,1]$, we take the corresponding sequence of operations $(O_1, O_2, \dots) \in W$ associated to the interval division $\{J_{j_1, j_2, \dots} \mid j_l = 0, 1, \dots, k\}$. Letting

$$\begin{aligned} x_n &= \begin{cases} k & (O_n = L) \\ k-j+1 & (O_n = R_j,\ j = 1, \dots, k) \end{cases} \\ p_1 &= 1, \quad p_n = n + \sharp\{k \le n-1 \mid O_k = L\}, \end{aligned}$$

we have

$$\theta = 1 + \sum_{n=1}^{\infty} x_n (-\alpha)^{p_n},$$

and applying the substitution

$$\begin{aligned} R_j &\mapsto (k-j+1), \quad j = 1, 2, \dots, k \\ L &\mapsto k\,0 \end{aligned}$$

to $(O_1, O_2, \dots) \in W$ gives a power series representation of θ in terms of $(-\alpha)$:

$$\theta = 1 + \sum_{j=1}^{\infty} y_j (-\alpha)^j,$$

which should be a definition of $(-\beta)_S$-expansion of θ. In $\{y_j\}_{j=1}^{\infty}$, 0 is always isolated and followed by k. In other words, $00, 10, 20, \dots, (k-1)0$ do not appear.

4.3. Shift Map

The map $T : [0,1] \to [0,1]$ sending $\theta = 1 + y_1(-\alpha) + \sum_{j=2}^{\infty} y_j (-\alpha)^j$ to $\theta' = 1 + y_1(-\alpha) + \sum_{j=2}^{\infty} y_{j+1}(-\alpha)^j$ is

$$T(\theta) = -\alpha^{-1}\theta + (k+1-y_1(\theta)) + (y_2(\theta) - y_1(\theta) + 1)\alpha,$$

where $y_1(\theta)$ is given by

$$y_1(\theta) = \begin{cases} k, & (\theta \in [0, \alpha^2)) \\ k, & (\theta \in \alpha^2 + [0, \alpha]) \\ k-j+1, & (\theta \in \alpha^2 + (j-1)\alpha + (0, \alpha], \quad j = 2, \dots, k) \end{cases}$$

and $y_2(\theta)$ is given by

(i) If $\theta \in \alpha^2 + (j-1)\alpha + (0, \alpha]$ $(j = 2, \ldots, k)$ or $\theta \in \alpha^2 + [0, \alpha]$,

$$y_2(\theta) = \begin{cases} l & (\theta \in \alpha^2 + (j-1)\alpha + (l-1)\alpha^2 + [0, \alpha^2), \quad l = 1, 2, \ldots, k-1) \\ k & (\theta \in \alpha^2 + (j-1)\alpha + (k-1)\alpha^2 + [0, \alpha^2]) \\ k & (\theta \in \alpha^2 + (j-1)\alpha + k\alpha^2 + (0, \alpha^3]) \end{cases}$$

(ii) If $\theta \in [0, \alpha^2)$, $y_2(\theta) = 0$.

Nevertheless, since 0 must be followed by k in the $(-\beta)_S$-expansion, if $y_1 \neq k$ and $y_3 = 0$, then $\theta' = T(\theta)$ is not the $(-\beta)_S$-expansion of θ' in the sense of the former subsection. Hence we restrict the domain of T to the interval

$$I_\alpha := [0, \alpha^2 + \alpha]$$

so that we always have $y_1(\theta) = k$, and $y_2 : [0, \alpha^2 + \alpha] \to \{0, 1, \ldots, k\}$ is now equal to

$$y_2(\theta) = \begin{cases} j & (\theta \in j\alpha^2 + [0, \alpha^2),\ j = 0, 1, \ldots, k) \\ k & (\theta \in k\alpha^2 + [0, \alpha^2]) \\ k & (\theta \in ((k+1)\alpha^2, \alpha + \alpha^2]). \end{cases}$$

We introduce the $(-\beta)_S$-transformation $T_{-\beta,S} : [0, \alpha^2 + \alpha] \to [0, \alpha^2 + \alpha]$:

$$T_{-\beta,S}(\theta) \quad = \quad -\alpha^{-1}\theta + 1 + (y_2(\theta) - k + 1)\alpha.$$

Definition 13. ($(-\beta)_S$-expansion) The power series expansion of $\theta \in [0, \alpha^2 + \alpha]$ in terms of $(-\alpha) = (-\beta)^{-1}$ given by

$$\theta = 1 + k(-\alpha) + \sum_{n=2}^{\infty} y_n(-\alpha)^n, \quad y_n = y_2((T_{-\beta,S})^{j-2}(\theta)), \quad n = 2, 3, \ldots \tag{7}$$

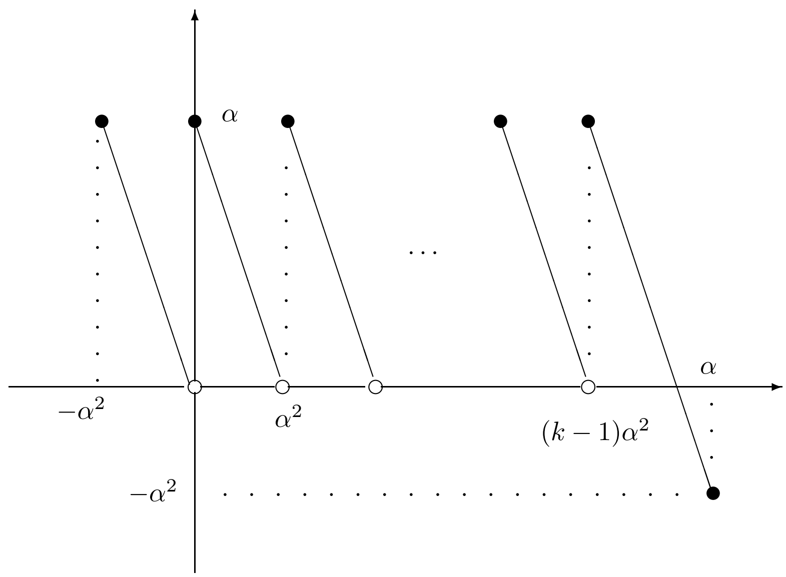
is called the $(-\beta)_S$-*expansion* of θ. We write $d_S(\theta, -\beta) = \{y_j\}_{j=1}^{\infty}$. As is done in the $(-\tau)_S$-expansion, we can also regard it as an expansion of $\theta - (1 + k(-\alpha)) \in [-\alpha^2, \alpha]$. In this case $T_{-\beta,S}$ is replaced by its translation

$$\begin{aligned} \hat{T}_{-\beta,S}(\theta) \quad &:= \quad T_{-\beta,S}(\theta + \alpha^2) - \alpha^2 \\ &= \quad -\alpha^{-1}\theta + \hat{y}_2(\theta)\alpha, \quad \theta \in [-\alpha^2, \alpha] \\ \hat{y}_2(\theta) \quad &:= \quad \begin{cases} j & (\theta \in [(j-1)\alpha^2, j\alpha^2),\ j = 0, 1, \ldots, k) \\ k & (\theta \in [k\alpha^2, \alpha]) \end{cases} \end{aligned}$$

and the definition of the $(-\beta)_S$-expansion becomes

$$\theta = \sum_{j=2}^{\infty} y_j(-\alpha)^j, \quad y_j := \hat{y}_2((\hat{T}_{-\beta,S})^{j-2}(\theta)), \quad j \geq 2, \quad \theta \in [-\alpha^2, \alpha]. \tag{8}$$

In this case any $\theta \in \mathbf{R}$ can be expanded as in the case for the $(-\beta)_{IS}$-expansion (Definition 23 in the Appendix).



The graph of $\hat{T}_{-\beta,S}$

Remark 14. $T_{-\beta,S}$ is discontinuous at $\theta = j\alpha^2$, $j = 1, 2, \ldots, k$. By the definition of $T_{-\beta,S}$, the orbits passing discontinuity points correspond to $(\ldots, R_1, R_{k-j+1}, \overline{L}) = (k, j, (\overline{k0}))$. Hence $(\ldots, L, R_k, \overline{L}) = (k, 0, 1, (\overline{k0}))$ and $(\ldots, R_1, R_{k-j+2}, R_k, \overline{L}) = (k, (j-1), 1, (\overline{k0}))$ $(j \geq 2)$ do not appear in the tails of sequences in the $(-\beta)_S$-expansion.

4.4. $(-\boldsymbol{\beta})_S$-Admissible Sequences

Definition 15. ($(-\beta)_S$-admissibility) We say $\{y_j\}_{j=2}^{\infty} \in \{0, 1, \ldots, k\}^{\mathbf{N}}$ is $(-\beta)_S$-*admissible* if and only if

$$y_j = y_2((T_{-\beta,S})^{j-2}(\theta)), \quad j = 2, 3, \ldots$$

for some $\theta \in [0, \alpha^2 + \alpha]$. In other words, $\theta = 1 + k(-\alpha) + \sum_{j=2}^{\infty} y_j(-\alpha)^j$ is the $(-\beta)_S$-expansion of θ.

By the argument in Section 3.3 and in Remark 14, we have the following simple characterization.

Proposition 16. *The set of $(-\beta)_S$-admissible sequences has the following characterization:*

$$\{\{y_j\}_{j=2}^{\infty} \,|\, \{y_j\}_{j=2}^{\infty} \textit{ is } (-\beta)_S\textit{-admissible } \} = X \setminus Y$$

where

$$\begin{aligned} X &= \{\{y_j\}_{j=2}^{\infty} \mid \ 00,\ 10,\ \ldots, (k-1)0 \ \textit{do not appear}\} \\ Y &= \{\{y_j\}_{j=2}^{\infty} \mid \ \textit{tail is equal to } (k,(j-1),1,\overline{(k0)}),\ j=1,2,\ldots,k\ \}. \end{aligned}$$

4.5. Shift Space

The shift space $S_{-\beta,S}$ is defined similarly as in Section 2.5.

Theorem 17. *$S_{-\beta,S}$ is SFT whose set of forbidden words is $\{00, 10, \ldots, (k-1)0\}$.*

Remark 18. For those β, the shift space S_β for the $(+\beta)$-expansion is also SFT whose set of forbidden words is $\{k1, k2, \ldots, k(k-1)\}$ with the same entropy. The shift space $S_{-\beta,IS}$ for the $(-\beta)_{IS}$-expansion is given in Example B in the Appendix.

4.6. Invariant Measure

The invariant measure of $\hat{T}_{-\beta,S}$ uniquely exists by [10].

Theorem 19. *The invariant measure of $\hat{T}_{-\beta,S}$ is given by $d\nu_{-\beta,S} = h_{-\beta,S}\,dx$ where*

$$h_{-\beta,S}(x) = \begin{cases} \alpha & (-\alpha^2 < x < 0) \\ 1 & (0 < x < \alpha). \end{cases}$$

It seems not to be possible to have the power series representation of $h_{-\beta,S}$, as it is in the $(+\beta)$-expansion, and in the $(-\beta)_{IS}$-expansion.

Proof. It suffices to check $\nu_{-\beta,S}(\hat{T}^{-1}_{-\beta,S}A) = \nu_{-\beta,S}(A)$ for intervals A. □ □

If $\beta = \tau$, the frequency of appearance of 0 and 1 is equal to $\frac{1}{\tau^2} : 1$, while it is $1 : \frac{1}{\tau^2}$ in the $(+\beta)$-expansion and $1 : \frac{2}{\tau}$ in the $(-\beta)_{IS}$-expansion.

5. General Case: $(-\beta)_S$-Expansion in an Extended Sense

In this section we extend the discussion in Sections 2, 3 to general irrational $\beta > 1$, which does not give us the power series expansion but those in an extended sense. Since the idea is the same as that in Sections 2, 3, we state the results without proofs. As in Section 2, let $\alpha = [a_1, a_2, \ldots]$, $a_n \in \mathbf{N}$ be the continued fraction expansion of $\alpha := \beta^{-1}$. We define recursively the sequence $\{\gamma_n\}_{n=1}^{\infty}$ by

$$\gamma_{n-1} = \frac{1}{a_n + \gamma_n}, \quad n \geq 1, \quad \gamma_0 = \alpha$$

and let

$$\alpha_n = \begin{cases} \gamma_{n-1} \cdot \gamma_{n-2} \cdots \gamma_1 \cdot \gamma_0 & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

We then have

$$a_n\alpha_n + \alpha_{n+1} = \alpha_{n-1}, \quad n \geq 1.$$

5.1. R, L-Construction

By (1), we have a_{n+1} operations $R_1, R_2, \dots, R_{a_{n+1}}$ to embed s_n into s_{n+1}; the set of embedding operations becomes $W := \prod_{n\geq 0} W_n$, $W_n := \{L^{(n)}, R_1^{(n)}, R_2^{(n)}, \dots, R_{a_{n+1}}^{(n)}\}$. The interval division is defined by dividing the interval of length α_k, from the former division point x, into a_{k+1} intervals of length α_{k+1} and an interval of length α_{k+2}. They correspond to the operations $O_{n+1} = R_{a_{k+1}}^{(k)}, R_{a_{k+1}-1}^{(k)}, \dots, R_2^{(k)}, R_1^{(k)}, L^{(k)}$ respectively where $k = k_n := n + \sharp\{l \leq n \,|\, O_l = L\}$.

5.2. $(-\beta)_S$-Expansion

For given $\theta \in [0, 1)$ let $(O_1, O_2, \dots) \in W$ be the corresponding sequence of operations. For $n \geq 0$ set $k = k_n$, and

$$\begin{aligned} x_{n+1} &= \begin{cases} a_{k+1} - j + 1 & (O_{n+1} = R_j^{(k)},\ j = 1, 2, \dots, a_{k+1}) \\ a_{k+1} & (O_{n+1} = L^{(k)}) \end{cases}, \quad n \geq 0 \\ p_n &:= \begin{cases} n + \sharp\{l \leq n-1 \mid O_l = L\}, & n \geq 2 \\ 1, & n = 1 \end{cases} \end{aligned}$$

and define $\{y_j\}_{j=1}^{\infty}$ by applying the substitution

$$\begin{aligned} &R_j^{(k)} \mapsto a_{k+1} + 1 - j \quad (j = 1, 2, \dots, a_{k+1}) \\ &L^{(k)} \mapsto a_{k+1}\ 0 \end{aligned}$$

to the sequence $(O_1, O_2, \dots)$. Then we have the following representation of θ, which may be viewed as the $(-\beta)_S$-expansion of θ in an extended sense.

$$\theta = 1 + \sum_{n=1}^{\infty} x_n (-1)^{p_n} \alpha_{p_n} = 1 + \sum_{n=1}^{\infty} y_n (-1)^n \alpha_n.$$

6. Local Flip Connectedness

In this section, β is the positive root of the equation $\beta^2 = k\beta + 1$ and we adopt $\hat{T}_{-\beta,S}$ and equation (8) as the definition of the $(-\beta)_S$-expansion. If $\theta \in \mathbf{R}$ has the following representation

$$\theta = \sum_{n\geq n_0} \frac{x_n}{(-\beta)^n}, \quad x_n \in \{0, 1, \dots, k\}, \quad n_0 \in \mathbf{Z}, \tag{9}$$

there may be many choices of $\{x_n\}_{n\geq n_0}$ in general. Whenever a single-sided sequence $\{x_n\}_{n\geq n_0}$ satisfies (9), we say this is a $(-\beta)$-*expansion* of θ, and distinguish it from the $(-\beta)_S$-expansion and the $(-\beta)_{IS}$-expansion. For simplicity, we call a sequence $\{x_n\}_{n\geq n_0}$ ($x_n \in \{0, 1, \dots, [\beta]\}$) in a $(-\beta)$-expansion a $(-\beta)$-*sequence*. Similarly, we also call the $(-\beta)_S$-admissible sequence (resp. the $(-\beta)_{IS}$-admissible sequence) the $(-\beta)_S$-*sequence* (resp. the $(-\beta)_{IS}$-*sequence*). We do not consider the case of finite expansions since the tail $\overline{0}$ can always be replaced by $\overline{k0}$. Since we have

$$\frac{1}{(-\beta)^n} + \frac{k}{(-\beta)^{n+1}} = \frac{1}{(-\beta)^{n+2}}, \quad n \in \mathbf{Z},$$

we may locally modify $1k0 \leftrightarrow 001$ in a $(-\beta)$-sequence. Therefore we introduce the following operations in the sequences $\{x_n\}_{n\geq n_0}$ ($x_n \in \{0, 1, \dots, k\}$):

(A) $\quad (l+1)k(j-1) \leftrightarrow l0j \quad (l = 0, 1, \dots, k-1, \quad j = 1, 2, \dots, k)$

(B) $\quad (k, (j-1), 0, 1, \overline{(k0)}) \leftrightarrow (k, j, \overline{(k0)}) \quad (j = 1, 2, \dots, k)$

(C) $\quad (0, 0, k, \overline{(k-1)}) \leftrightarrow (1, k, \overline{(k-1)})$.

Operation (B) turns non-$(-\beta)_S$-admissible sequences into admissible ones at $\theta \in D_-$ while (C) modifies the sequences whose tails are equal to that of $d^*_{IS}(r_\beta, -\beta)$ into $(-\beta)_{IS}$-admissible ones ($d^*_{IS}(r_\beta, -\beta)$ is defined in the Appendix). Then we can prove that any $(-\beta)$-sequences corresponding to the same number θ are connected by these operations.

Theorem 20. *Any $(-\beta)$-sequence can be transformed into the $(-\beta)_S$-sequence, and the $(-\beta)_{IS}$-sequence, via the operations (A), (B), (C).*

Proof. (1) If the sequence in question $\{x_n\}_{n\geq n_0}$ is not $(-\beta)_S$-admissible, it should contain $l0j$ ($l = 0, 1, \dots, k-1, j = 1, 2, \dots, k$). A successive application of (A) turns it into a $(-\beta)_S$-admissible one.

(2) According to Example B in the Appendix, the $(-\beta)_{IS}$-admissibility is expressed by two rules (i), (ii). We say that $\{x_n\}_{n\geq n_0}$ *is $(-\beta)_{IS}$-admissible up to x_d* if when we look at $x_{n_0}, x_{n_0+1}, \dots, x_d$ only, these two rules are not broken. Suppose that $\{x_n\}$ is $(-\beta)_{IS}$-admissible up to x_d and is not $(-\beta)_{IS}$-admissible at x_{d+1}. We would like to modify $\{x_n\}$ into a $(-\beta)_{IS}$-admissible sequence using (A). We proceed by a case-by-case analysis.

`Case (I)` $(x_d, x_{d+1}) = (k, (j-1))$, $j = 1, 2, \dots, (k-1)$: we further divide our discussion into some cases according to x_{d-1}.

(i) $x_{d-1} = 1, 2, \dots, k$: we apply (A) to (x_{d-1}, x_d, x_{d+1}). Then (x_{d-1}, x_d, x_{d+1}) is transformed to

$$(x_{d-1}, x_d, x_{d+1}) = ((l+1), k, (j-1)) \mapsto (l, 0, j)$$

$(j = 1, 2, \ldots, (k-1), l = 0, 1, \ldots, (k-1))$ which is $(-\beta)_{IS}$-admissible up to x_{d+1}.

Remark 21. Since we modified x_{d-1}, we have to check whether the $(-\beta)_{IS}$-admissibility is maintained up to x_{d-1} after applying this operation. If $l+1=k$ for instance, the only possibility where $(-\beta)_{IS}$-admissibility may be broken is that $(\ldots, x_{d-2}) = (k, \overbrace{(k-1), \ldots, (k-1)}^{even})$. After applying (A) to (x_{d-1}, x_d, x_{d+1}), we have

$$\begin{aligned}(\ldots, x_{d-2}, x_{d-1}, x_d, x_{d+1}) &= (k, \overbrace{(k-1), \ldots, (k-1)}^{even}, k, k, (j-1)) \\ &\mapsto (k, \overbrace{(k-1), \ldots, (k-1)}^{even}, (k-1), 0, j)\end{aligned}$$

$(j = 1, 2, \ldots, (k-1), l = 0, 1, \ldots, (k-1))$. This is $(-\beta)_{IS}$-admissible. If $l+1 = 1, \ldots, k-1$, and in those arguments below, we can similarly check that $(-\beta)_{IS}$-admissibility is maintained.

(ii) $x_{d-1} = 0$: since we assumed $\{x_n\}$ is $(-\beta)_{IS}$-admissible up to x_d, we should have $x_{d-2} = 0, 1, \ldots, k-1$. We apply (A) to (x_{d-2}, x_{d-1}, x_d) and obtain

$$(x_{d-2}, x_{d-1}, x_d, x_{d+1}) = (l, 0, k, (j-1)) \mapsto ((l+1), k, (k-1), (j-1))$$

$(l = 0, 1, 2, \ldots, k-1)$ which is $(-\beta)_{IS}$-admissible.

`Case (II)` $(x_d, \ldots) = k\,\overbrace{(k-1)\ldots(k-1)}^{odd}\,k$: we modify $(x_{d-1}, x_d, x_{d+1}) = (*, k, (k-1))$ to have the $(-\beta)_{IS}$-admissibility. In the argument below, the number of succession of $(k-1)$ changes by one so that we obtain the $(-\beta)_{IS}$-admissibility.

(i) $x_{d-1} = 1, 2, 3, \ldots, k$: we apply (A) as follows.

$$(x_{d-1}, x_d, x_{d+1}) = ((l+1), k, (k-1)) \mapsto (l, 0, k) \quad l = 0, 1, \ldots, (k-1).$$

(ii) $x_{d-1} = 0$: since we assumed $\{x_n\}$ is $(-\beta)_{IS}$-admissible up to x_d, we should have $x_{d-2} = 0, 1, \ldots, k-1$. We apply (A) to (x_{d-2}, x_{d-1}, x_d) and obtain

$$(x_{d-2}, x_{d-1}, x_d) = (l, 0, k) \mapsto ((l+1), k, (k-1)), \quad l = 0, 1, \ldots, (k-1).$$

`Case (III)` $(x_d, \ldots) = k\,\overbrace{(k-1)\ldots(k-1)}^{even}\,j$, $j \neq k$: As in Case(II), we modify $(x_{d-1}, x_d, x_{d+1}) = (*, k, (k-1))$ and obtain the $(-\beta)_{IS}$-admissibility up to x_{d+1}. $\square$

In the proof above, to have a $(-\beta)_S$-sequence, we need the operation $l0j \mapsto (l+1)k(j-1)$ only, but to have a $(-\beta)_{IS}$-sequence we need both $l0j \mapsto (l+1)k(j-1)$ and $(l+1)k(j-1) \mapsto l0j$. Indeed, we have

Proposition 22. *For a sequence* $\{x_n\}_{n=1}^{\infty} \in \{0, 1, \ldots k\}^{\mathbf{N}}$*, the following conditions are equivalent.*
(i) *The operation* $l0j \mapsto (l+1)k(j-1)$ $(l = 0, 1, \ldots, k-1,\ j = 1, 2, \ldots, k)$ *is impossible*[2],
(ii) 0 *is isolated and followed by* k.

This property of $(-\beta)_S$-sequences should have something to do with that of the $(-\beta)_S$-admissibility discussed in Section 2.3.

7. Appendix : Ito-Sadahiro's $(-\beta)$-Expansion

We briefly review the basic properties of the $(-\beta)_{IS}$-expansion to compare it with those in this paper[3]. Let

$$I_\beta \quad := \quad [l_\beta, r_\beta), \quad l_\beta := -\frac{\beta}{\beta+1}, \quad r_\beta := \frac{1}{\beta+1}.$$

Definition 23. ($(-\beta)_{IS}$-expansion) Let $T_{-\beta,IS} : I_\beta \to I_\beta$ be the $(-\beta)$-transformation defined by

$$T_{-\beta,IS}(x) := -\beta x - [-\beta x - l_\beta].$$

The power series representation of $x \in I_\beta$ in terms of $(-\beta)^{-1}$ given by

$$x \quad = \quad \sum_{k=1}^{\infty} \frac{x_k}{(-\beta)^k}, \quad x_k = [-\beta (T_{-\beta,IS})^{k-1}(x) - l_\beta]$$

is called the $(-\beta)_{IS}$- *expansion* of $x \in I_\beta$. We write $d_{IS}(x, -\beta) = \{x_n\}_{n=1}^{\infty}$. For $x \notin I_\beta$ we take $k \in \mathbf{N}$ such that $(-\beta)^{-k}x \in I_\beta$ and multiply the $(-\beta)_{IS}$-expansion of $(-\beta)^{-k}x$ by $(-\beta)^k$.

The $(-\beta)_{IS}$-admissibility is defined similarly as in the $(-\beta)_S$-expansion.

Definition 24. ($(-\beta)_{IS}$-admissibility) We say $\{x_n\}_{n=1}^{\infty} \in \{0, 1, \ldots, [\beta]\}^{\mathbf{N}}$ is $(-\beta)_{IS}$-admissible if and only if we can find $x \in I_\beta$ such that $\{x_n\}_{n=1}^{\infty}$ is the $(-\beta)_{IS}$-expansion of x.

Let $d_{IS}(l_\beta, -\beta) = (b_1, b_2, \ldots)$, $d_{IS}(r_\beta, -\beta) = (0, b_1, b_2, \ldots)$ be the $(-\beta)_{IS}$-expansions of l_β, r_β respectively. Then $d_{IS}(r_\beta, -\beta)$ can at least formally be defined as

[2] For a sequence $(0, j, \ldots)$ $(j = 1, 2, \ldots, k)$, we regard it as $(0, 0, j, \ldots)$ so that operation (i) is possible.

[3] [13] contains a review of the $(-\beta)_{IS}$- and the $(-\beta)_S$-expansions as well as a discussion of some unsolved problems.

above. We set

$$d^*_{IS}(r_\beta,-\beta) \quad := \quad \begin{cases} \overline{(0,b_1,\ldots,b_{q-1},b_q-1)} & (d_{IS}(l_\beta,-\beta)=(\overline{b_1,\ldots,b_q}),\ q:\text{odd}) \\ d_{IS}(r_\beta,-\beta) & (\text{otherwise}). \end{cases}$$

Because the orbit of $(T_{-\beta,IS})^n(l_\beta)$ passes the discontinuity points of $T_{-\beta,IS}$ when $d_{IS}(l_\beta,-\beta)=\overline{(b_1,\ldots,b_q)}$, we set the definition of $d^*_{IS}(r_\beta,-\beta)$ as above. The condition for the $(-\beta)_{IS}$-admissibility of a given sequence $\{x_n\}$ is expressed as follows.

Theorem 25. *$\{x_n\}_{n=1}^\infty$ is $(-\beta)_{IS}$-admissible if and only if*

$$\textit{for any}\ \ n\geq 1,\quad d_{IS}(l_\beta,-\beta)\preceq_{IS}(x_n,x_{n+1},\ldots)\prec_{IS} d^*_{IS}(r_\beta,-\beta).$$

The IS-ordering $\prec_{IS}$ is defined in Definition 9.

Example A. Let $\beta=\tau$. Then $d_{IS}(l_\beta,-\beta)=(10\ldots)=(1\overline{0}), d_{IS}(r_\beta,-\beta)=(0100\ldots)$. Hence, $\{x_n\}$ is $(-\tau)_{IS}$-admissible if and only if (i) after the first 1 appears, all subsequent blocks of consecutive 0s have even length, and (ii) its tail is not equal to $1\overline{0}$.

Example B. Let β be the positive root of $\beta^2=k\beta+1$, $k\geq 2$. Then $d_{IS}(l_\beta,-\beta)=(k,\overline{(k-1)})$. Hence $\{x_n\}$ is $(-\beta)_{IS}$-admissible if and only if its tail is not equal to $k\overline{(k-1)}$ and it satisfies the following rules.

(i) Whenever k appears, it should be followed by k or $(k-1)$.

(ii) When we have $k\ldots k\overbrace{(k-1)\ldots(k-1)}^{j}x$, then

$$x=\begin{cases} 0,1,\ldots,(k-2) & (j:\text{odd}) \\ k & (j:\text{even}). \end{cases}$$

The shift space $S_{-\beta,IS}$ is defined similarly to $S_{-\beta,S}$.

Theorem 26. *$\{x_n\}_{n=1}^\infty\in S_{-\beta,IS}$ if and only if*

$$\textit{for any}\ \ n\in\mathbf{Z},\quad d_{IS}(l_\beta,-\beta)\preceq_{IS}(x_n,x_{n+1},\ldots)\preceq_{IS} d^*_{IS}(r_\beta,-\beta).$$

Theorem 27. *$S_{-\beta,IS}$ is a Sofic shift if and only if $d_{IS}(l_\beta,-\beta)$ is eventually periodic.*

The invariant measure of $T_{-\beta,IS}$ has a power series representation like the β-expansion does [14].

Theorem 28. *The invariant measure is given by $d\nu_{-\beta,IS}=h_{-\beta,IS}\,dx$ where*

$$h_{-\beta,IS}(x)=\sum_{n=0}^\infty\frac{1}{(-\beta)^n}1_{\{x>(T_{-\beta,IS})^n(l_\beta)\}}.$$

Example C. Let β be the positive root of $\beta^2 = k\beta + 1$. Then

$$h_{-\beta,IS}(x) = \begin{cases} 1 & (l_\beta < x < -\frac{k-1}{\beta+1}) \\ \frac{\beta}{\beta+1} & (-\frac{k-1}{\beta+1} < x < r_\beta). \end{cases}$$

Remark 29. As is done for the $(-\beta)_S$-expansion, it is possible to determine the sequence $d_{IS}(x, -\beta)$ by interval division, whose construction is not simple however (this fact would imply that generically we may not have a simple formula to relate $d(x, \beta)$, $d_{IS}(x, -\beta)$ and $d_S(x, -\beta)$). For instance, we let $\beta = \tau$. We first divide I_β into the two intervals with ratio $1 : \tau$:

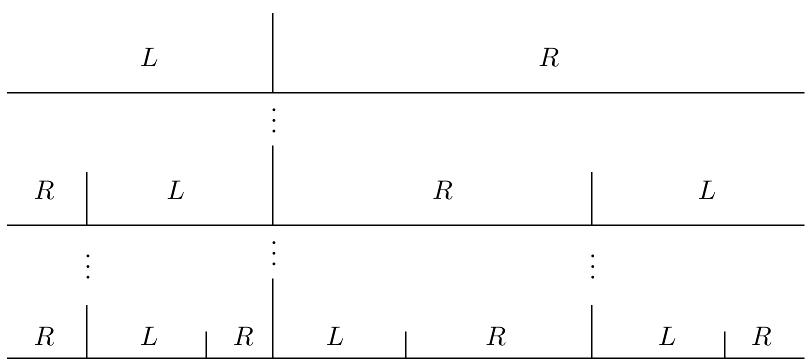
$$I = \left[-\frac{1}{\tau}, -\frac{1}{\tau^3}\right) \cup \left[-\frac{1}{\tau^3}, \frac{1}{\tau}\right) =: I_1 \cup I_2.$$

We label them as L, R. We define inductively the division and labeling of intervals:

(i) If we divide an interval labelled R, we divide it in the same way as in the $(-\beta)_S$-expansion. That is, we divide into two intervals with ratio $1 : \tau$ and label the longer one (resp. shorter one) R (resp. L).

(ii) If we divide an interval labelled L, we divide it into two intervals with ratio $1 : \tau$, but label the longer one (resp. shorter one) L (resp. R). And we do not divide the shorter interval in the next step.

If x lies in an interval labelled R (resp. L), then we set $x_n = 0$ (resp., $x_n = 1$).



The interval division corresponding to $(-\tau)_{IS}$-expansion

Acknowledgement This work is partially supported by JSPS grant Kiban-C no.22540140.

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