A CORRELATION IDENTITY FOR STERN'S SEQUENCE

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Abstract

The *Stern sequence* (also called *Stern's diatomic sequence*), is defined by the recurrence relations $s(0) = 0$, $s(1) = 1$, and in general by $s(2n) = s(n)$, and $s(2n + 1) = s(n) + s(n + 1)$. In this note we prove a new identity for Stern's sequence. In particular, we show that if e and a are nonnegative integers, then for any integer r with $0 \leq r \leq 2^e$, we have

$$s(r)s(2a+5) + s(2^e - r)s(2a+3) = s(2^e(a+2)+r) + s(2^e(a+1)+r).$$

It seems that this is the first correlation–type identity concerning Stern's sequence in the literature.

1. Introduction

The *Stern sequence* (also called *Stern's diatomic sequence*; A002487 in Sloane's list), is defined by the recurrence relations $s(0) = 0$, $s(1) = 1$, and in general by $s(2n) = s(n)$ and $s(2n + 1) = s(n) + s(n + 1)$.

In this note, we prove the following result.

Theorem 1. *Let e and a be nonnegative integers. Then for any integer r with $0 \leq r \leq 2^e$, we have*

$$s(r)s(2a+5) + s(2^e - r)s(2a+3) = s(2^e(a+2)+r) + s(2^e(a+1)+r). \qquad (1)$$

Recently, Bacher introduced a twisted version of the Stern sequence. He defined the sequence $\{t(n)\}_{n\geq 0}$ given by the recurrence relations $t(0) = 0$, $t(1) = 1$, and in general by $t(2n) = -t(n)$ and $t(2n + 1) = -t(n) - t(n + 1)$.

Proving a conjecture of Bacher [1] connecting the Stern sequence $\{s(n)\}_{n\geq 0}$ with it's twist $\{t(n)\}_{n\geq 0}$ we gave the following result [2].

Theorem 2. (Theorem 1.1 of [2]]) *There exists an integral sequence $\{u(n)\}_{n\geqslant 0}$ such that for all $e \geqslant 0$ we have*

$$\sum_{n\geqslant 0} t(3 \cdot 2^e + n)z^n = (-1)^e S(z) \sum_{n\geqslant 0} u(n)z^{n\cdot 2^e}, \qquad (2)$$

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where $S(z)$ denotes the generating function of the Stern sequence.

Note that in this theorem (as in the original conjecture), it is implicit that the sequence $\{u(n)\}_{n\geqslant 0}$ is defined by the relationship

$$U(z) := \sum_{n\geqslant 0} u(n)z^n = \frac{\sum_{n\geqslant 0} t(3+n)z^n}{S(z)}.$$

We will prove Theorem 1 as a corollary of Theorem 2.

2. Preliminaries

Lemma 3. *The generating series $S(z) = \sum_{n\geqslant 0} s(n)z^n$ satisfies the functional equation*

$$S(z^2) = \left(\frac{z}{1+z+z^2}\right) S(z).$$

We will need the following lemmas (see [1, Theorem 1.2]).

Lemma 4. *For all $e \geq 0$ and for all n such that $0 \leq r \leq 2^e$ we have $s(2^e + r) = s(2^e - r) + s(r)$.*

Lemma 5. (Bacher [1]]) *We have $t(3 \cdot 2^e + r) = t(6 \cdot 2^e - r) = (-1)^e s(r)$ for all $e \geqslant 0$ and for all r such that $0 \leq r \leq 2^{e+1}$.*

Theorem 6. (Bacher [1]]) *For all $e \geqslant 1$, we have*

$$\prod_{i=0}^{e-1}\left(1 + z^{2^i} + z^{2^{i+1}}\right) = \frac{(-1)^e}{z(1+z^{2^e})}\sum_{n=0}^{3\cdot 2^e} t(3\cdot 2^e + n)z^n.$$

Using the definitions above, Theorem 6 and the functional equation for $S(z)$, we have that the right–hand side of (2) is

$$\begin{aligned}
(-1)^e S(z)U(z^{2^e}) &= (-1)^e \frac{S(z)}{S(z^{2^e})} \sum_{n\geqslant 0} t(3+n)z^{2^e n} \\
&= (-1)^e \prod_{i=0}^{e-1}\left(\frac{1+z^{2^i}+z^{2^{i+1}}}{z^{2^i}}\right) \cdot \sum_{n\geqslant 0} t(3+n)z^{2^e n} \\
&= (-1)^e \frac{z}{z^{2^e}} \prod_{i=0}^{e-1}\left(1+z^{2^i}+z^{2^{i+1}}\right) \cdot \sum_{n\geqslant 0} t(3+n)z^{2^e n} \\
&= \frac{1}{z^{2^e}(1+z^{2^e})}\left(\sum_{n\geqslant 0} t(3+n)z^{2^e n}\right)\left(\sum_{n=0}^{3\cdot 2^e} t(3\cdot 2^e+n)z^n\right).
\end{aligned}$$

This calculation combined with Theorem 2 gives the following corollary.

Corollary 7. *For all $e \geqslant 0$, we have*

$$z^{2^e}(1+z^{2^e})\sum_{n\geqslant 0} t(3\cdot 2^e+n)z^n = \left(\sum_{n\geqslant 0} t(3+n)z^{2^e n}\right)\left(\sum_{n=0}^{3\cdot 2^e} t(3\cdot 2^e+n)z^n\right). \quad (3)$$

Now write the left–hand side of (3) as $G(z)=\sum_{n\geq 0} g(n)z^n$ and write the right–hand side of (3) as $H(z)=\sum_{n\geq 0} h(n)z^n$; that is,

$$G(z)=\sum_{n\geq 0} g(n)z^n = z^{2^e}(1+z^{2^e})\sum_{n\geqslant 0} t(3\cdot 2^e+n)z^n,$$

and

$$H(z)=\left(\sum_{n\geqslant 0} t(3+n)z^{2^e n}\right)\left(\sum_{n=0}^{3\cdot 2^e} t(3\cdot 2^e+n)z^n\right).$$

Then we have

$$g(n)=\begin{cases} 0 & \text{if } n\in[0,2^e) \\ t(2^{e+1}+n) & \text{if } n\in[2^e,2^{e+1}) \\ t(2^{e+1}+n)+t(2^e+n) & \text{if } n\in[2^{e+1},\infty) \end{cases} \quad (4)$$

and

$$h(n)=\sum_{\substack{2^e k+j=n \\ 0\leqslant j\leqslant 3\cdot 2^e \\ k\in\mathbb{N}}} t(3+k)t(3\cdot 2^e+j). \quad (5)$$

Note that if we let $k=0$, then $t(3+k)=0$, so that we can assume that $k\in\mathbb{N}$. Also, it is easy to see that we can rewrite (5) as

$$h(n)=\sum_{\frac{n}{2^e}-3\leqslant k\leqslant \frac{n}{2^e}} t(3+k)t(3\cdot 2^e+n-2^e k). \quad (6)$$

Note that $n-2^e\left\lfloor \frac{n}{2^e}\right\rfloor = r \mod 2^e$, where r is the residue of n modulo 2^e which lies in the set $\{0,1,\ldots,2^e-1\}$. Thus,

$$\begin{aligned} n-2^e\left\lfloor \frac{n}{2^e}\right\rfloor &= r, \\ n-2^e\left(\left\lfloor \frac{n}{2^e}\right\rfloor -1\right) &= r+2^e, \\ n-2^e\left(\left\lfloor \frac{n}{2^e}\right\rfloor -2\right) &= r+2\cdot 2^e, \end{aligned}$$

where r is defined as above. Thus all the values of $3\cdot 2^e+n-2^e k$ for $k\in\left(\frac{n}{2^e}-3,\frac{n}{3}\right)$ lie in the interval $(3\cdot 2^e+1, 6\cdot 2^e-1)$, and thus (6) becomes

$$h(n)=t\left(3+\left\lfloor \frac{n}{2^e}\right\rfloor\right)t(3\cdot 2^e+r)+t\left(2+\left\lfloor \frac{n}{2^e}\right\rfloor\right)t(4\cdot 2^e+r)+t\left(1+\left\lfloor \frac{n}{2^e}\right\rfloor\right)t(5\cdot 2^e+r).$$

Applying Lemma 5 we have that

$$t(5\cdot 2^e+r)=t(6\cdot 2^e-(2^e-r))=t(3\cdot 2^e+(2^e-r))=t(4\cdot 2^e-r),$$

and thus the above equality becomes

$$h(n)=t\left(3+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(3\cdot 2^e+r)+t\left(2+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(4\cdot 2^e+r)+t\left(1+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(4\cdot 2^e-r),$$

or rather

$$\begin{aligned}h(n)=t\left(3+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(3\cdot 2^e+r)+t\left(2+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(3\cdot 2^e+r+2^e)\\+t\left(1+\left\lfloor\frac{n}{2^e}\right\rfloor\right)t(3\cdot 2^e-r+2^e).\end{aligned}\qquad(7)$$

With this identity for $h(n)$ in hand, we are ready to prove Theorem 1.

3. Proof of Theorem 1

Proof of Theorem 1. Let e and a be nonnegative integers, and let r be an integer with $0\leq r\leq 2^e$. To prove Theorem 1 we will show that the right–hand side of (1) is equal to the left–hand side of (1). To this end, define $b:=2^ea+r$. Then the right–hand side of (1) becomes

$$\begin{aligned}s(2^e(a+2)+r)+s(2^e(a+1)+r)&=s(2^ea+r+2\cdot 2^e)+s(2^ea+r+2^e)\\&=s(b+2\cdot 2^e)+s(b+2^e).\end{aligned}$$

Let k be any positive integer satisfying $k\geq\log_2(a+3)+e$, and note that this implies that $b\leq 2^k$ so that both $b+2\cdot 2^e$ and $b+2^e$ are less than or equal to 2^{k+1}. Then applying Lemma 5 we have

$$s(2^e(a+2)+r)+s(2^e(a+1)+r)=(-1)^k\left[t(3\cdot 2^k+b+2\cdot 2^e)+t(3\cdot 2^k+b+2^e)\right].$$

Since $k\geq e$ and $b\geq 0$ we have $3\cdot 2^k+b\geqslant 2^{e+1}$, so that the previous equality yields

$$(-1)^kg(3\cdot 2^k+b)=s(2^e(a+2)+r)+s(2^e(a+1)+r),\qquad(8)$$

where $g(n)$ is as in (4).

By Theorem 2 we have that $g(n)=h(n)$ for all $n\geq 0$, so that using identity (7) and the definition of b yields

$$\begin{aligned} g(3\cdot 2^k+b) &= t\left(3+\left\lfloor\frac{3\cdot 2^k+b}{2^e}\right\rfloor\right)t(3\cdot 2^e+r)+t\left(2+\left\lfloor\frac{3\cdot 2^k+b}{2^e}\right\rfloor\right)t(3\cdot 2^e+r+2^e)\\ &\quad +t\left(1+\left\lfloor\frac{3\cdot 2^k+b}{2^e}\right\rfloor\right)t(3\cdot 2^e-r+2^e)\\ &= t\left(3+\left\lfloor\frac{3\cdot 2^k+2^e a+r}{2^e}\right\rfloor\right)t(3\cdot 2^e+r)\\ &\quad +t\left(2+\left\lfloor\frac{3\cdot 2^k+2^e a+r}{2^e}\right\rfloor\right)t(3\cdot 2^e+r+2^e)\\ &\quad +t\left(1+\left\lfloor\frac{3\cdot 2^k+2^e a+r}{2^e}\right\rfloor\right)t(3\cdot 2^e-r+2^e)\\ &= t(3\cdot 2^{k-e}+a+3)t(3\cdot 2^e+r)+t(3\cdot 2^{k-e}+a+2)t(3\cdot 2^e+r+2^e)\\ &\quad +t(3\cdot 2^{k-e}+a+1)t(3\cdot 2^e-r+2^e). \end{aligned}$$

Since $k\geq \log_2(a+3)+e$ we have that $a+j\leq 2^{k-e}$ for $j=1,2,3$, and so we may apply Lemma 5. Thus continuing the proceeding equalities by applying Lemma 5 several times, we have

$$g(3\cdot 2^k+b)=(-1)^k\Big[s(a+3)s(r)+s(a+2)s(2^e+r)+s(a+1)s(2^e-r)\Big].$$

We now use Lemma 4 and gather the terms with $s(r)$ and $s(2^e-r)$ to get

$$g(3\cdot 2^k+b)=(-1)^k\Big[s(r)(s(a+3)+s(a+2))+s(2^e-r)(s(a+2)+s(a+1)\Big],$$

so that using the definition of the Stern sequence we have

$$g(3\cdot 2^k+b)=(-1)^k\Big[s(r)s(2a+5)+s(2^e-r)s(2a+3)\Big].$$

To complete the proof we apply identity (8) to the left–hand side to give

$$s(2^e(a+2)+r)+s(2^e(a+1)+r)=(-1)^k g(3\cdot 2^k+b)=s(r)s(2a+5)+s(2^e-r)s(2a+3),$$

which is the desired result. □

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