EDGE ANIMAL WEAK (1,2)-ACHIEVEMENT GAMES

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Received: 12/28/11, Accepted: 11/12/12, Published: 11/16/12

Abstract

A variation of polyform achievement games is studied, in which the cells the players mark are the edges of the three tilings of the plane by regular polygons. Planar game boards whose faces have a bijective correspondence to the edges of the tilings by regular polygons are presented, and all but one of the edge animals on each tiling are characterized as either a winner or loser.

1. Introduction

Abstractions of the game Tic-Tac-Toe called *achievement games* were first introduced by Harary in [6]. The playing *board* is usually an infinite set of *cells*, which is often a regular tiling of the plane by squares [7], by triangles [4] or by hexagons [3, 8, 10]. Other boards such as the platonic solids [2], tilings of the hyperbolic plane [1], and higher dimensional boards [12] have also been studied.

An *animal* is a finite set of connected cells of the board, considered up to congruence. Thus, an animal can be translated, reflected, or rotated on the board and is still considered to be the same animal. In a *weak achievement* game, two players alternate marking empty cells of a *board* with their own marks. The first player (the *maker*) is trying to mark a copy of the goal animal on the board. The second player (the *breaker*) tries to prevent the maker achieving his goal. An animal is called a *winner* if the maker can win the achievement game. The animal is called a *loser* otherwise.

Our goal is to study achievement games on boards where the cells are the edges and not the faces of a tiling of the plane. We call these boards the *edge boards* and call the usual boards with polygonal cells *face boards*. In particular, we are interested in the three *regular edge boards* built from the regular tilings of the plane.

The regular edge boards are fairly complex. On complex playing boards the number of winning animals is usually too large, so we study a *biased* version of the game where the maker marks one cell while the breaker marks two cells each turn. This is called the *weak* $(1, 2)$-*achievement* game. Biased $(1, 2)$ games were studied for example in [5].

Presenting strategies on the edge boards is not ideal because it is hard to attach information to the cells. To avoid this difficulty, we find a face board for each edge board that has equivalent game play.

2. Game Boards

Two cells of an edge board are *adjacent* if they share a common vertex. The situation is not so simple on face boards. We say two cells of a face board are *adjacent* if they share a common edge, and *wildly adjacent* if they share a common edge or a common vertex.

A *wild animal* is a finite wildly connected set of cells. In a wild animal we can get from any cell to any other cell by jumping through cells that are wildly adjacent. Note that every regular animal is also a wild animal.

Our first goal is to switch to face boards. Figure 2.1 shows the three regular edge boards and their corresponding face boards with equivalent game play. The gray squares are holes in the board such that cells are not adjacent on opposite sides of the holes. A corresponding face board is found by drawing faces around the vertices of a representation of the line graph of the tiling.

Adjacency on the hexagonal edge board corresponds to adjacency on the corresponding face board. Adjacency on the triangular and rectangular edge boards correspond to wild adjacency on the corresponding face boards. As a result, regular animals on the triangular and rectangular boards become wild animals on the corresponding face boards. Note that the face boards corresponding to the hexagonal and triangular edge boards seem to be the same but the adjacency relationship is interpreted differently. We call this common face board the *tumbling blocks* board.

We use the notations $E_i^{\triangle}$ for triangular, $E_i^{\square}$ for rectangular and $E_i^{\hexagon}$ for hexagonal edge animals. We use the notation $F_i^{\Diamond}$ for wild face animals on the tumbling blocks board. The regular face animals corresponding to rectangular edge animals are denoted by $F_i^{\hexagon}$. In all cases, the indices get larger with the size of the animals. Figure 2.2 shows all the edge animals with their corresponding face animals up to size two.

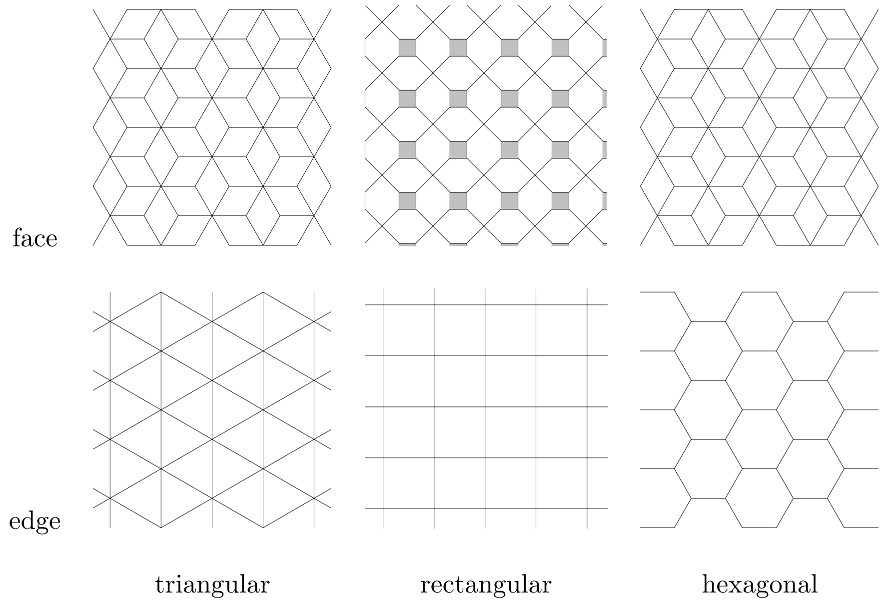

Figure 2.1: Edge boards and their corresponding face boards.

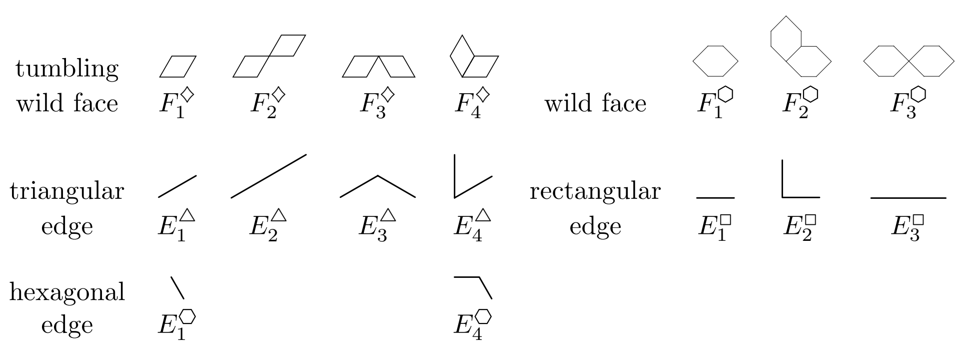

Figure 2.2: All edge animals up to size two and their corresponding face animals. Triangular and rectangular edge animals have wild corresponding face animals. Hexagonal edge animals have regular corresponding face animals. Note that the numbering of the hexagonal edge animals has gaps.

3. Winning Strategies

This section will describe the strategies used by the maker and the breaker. A strategy for the maker can be captured by a *proof sequence* $(s_0, \ldots, s_n)$ of situations [3, 11]. A *situation* $s_i = (C_i, N_i)$ is an ordered pair of disjoint sets of cells. We think of the *core* C_i as a set of cells marked by the maker and the *neighborhood* N_i as a set of cells not marked by the breaker. A situation is the part of the playing board that is important for the maker. A situation does not contain any of the breaker's marks. Those marks are not important as long as the situation contains enough empty cells in the neighborhood. As with animals, congruent situations are considered to be the same. In the situations of a proof sequence, it is always the breaker who is about to mark cells. The game progresses from s_n towards s_0. We require that C_0 is the goal animal and $N_0 = \emptyset$. This means that the maker has already won by marking the cells in C_0 and there is no need for any free cells on the board in N_0. For each $i \in \{1, \ldots, n\}$ we also require that if the breaker marks any two cells in N_i, then the maker can mark a different cell of N_i to reach a position s_j closer to his goal, that is, satisfying $j < i$. More precisely, for all $\{x, y\} \subseteq N_i$ there must be an $\tilde{x} \in N_i \setminus \{x, y\}$ and a $j \in \{0, \ldots, i-1\}$ such that

$$C_j \subseteq C_i \cup \{\tilde{x}\} \text{ and } N_j \subseteq C_i \cup N_i \setminus \{x, y\}.$$

We present proof sequences graphically. Figure 4.3 shows an example. On the figures, filled cells represent the marks of the maker in C_i. Cells with letters in them are the neighborhood cells in N_i that must be unmarked. Each letter represents a possible continuation for the maker. After the marks of the breaker, the maker picks a letter unaffected by the breaker marks. The maker marks the cell with the capital version of this letter. The cells with the lower case version of the chosen letter become the neighborhood cells of the new situation. Each situation is constructed to make sure that the breaker cannot mark two cells which contain every single letter. We include a flow chart for each proof sequence. The letter on the arrows of the flow chart is used to determine which situation the maker can reach by picking that letter. The lack of letters indicate that all choices lead to the same situation.

The most useful strategies for the breaker are based on pairings of the cells of the board. A *double paving* of the board is a symmetric and irreflexive relation on the set of cells where each cell is related to at most 2 other cells. In the visual representation of a double paving, related cells are connected by a line segment. A double paving is said to *kill* an animal if every translation, reflection, and rotation of that animal contains at least one pair of related cells. A killing double paving determines a winning *paving strategy* for the breaker in the $(1, 2)$ game as follows. In each turn, the breaker marks the unmarked cells related to the cell last marked by the maker. If there are fewer than two such cells then she uses her remaining marks randomly. The breaker wins following the paving strategy, since every placement

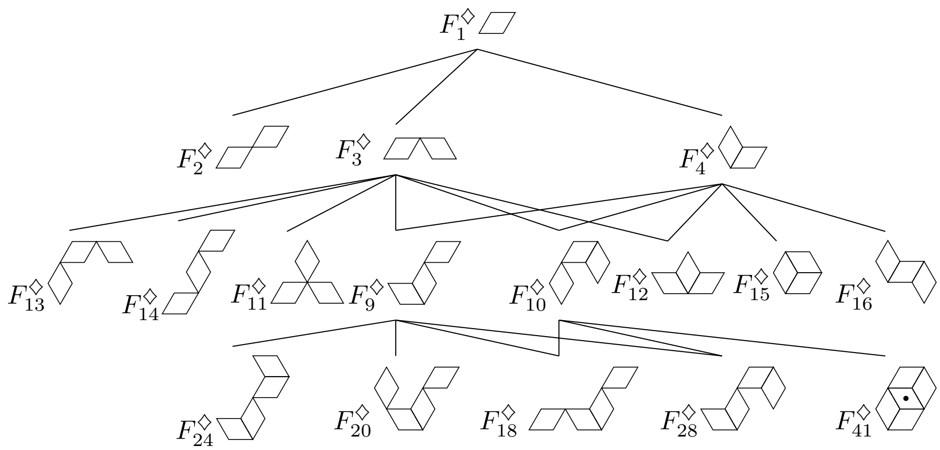

Figure 4.1: The tree of the winners and losers on the tumbling blocks board. Children of losers are not drawn. Animal $F_{28}^\diamondsuit$ remains a mystery but all of its children are losers. The middle cell containing the dot in animal $F_{41}^\diamondsuit$ is an empty cell.

of the goal animal contains a pair of related cells and the maker cannot mark any two related cells.

4. Tumbling Blocks Games

We now turn to the game played with face animals on the tumbling blocks board. We will study all possible animals, regular and wild, in order to find the hexagonal and triangular edge winners. We use the terminology *child* for an animal created from a *parent* animal by adding an extra cell. We collect the size i winning animals in $\mathcal{W}_i$, the size i losing animals in $\mathcal{L}_i$ and the size i potential winning animals in $\mathcal{V}_i$. We start with the animal $F_1^\diamondsuit$ containing only one cell which is clearly a winner. So we let $\mathcal{V}_1 = \mathcal{W}_1 = \{F_1^\diamondsuit\}$ and $\mathcal{L}_1 = \emptyset$. Now we proceed inductively. Any animal containing a losing animal is a loser as well. So the set $\mathcal{V}_{i+1}$ of potential winning animals of size $i+1$ contains all the children of animals in $\mathcal{W}_i$ which are not descendants of any animal in $\mathcal{L}_j$ for $j \leq i$. We analyze the animals in $\mathcal{V}_{i+1}$ and collect the winners in $\mathcal{W}_{i+1}$ and losers in $\mathcal{L}_{i+1}$ so that $\mathcal{V}_{i+1} = \mathcal{W}_{i+1} \cup \mathcal{L}_{i+1}$. The procedure is summarized in Figure 4.1. Each level of the tree shows the elements of $\mathcal{V}_i$. The known winners are the animals with children.

There are three animals in $\mathcal{V}_2 = \{F_2^\diamondsuit, F_3^\diamondsuit, F_4^\diamondsuit\}$. It is clear that $F_3^\diamondsuit$ and $F_4^\diamondsuit$ are winners.

Proposition 4.1. *The animal $F_2^\diamondsuit$ is a loser.*

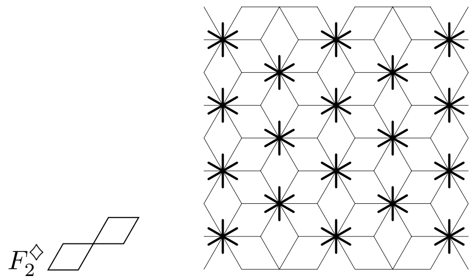

Figure 4.2: The animal $F_2^\diamondsuit$ and the paving that kills it.

Proof. The breaker wins using the paving strategy shown in Figure 4.2. □

There are 8 animals in $\mathcal{V}_3$, shown in the third row of Figure 4.1. Only two of them are winners.

Proposition 4.2. *The animal $F_9^\diamondsuit$ and $F_{10}^\diamondsuit$ are winners.*

Proof. The maker wins using the proof sequences shown in Figures 4.3 and 4.4, respectively. □

Proposition 4.3. *The animals $F_{11}^\diamondsuit$, $F_{12}^\diamondsuit$, $F_{13}^\diamondsuit$, $F_{14}^\diamondsuit$, $F_{15}^\diamondsuit$, and $F_{16}^\diamondsuit$ are losers.*

Proof. The breaker wins using a strategy based on double pavings shown in Figure 4.5. □

We include an alternate breaker strategy for $F_{13}^\diamondsuit$ called *priority strategy.* Priority strategies are new and we believe they have great potential. A more extensive description and several variations with examples are presented in [9].

Proposition 4.4. *The animal $F_{13}^\diamondsuit$ is a loser.*

Proof. Figure 4.6(a) shows a priority strategy for the breaker. The diagrams show the three possible orientations of the current mark of the maker. The cells with numbers in them are the possible response cells. The numbers are the priorities of the response cells. A smaller number represents a higher priority response cell. In each case, the breaker marks two of the unmarked response cells with the highest priorities. If all the response cells are already marked, then the breaker marks random cells.

Figure 4.6(b) shows the six different placements of the goal animal on the board together with their dependency digraphs. The vertex set of the digraph is the set of cells of the goal animal. We use three types of arrows:

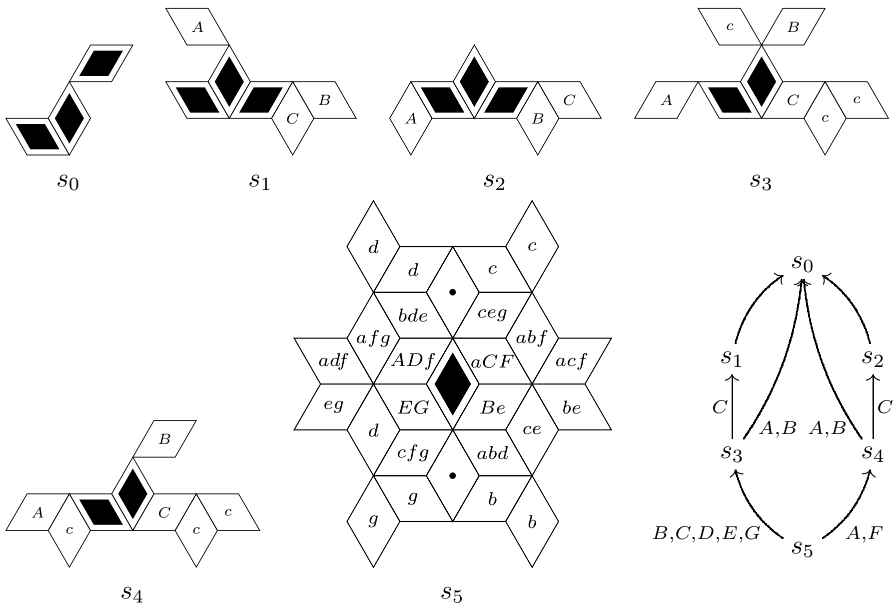

Figure 4.3: A proof sequence for $F_9^\diamond$.

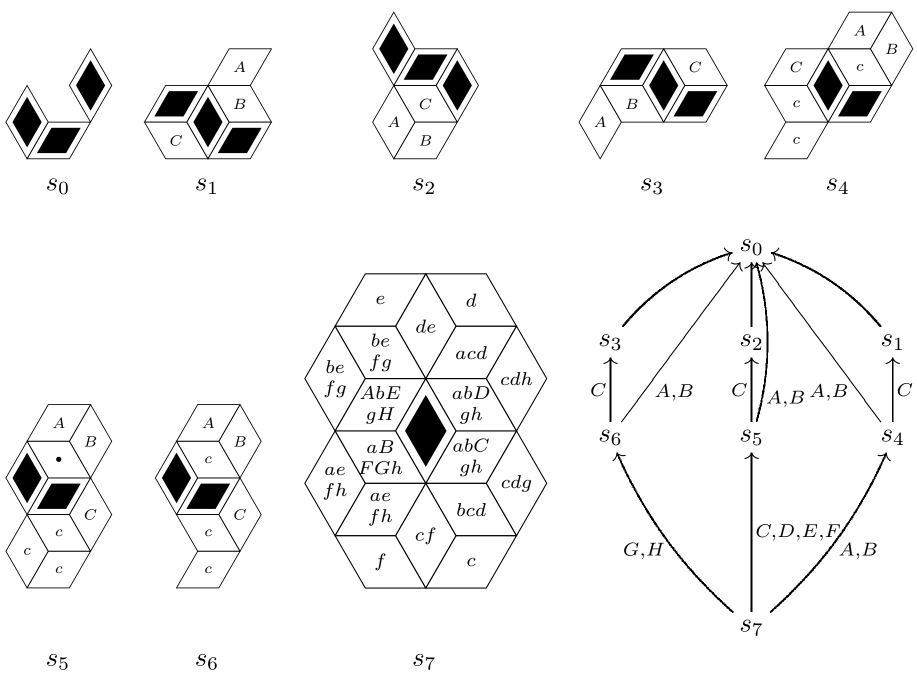

Figure 4.4: A proof sequence for $F_{10}^\diamond$.

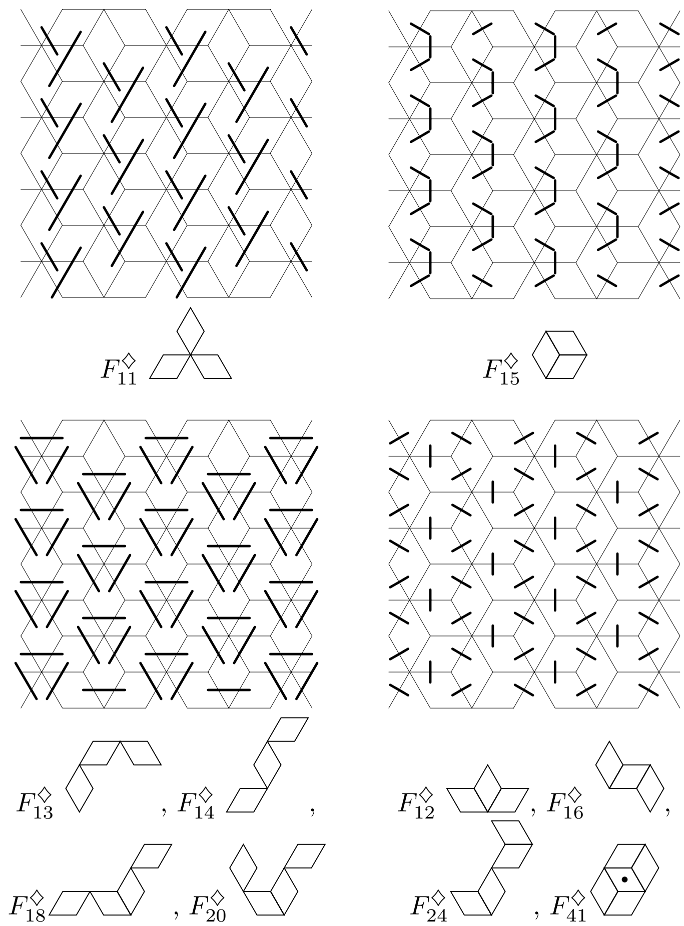

Figure 4.5: Animals and the pavings that kill them.

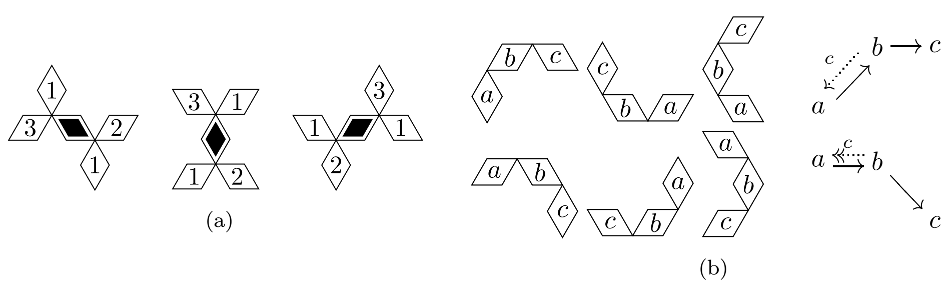

Figure 4.6: (a) The priority strategy for the breaker to prevent the maker from marking $F_{13}^{\diamondsuit}$. (b) Dependency digraphs of the cells in the orientations of $F_{13}^{\diamondsuit}$.

- The *unconditional arrow* $a \longrightarrow b$ indicates that cell b cannot be marked by the maker after cell a because an unmarked cell b is going to be marked by the breaker right after the maker marks cell a. The other unconditional arrow from b to c indicates that cell c cannot be marked by the maker after cell b.

- The *conditional arrow* $b \overset{c}{\dashrightarrow} a$ in the first digraph indicates that cell a cannot be marked by the maker after cell b if cell c has already been marked by either the maker or the breaker in an earlier turn. In this situation, cell a is going to be marked by the breaker right after the maker marks cell b because the priority 1 response cell c is not available so the breaker marks the priority 2 response cell a.

- The *secondary arrow* $b \overset{c}{\cdots\twoheadrightarrow} a$ in the second digraph indicates that cell a cannot be marked by the maker after cell b if cell c is already marked by the maker in an earlier turn. To see this, note that cell c is the priority 1 response cell to cell b and the priority 1 response cell to cell c is the priority 2 response cell to cell b. So if cell c is already marked by the maker then cell a, the priority 3 response cell to b, is going to be marked by the breaker.

It is clear from the digraphs that the maker needs to mark cells a and b in the same turn if he wants to mark the goal animal in any orientation. This is not possible since the maker can only mark one cell in a turn. Thus the goal animal must be a loser. □

There are 5 animals in $\mathcal{V}_4$, shown in the fourth row of Figure 4.1. Four of them are losers.

Proposition 4.5. *The animals* $F^{\Diamond}_{18}$, $F^{\Diamond}_{20}$, $F^{\Diamond}_{24}$, *and* $F^{\Diamond}_{41}$ *are losers.*

Proof. The breaker wins using a strategy based on corresponding double pavings shown in Figure 4.5. □

The animal $F^{\Diamond}_{28}$ remains a mystery. All of its five cell children are losers as shown in Figure 4.7, so we know that no animals with more than four cells can be winners. Figure 4.8 shows a proof sequence in the $(1,1)$-game. This strategy is fairly complex which suggests that $F^{\Diamond}_{28}$ is likely a $(1,2)$-loser. Perhaps a priority strategy could be used to prove this. We used a backtracking search on a computer cluster to show that there is no paving strategy for the breaker in the $(1,2)$-game.

5. Face Board Games Corresponding to Rectangular Edge Games

We now turn to the game played with face animals corresponding to rectangular edge animals. We carry out the procedure described in Section 4 using $\mathcal{V}_i$, $\mathcal{W}_i$

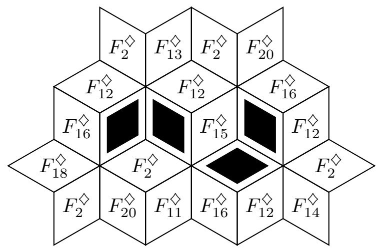

Figure 4.7: Each of the 19 distinct five cell children of $F_{28}^\diamond$ are losers. Adding one of the unmarked cells results in an animal that is the descendant of the losing animal whose name is written in the cell.

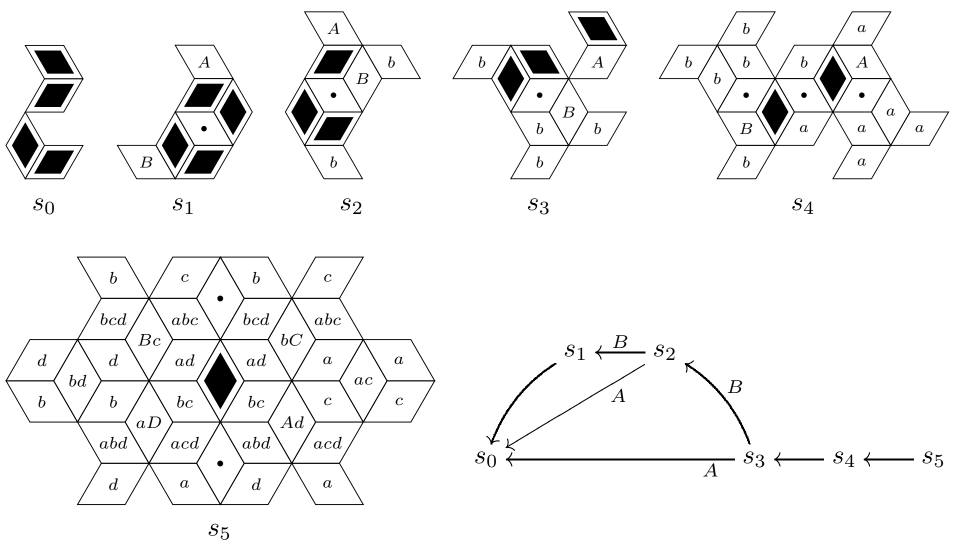

Figure 4.8: A $(1, 1)$-proof sequence for $F_{24}^\diamond$.

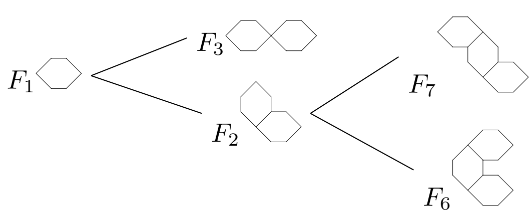

Figure 5.1: The tree of the winners and losers on the face board corresponding to the rectangular edge board. Children of losers are not drawn.

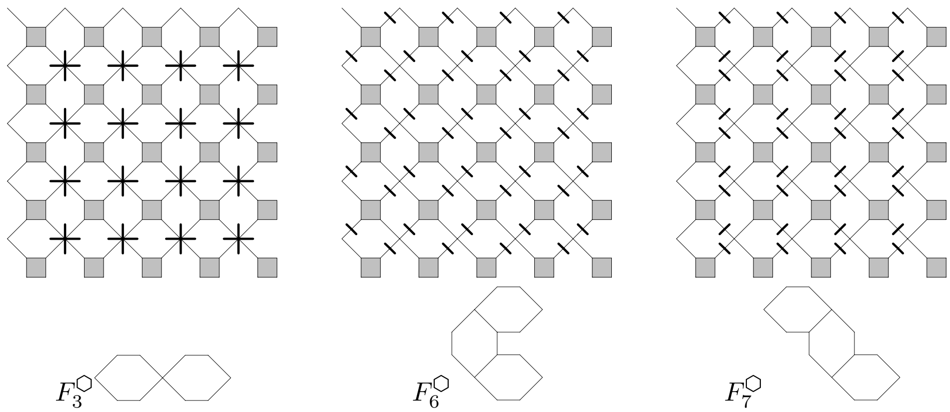

Figure 5.2: Animals and the paving that kills them.

and $\mathcal{L}_i$. The one cell animal is clearly a winner. There are two animals in $\mathcal{V}_2 = \{F_2^{\hexagon}, F_3^{\hexagon}\}$, shown in the second level of Figure 5.1. It is easy to see that $F_2^{\hexagon}$ is a winner.

Proposition 5.1. *The animal $F_3^{\hexagon}$ is a loser.*

Proof. The breaker wins following the strategy based on the double paving shown in Figure 5.2. □

There are two animals in $\mathcal{V}_3 = \{F_6^{\hexagon}, F_7^{\hexagon}\}$, shown in the third level of Figure 5.1.

Proposition 5.2. *The animal $F_6^{\hexagon}$ and $F_7^{\hexagon}$ are losers.*

Proof. The breaker wins following the strategy based on the double paving shown in Figure 5.2. □

Since $\mathcal{V}_4 = \emptyset$, the largest winner has two cells.

6. Edge Winners and Losers

Now we can translate our results to classify the edge animals. Figure 6.1 shows the triangular edge winners and their corresponding face animals. We were not able to classify $F_{28}^{\diamond}$.

Proposition 6.1. *The only winning triangular edge animals in the weak $(1,2)$-achievement game are the animals $E_1^{\triangle}$, $E_3^{\triangle}$, $E_4^{\triangle}$, $E_9^{\triangle}$, $E_{10}^{\triangle}$, and possibly $E_{28}^{\triangle}$.*

Figure 6.2 shows the hexagonal edge winners and their corresponding face animals.

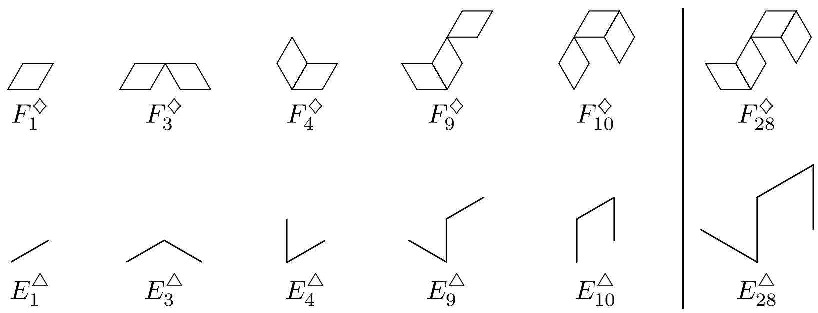

Figure 6.1: Known triangular edge winners and the mystery animal $F_{28}^{\diamond}$ together with their corresponding face animals.

Figure 6.2: Hexagonal edge winners and their corresponding face animals.

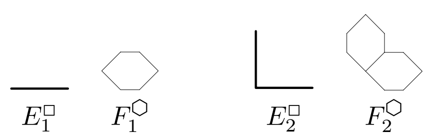

Figure 6.3: Rectangular edge winners and their corresponding face animals.

Proposition 6.2. *The only winning hexagonal edge animals in the weak* $(1,2)$*-achievement game are the animals* $E_1^{\hexagon}$ *and* $E_4^{\hexagon}$.

Figure 6.3 shows the rectangular edge winners and their corresponding face animals.

Proposition 6.3. *The only winning rectangular edge animals in the weak* $(1,2)$*-achievement game are the animals* $E_1^{\square}$ and $E_2^{\square}$.

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