A NEW APPROACH TO THE RESULTS OF KÖVARI, SÓS, AND TURÁN CONCERNING RECTANGLE-FREE SUBSETS OF THE GRID

Jeremy F. Alm
Department of Mathematics, Illinois College, Jacksonville, Illinois
alm.academic@gmail.com

Jacob Manske[1]
Department of Mathematics, Texas State University, San Marcos, Texas
jmanske@txstate.edu

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Abstract

For positive integers m and n, define $f(m,n)$ to be the smallest integer such that any subset A of the $m \times n$ integer grid with $|A| \geq f(m,n)$ contains a rectangle; that is, there are $x \in [m]$ and $y \in [n]$ and $d_1, d_2 \in \mathbb{Z}^+$ such that all four points (x,y), $(x+d_1,y)$, $(x,y+d_2)$, and $(x+d_1,y+d_2)$ are contained in A. In 1954, Kövari, Sós, and Turán showed that $\lim_{k\to\infty} \frac{f(k,k)}{k^{3/2}} = 1$. They also showed that $f(p^2, p^2+p) = p^2(p+1)+1$ whenever p is a prime number. We recover their asymptotic result and strengthen the second, providing cleaner proofs which exploit a connection to projective planes, first noticed by Mendelsohn. We also provide an explicit lower bound for $f(k,k)$ which holds for all k.

1. Introduction and Motivation

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For $m, n \in \mathbb{Z}^+$, define $f(m,n)$ to be the least integer such that if $A \subseteq [m] \times [n]$ with $|A| \geq f(m,n)$, then A contains a rectangle; that is, there is $x \in [m], y \in [n]$, and $d_1, d_2 \in \mathbb{Z}^+$ such that all four points (x,y), $(x+d_1,y)$, $(x,y+d_2)$, and $(x+d_1,y+d_2)$ are contained in A. For ease in notation, let $f(k) = f(k,k)$. For $c \in \mathbb{Z}^+$, a *c-coloring* of a set S is a surjective map $\chi : S \to [c]$. If χ is constant on a set $A \subset S$, we say that A is *monochromatic*.

We will write $g(k) \sim h(k)$ to mean that functions g and h are *asymptotically equal*; that is, $\lim_{k\to\infty} \frac{g(k)}{h(k)} = 1$. Also, notice that $f(m,n) = f(n,m)$ for any choice of

[1]Corresponding author.

n and m.

The problem of finding bounds or exact values of $f(m,n)$ finds its roots in the famous theorem of van der Waerden from [21], which states that given any positive integers c and d, there exists an integer N such that any c-coloring of $[N]$ contains a monochromatic arithmetic progression of length d. Szemerédi proved a density version of this theorem in [20], using the now well-known Regularity Lemma. Progress in this area is still being made. For instance, in [3], Axenovich and the second author try to find the smallest k so that in any 2-coloring of $[k] \times [k]$ there is a monochromatic *square*; i.e., a rectangle with $d_1 = d_2$. While the upper bounds are enormous, they proved $k \geq 13$; in [4], Bacher and Eliahou show that $k = 15$. In [10], the authors are interested in finding OBS_c, which is the collection of $[m] \times [n]$ grids which cannot be colored in c colors without a monochromatic rectangle, but every proper subgrid can be; see also [7]. For a more complete survey on van der Waerden type problems, see [11].

Zarankiewicz introduced the problem of finding $f(m,n)$ in [22] using the language of minors of (0,1)-matrices. In [12], Kövari, Sós, and Turán show that $f(k) \sim k^{3/2}$ and that whenever p is a prime number, we have $f(p^2+p, p^2) = p^2(p+1)+1$. In this manuscript, we will recover this asymptotic result and strengthen the second result.

In [17], Reiman achieved the bound of

$$f(m,n) \leq \frac{1}{2}\left(m + \sqrt{m^2 + 4mn(n-1)}\right) + 1. \tag{1}$$

Notice that by setting $m = p^2 + p$ and $n = p^2$, the right hand side of (1) becomes $p^2(p+1)+1$, so the result of Kövari, Sós, and Turán implies that the inequality is sharp. Reiman showed equality in (1) in the case that $m = n = q^2+q+1$, provided q is a prime power. In [14], Mendelsohn recovers and strengthens the equality result of Reiman by noticing the connection of the Zarankiewicz problem to projective planes.

A $k \times k$ $(0,1)$-matrix A corresponds to a subset $S_A \subset [k] \times [k]$ by

$$(i,j) \in S \text{ if and only if the } (i,j) \text{ entry of } A \text{ is } 1.$$

Notice that the set S_A contains a rectangle if and only if the matrix $A^T A$ has an entry off the main diagonal which is not equal to 0 or 1. Also notice that $tr(A^T A) = |S_A|$.

Such $(0,1)$-matrices arise in the study of projective planes. A projective plane of order n is an incidence structure consisting of n^2+n+1 points and n^2+n+1 lines such that

(i) any two distinct points lie on exactly one line;

(ii) any two distinct lines intersect in exactly one point;

(iii) each line contains exactly $n+1$ points; and

(iv) there is a set of 4 points such that no 3 of these points lie on the same line.

It is not known for which positive integers n there exists a projective plane of order n; projective planes have been constructed for all prime-power orders, but for no others. In the well-known paper [5], Bruck and Ryser show that if the square-free part of n is divisible by a prime of the form $4k+3$, and if n is congruent to 1 or 2 modulo 4, then there is no projective plane of order n; see also [6]. More recently, the authors in [8] draw a connection between the existence of projective planes of order greater than or equal to 157 and the number of cycles in $n \times n$ bipartite graphs of girth at least 6. In 1989, a computer search conducted by the authors in [13] showed that there is no projective plane of order 10. The smallest order for which it is still not known whether there is a projective plane is 12, although the results in [15, 19, 16, 1, 2] suggest that there is no such structure.

Next we state a lemma which appears in [14] connecting projective planes to the Zarankiewicz problem.

Lemma 1. *If n is a positive integer such that there exists a projective plane of order n, then $f(n^2+n+1) = (n+1)(n^2+n+1)+1$.*

We will include a proof of Lemma 1 both for completeness and since we will reference the lower bound construction in the proof of Theorem 4.

Proof of Lemma 1. Let n be a positive integer such that there is a projective plane of that order. For ease in notation, set $N = n^2+n+1$. First we will show that $f(N) \geq (n+1)N+1$.

We begin by constructing a $N \times N$ $(0,1)$-matrix A. There exists a projective plane P of order n; so let A be the $N \times N$ matrix whose rows correspond to the points of P and whose columns correspond to the lines of P where the (i,j) entry of A is equal to 1 if and only if the point indexed by i lies on the line indexed by j. Since any two distinct lines have exactly one point in common, the scalar product of any two distinct columns must be 1; hence, S_A does not contain a rectangle. Since each line contains exactly $(n+1)$ points, $|S_A| = tr(A^TA) = (n+1)N$, so $f(N) \geq (n+1)N+1$.

Now, suppose A is any $N \times N$ $(0,1)$-matrix with $(n+1)N+1$ nonzero entries, and let a_i denote the number of 1s in row i. The number of pairs of 1s in row i is $\binom{a_i}{2}$, so the total number of pairs of 1s from each row is $\sum_{i=1}^{N}\binom{a_i}{2}$. The number of pairs of distinct column indices is $\binom{N}{2}$. If $\sum_{i=1}^{N}\binom{a_i}{2} > \binom{N}{2}$, the pigeonhole principle implies that there is a pair of column indices such that there are two distinct rows which have 1s in both of those columns; i.e., S_A contains a rectangle.

To see that $\sum_{i=1}^{N}\binom{a_i}{2} > \binom{N}{2}$, recall that the Cauchy-Schwarz inequality gives

$$\left(\sum_{i=1}^{N} a_i\right)^2 \leq \sum_{i=1}^{N} a_i^2 \sum_{i=1}^{N} 1^2. \tag{2}$$

Since $\sum_{i=1}^{N} a_i = (n+1)N + 1$ by assumption, the bound in (2) gives

$$(n+1)^2 N + 2(n+1) + \frac{1}{N} \leq \sum_{i=1} a_i^2. \tag{3}$$

Since $\sum_{i=1}^{N} a_i^2 = \sum_{i=1}^{N} a_i(a_i - 1) + \sum_{i=1}^{N} a_i = 2\sum_{i=1}^{N}\binom{a_i}{2} + (n+1)N + 1$, inequality (3) gives

$$N\left((n+1)^2 - (n+1)\right) + 2(n+1) + \frac{1}{N} - 1 \leq 2\sum_{i=1}^{N}\binom{a_i}{2}. \tag{4}$$

Since $(n+1)^2 - (n+1) = n^2 + n + 1 - 1 = N - 1$, inequality (4) can be rewritten as

$$\frac{N(N-1)}{2} + n + \frac{1}{N} + \frac{1}{2} \leq \sum_{i=1}^{N}\binom{a_i}{2}, \tag{5}$$

and since $n > 0$, the left hand side of (5) is bound from below by $\binom{N}{2}$, as desired. □

It is interesting to note that we have equality in (2) just in case all of the a_i are equal; that is, each row and column contain the same number of 1s.

2. Main Results

Our main lemma is below, a useful proposition for dealing with asymptotic behavior of functions when some explicit values of the functions are known. A version of this lemma is used in [12], but it is neither proved nor explicitly stated.

Lemma 2. *Suppose g and h are monotonically increasing functions. If a_n is a strictly increasing sequence of positive integers such that*

(i) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$;

(ii) $\lim_{n\to\infty} \frac{h(a_{n+1})}{h(a_n)} = 1$; *and*

(iii) $g(a_n) = h(a_n)$ *for all* n,

all hold, then $g \sim h$.

Theorem 3 recovers the asymptotic result of Kövari, Sós, and Turán. Theorem 4 strengthens another of their results. The proofs exploit the connection to projective planes, cleaning up the arguments found in [12]. Theorem 5 is an explicit lower bound for $f(k)$, which holds for all k.

Theorem 3. $f(k) \sim k^{3/2}$.

Theorem 4. *Let* n *be a positive integer. If there is a projective plane of order* n, *then* $f(n^2, n^2+n) = n^2(n+1)+1$.

Theorem 5. *If* $k \in \mathbb{Z}$ *with* $k \geq 3$, *then* $f(k) \geq \frac{1}{16}\left((k+4)\sqrt{4k-3}+5k+22\right)$.

3. Proof of Lemma 2

Now we prove Lemma 2.

Proof. Let g and h be monotonically increasing functions. Suppose a_n is a strictly increasing sequence of positive integers such that $\lim_{n\to\infty} \frac{h(a_{n+1})}{h(a_n)} = 1$ and that $g(a_n) = h(a_n)$ for all n. Let $\varepsilon > 0$. Choose N so that

$$\left|\frac{h(a_{n+1})}{h(a_n)} - 1\right| < \varepsilon \text{ and } \left|\frac{h(a_n)}{h(a_{n+1})} - 1\right| < \varepsilon \tag{6}$$

whenever $n > N$. Next, choose m large enough so that for some $n > N$, we have $a_n \leq m \leq a_{n+1}$. Since g is increasing and g and h agree on the sequence a_n, we have

$$h(a_n) = g(a_n) \leq g(m) \leq g(a_{n+1}) = h(a_{n+1}). \tag{7}$$

Since h is monotone increasing, $h(a_n) \leq h(m) \leq h(a_{n+1})$, so we may transform (7) into

$$\frac{h(a_n)}{h(a_{n+1})} \leq \frac{g(m)}{h(m)} \leq \frac{h(a_{n+1})}{h(a_n)}. \tag{8}$$

Subtracting 1 from every term in (8) and taking absolute values gives that either

$$\left|\frac{g(m)}{h(m)} - 1\right| \leq \left|\frac{h(a_{n+1})}{h(a_n)} - 1\right| \text{ or } \left|\frac{g(m)}{h(m)} - 1\right| \leq \left|\frac{h(a_n)}{h(a_{n+1})} - 1\right|.$$

Without loss of generality, say $\left|\frac{g(m)}{h(m)} - 1\right| \leq \left|\frac{h(a_{n+1})}{h(a_n)} - 1\right|$. By (6), we have

$$\left|\frac{g(m)}{h(m)} - 1\right| < \varepsilon,$$

so $\frac{g}{h} \to 1$ and $g \sim h$, as desired. □

4. Proof of Theorem 3

Now we prove Theorem 3.

Proof. For a positive integer k, set

$$h(k) = \left(\sqrt{k - \frac{3}{4}} + \frac{1}{2}\right) k + 1.$$

Notice that $h(k) \sim k^{3/2}$ and that $h(n^2 + n + 1) = (n + 1)(n^2 + n + 1) + 1$, so by Lemma 1, we have $f(n^2 + n + 1) = h(n^2 + n + 1)$ whenever there is a projective plane of order n. Since there a projective plane of order p for every prime p, we have that f and h agree on an infinite sequence of integers a_n for which $\frac{a_{n+1}}{a_n} \to 1$ (see [18, 9]). Notice that $\frac{h(a_{n+1})}{h(a_n)} \to 1$, so we may apply Lemma 2 to achieve $f \sim h$, and thus $f \sim k^{3/2}$, as desired. □

5. Proof of Theorem 4

Proof. Let n be a positive integer such that there is a projective plane of order n. Set $N = n^2 + n + 1$. As in the proof of Lemma 1, we can construct an $N \times N$ matrix A such that $tr\left(A^T A\right) = (n+1)N$ and that $A^T A$ has only 1s off the main diagonal; hence, the corresponding subset S_A of the $N \times N$ grid has no rectangle.

To construct an $n^2 \times \left(n^2 + n\right)$ matrix B from A, we delete the first column of A along with all rows having a 1 in the first column. Since each row and column of A contains exactly $n + 1$ nonzero entries, we have deleted $n + 1$ rows and 1 column. The resulting matrix B is thus an $n^2 \times \left(n^2 + n\right)$ matrix. Since $A^T A$ has no entries off the main diagonal greater than 1, $B^T B$ has no entries off the main diagonal greater than 1. Since we have deleted $(n + 1)^2$ nonzero entries from A, we have that

$$|S_B| = (n + 1)N - (n + 1)^2 = (n + 1)\left(n^2 + n + 1\right) - (n + 1)^2 = n^2(n + 1),$$

so $f\left(n^2, n^2+n\right) \geq n^2(n+1)+1$.

Using the inequality from Reiman (1),

$$f\left(n^2, n^2+n\right) \leq n^2(n+1)+1,$$

and hence $f\left(n^2, n^2+n\right) = n^2(n+1)+1$, as desired. □

The structure obtained by taking a projective plane and deleting a line together with all of the points on that line is called an *affine plane*. Our result is stronger than that of the authors in [12], since we need only that there is a projective plane of order n, not that n is a prime number.

6. Proof of Theorem 5

Proof. Suppose k is an integer with $k \geq 3$. There exists a nonnegative integer α such that

$$2^{2\alpha} + 2^{\alpha} + 1 \leq k \leq 2^{2\alpha+2} + 2^{\alpha+1} + 1. \tag{9}$$

By focusing on the upper bound from (9), this gives $k \leq \left(2^{\alpha+1}+1/2\right)^2 + 3/4$, or

$$\frac{\sqrt{k-3/4}-1/2}{2} \leq 2^{\alpha}. \tag{10}$$

Let $g(n) = (n+1)(n^2+n+1)+1$, and let $h(k) = \dfrac{\sqrt{k-3/4}-1/2}{2}$. Since g is an increasing function, inequality (10) gives

$$g\left(h(k)\right) \leq g\left(2^{\alpha}\right). \tag{11}$$

By Lemma 1, we have $g(n) = f(n^2+n+1)$ whenever there exists a projective plane of order n. Since there is a projective plane of any prime power order, (11) gives

$$g\left(h(k)\right) \leq f\left(2^{2\alpha} + 2^{\alpha} + 1\right). \tag{12}$$

But since f is increasing, the lower bound in (9) gives $g\left(h(k)\right) \leq f(k)$, and since $g\left(h(k)\right) = \dfrac{1}{16}\left((k+4)\sqrt{4k-3} + 5k + 22\right)$, we have the desired result.

We also note that while $g\left(h(k)\right) \sim \dfrac{1}{8}k^{3/2}$, which is worse than the result in Theorem 3, this lower bound holds for every choice of k, and not just those k for which there exists a projective plane of order k. □

7. Further Research

Trying to find the exact value of $f(m,n)$ without conditions on m and n (that is, removing the extra hypotheses from the results in [12]) would be attractive, although this problem has been open for years, and likely requires a new idea.

The next attractive direction is to take the approach of the authors in [10], and consider colorings of rectangular grids.

Recall that OBS_c is the collection of $[m] \times [n]$ grids which cannot be colored in c colors without a monochromatic rectangle, but every proper subgrid can be. An open problem from [10] is the *rectangle-free conjecture*: if there exists a rectangle-free subset of $[m] \times [n]$ of size $\lceil mn/c \rceil$, then it is possible to color $[m] \times [n]$ in c colors so there is no monochromatic rectangle. Since the authors in [10] have theorems which depend on the rectangle-free conjecture, resolving this conjecture either in the affirmative or the negative would result in progress for obtaining $|\mathrm{OBS}_c|$ or even OBS_c.

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